A randomly weighted minimum spanning tree with a random cost constraint

Alan Frieze[*]and Tomasz Tkocz
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA15217
U.S.A.

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Abstract

We study the minimum spanning tree problem on the complete graph where an edge e has a weight W_e and a cost C_e, each of which is an independent uniform $[0,1]$ random variable. There is also a constraint that the spanning tree T must satisfy $C(T) \leq c_0$. We establish the asymptotic value of the optimum weight via the consideration of a dual problem. The proof is therefore constructive i.e. can be thought of as the analysis of a polynomial time algorithm.

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1 Introduction

We consider the minimum spanning tree problem in the context of the complete graph K_n where each edge has an independent uniform $[0,1]$ weight W_e and an independent uniform $[0,1]$ cost C_e. Let $\mathcal{T}$ denote the set of spanning trees of K_n. The weight of a spanning tree T is given by $W(T) = \sum_{e\in T} W_e$ and its cost $C(T)$ is given by $C(T) = \sum_{e\in T} C_e$. The problem we study is

$$\text{Minimise } W(T) \text{ subject to } T \in \mathcal{T},\ C(T) \leq c_0, \tag{1}$$

where c_0 may depend on n. We let $W^* = W^*(c_0) = W(T^*)$ denote the optimum value to (1).

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The unconstrained case of this question ($c_0 = \infty$) has been well studied: Frieze [6], Steele [14], Janson [11], Penrose [13], Frieze and McDiarmid [7], Frieze, Ruszinkó and Thoma [8], Beveridge, Frieze and McDiarmid [2], Li and Zhang [12] and Cooper, Frieze, Ince, Janson and Spencer [5] and is well understood. For example, [5] proves that if L_n denotes the expected minimum weight of a spanning tree then

$$L_n = \zeta(3) + \frac{c_1}{n} + \frac{c_2 + o(1)}{n^{4/3}}$$

for explicitly defined c_1, c_2.

Equation (1) defines a natural problem that has been considered in the literature, in the worst-case rather than the average case. See for example Aggarwal, Aneja and Nair [1] and Guignard and Rosenwein [10] (for a directed version) and Goemans and Ravi [9].

We tackle (1) by considering the dual problem:

$$\text{Maximise } \phi(\lambda) \text{ over } \lambda \geq 0, \text{ where } \phi(\lambda) = \min \{W(T) + \lambda(C(T) - c_0) : T \in \mathcal{T}\}. \tag{2}$$

We note that

$$\text{if } \lambda \geq 0 \text{ and } T \text{ is feasible for (1) then } \phi(\lambda) \leq W(T). \tag{3}$$

We will show that w.h.p.

$$\text{that if } \lambda^* \text{ solves (2) and } T^* \text{ solves (1) then } \phi(\lambda^*) \approx W(T^*). \tag{4}$$

Here $A \approx B$ is an abbreviation for $A = (1 + o(1))B$ as $n \to \infty$, assuming that $A = A(n), B = B(n)$.

We need to make the following definitions:

$$c_1 = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \frac{\Gamma\left(k - \frac{1}{2}\right)}{k!}. \tag{5}$$

$$f(\beta) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} f_k(\beta) \tag{6}$$

where

$$f_k(\beta) = \beta^{1/2} \int_{x=0}^{\beta} x^{k-3/2} e^{-kx} \mathrm{d}x + \int_{x=\beta}^{\infty} x^{k-1} e^{-kx} \mathrm{d}x. \tag{7}$$

Theorem 1. *The following hold w.h.p.:*

(1) If

$$c_0 \in \left[c_1(500 \log n)^{1/2}, \frac{c_1 n}{(8000 \log n)^{1/2}}\right] \tag{8}$$

then

$$W^* \approx \frac{c_1^2 n}{4c_0}. \tag{9}$$

(2) Suppose now that $c_0 = \alpha n$ where $\alpha = O(1)$.

(i) If $\alpha > 1/2$ then

$$W^* \approx \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

(ii) If $0 < \alpha \leq 1/2$ and if $\beta^* = \beta^*(\alpha)$ is the solution to

$$f'(\beta) = 2\alpha, \tag{10}$$

then

$$W^* \approx f(\beta^*) - 2\alpha\beta^*. \tag{11}$$

(3) Suppose now that $c_0 = \alpha$ where $\alpha = O(1)$.

(i) If $\alpha < \zeta(3)$ then there is no feasible solution to (1).

(ii) If $\alpha > \zeta(3)$ and if $\beta^* = \beta^*(\alpha)$ is the solution to

$$f(\beta) - \beta f'(\beta) = \alpha, \tag{12}$$

then

$$W^* \approx \frac{f(\beta^*) - \alpha}{2\beta^*} n. \tag{13}$$

Before proceeding to the proof of Theorem 1, we will check that the claims in (2) and (3) are intuitively reasonable. First consider Case (2). If $\alpha > 1/2$ and if T^* is the tree minimising $W(T)$ then w.h.p. $W(T^*) \approx \zeta(3)$ and $C(T^*) \leq (1 + o(1))n/2$.

We observe next that $f'(\beta) > 0$. This follows directly from

$$f'(\beta) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \beta^{-1/2} \int_0^\beta x^{k-3/2} e^{-kx} \mathrm{d}x. \tag{14}$$

It is shown in an appendix that

$$f'(\beta) \text{ is a strictly monotone decreasing function.} \tag{15}$$

By inspection we see that $f'(\infty) = 0$.

Note also that $f'(0) = 1$ (use L'Hôpital's rule) and

$$f(0) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \int_{x=0}^{\infty} x^{k-1} e^{-kx} \mathrm{d}x = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \cdot \frac{(k-1)!}{k^k} = \zeta(3)$$

and so (10) and (11) are consistent with (i) when $\alpha = 1/2$.

If $\alpha < 1/2$ then from the above properties of f' we see that (10) has a unique positive solution. We derive expression (11) below.

Now consider Case (3). If $\alpha < \zeta(3)$ then w.h.p. there is no tree T with $C(T) < \alpha$. If $g(\beta) = f(\beta) - \beta f'(\beta)$, then $g(0) = \zeta(3)$, $g'(\beta) = -\beta f''(\beta) > 0$ and

$$g(\beta) \geq \frac{\beta^{1/2}}{2} \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \int_{x=0}^{\beta} x^{k-3/2} e^{-kx} \mathrm{d}x \to \infty \text{ as } \beta \to \infty.$$

This implies that (12) has a unique positive solution. We derive expression (13) below.

2 Outline Proof

We use a standard integral formula to compute $\phi(\lambda)$ in Section 3.1. This is straightforward, but lengthy. We then prove conscentration around the mean in Section 3.2. We then use a result of [10] to show in Section 4 that in the cases discussed, the duality gap is negligible w.h.p.

3 Evaluation of the dual problem

3.1 Expectation

Lemma 2. *Let $\lambda \geq 0$ and let L_n be the total weight of a minimum spanning tree in the complete graph on n vertices with each edge e having weight $Z_e = W_e + \lambda C_e$, where W_e and C_e are i.i.d. random variables uniform on $[0,1]$. We have*

1. *If $\frac{2000 \log n}{n} \leq \lambda \leq \frac{n}{2000 \log n}$, then*
$$\mathbf{E} L_n \approx c_1 \sqrt{\lambda n}. \tag{16}$$
2. *If $\lambda < \frac{2000 \log n}{n}$, then*
$$\mathbf{E} L_n \approx \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \Big[\sqrt{\frac{\lambda n}{2}} \int_0^{\frac{\lambda n}{2}} x^{k-3/2} e^{-kx} \mathrm{d}x + \int_{\frac{\lambda n}{2}}^{\infty} x^{k-1} e^{-kx} \mathrm{d}x \Big]. \tag{17}$$
3. *If $\lambda > \frac{n}{2000 \log n}$, then*
$$\mathbf{E} L_n \approx \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \lambda \Big[\sqrt{\frac{n}{2\lambda}} \int_0^{\frac{n}{2\lambda}} x^{k-3/2} e^{-kx} \mathrm{d}x + \int_{\frac{n}{2\lambda}}^{\infty} x^{k-1} e^{-kx} \mathrm{d}x \Big]. \tag{18}$$

The implied $o(1)$ terms in the above expressions can be taken to be independent of λ. Also, we have not optimised all constants.

Proof. Let T be a minimum spanning tree. The starting point is Janson's formula [11],
$$\mathbf{E} L_n = \mathbf{E} \sum_{e \in T} Z_e = \mathbf{E} \sum_{e \in T} \int_0^{\infty} \mathbf{1}_{\{Z_e \geq p\}} \mathrm{d}p = \int_0^{\infty} \mathbf{E} |\{e \in T,\ Z_e \geq p\}| \mathrm{d}p = \int_0^{\infty} \mathbf{E}\big(\kappa(G) - 1\big) \mathrm{d}p, \tag{19}$$
where $\kappa(G)$ is the number of components in the graph G on n vertices with the edge set $\{e :\ Z_e < p\}$. Since the Z_e are i.i.d., this is the random graph $G_{n,\hat{p}}$, with $\hat{p} = \mathbf{Pr}\,(Z_e < p)$. Since $Z_e \leq 1 + \lambda$, $\hat{p} = 1$ for $p > 1 + \lambda$, so the last integral can be taken from 0 to $1 + \lambda$ and after a change of variables $p \leftarrow \frac{p}{1+\lambda}$, we get
$$\mathbf{E} L_n = (1 + \lambda) \int_0^1 \mathbf{E}\big(\kappa(G_{n,\hat{p}(p)}) - 1\big) \mathrm{d}p, \tag{20}$$
where
$$\begin{aligned} \hat{p}(p) = \mathbf{Pr}(Z_e < (1+\lambda)p) &= \mathbf{Pr}\left(\frac{1}{1+\lambda} W_e + \frac{1}{1+\lambda^{-1}} C_e < p \right) \\ &= \left| \left\{ (u,v) \in [0,1]^2,\ \frac{1}{1+\lambda} u + \frac{1}{1+\lambda^{-1}} v \leq p \right\} \right| \end{aligned}$$

where in the last expression $|\cdot|$ denotes Lebesgue measure. An elementary computation (given in an appendix) yields

$$\hat{p}(p) = \begin{cases} \frac{(1+\lambda)(1+\lambda^{-1})}{2}p^2, & 0 \le p \le \frac{1}{1+\max\{\lambda,\lambda^{-1}\}} \\ -\frac{1}{2}\min\{\lambda,\lambda^{-1}\} + p(1+\min\{\lambda,\lambda^{-1}\}), & \frac{1}{1+\max\{\lambda,\lambda^{-1}\}} < p \le \frac{1}{1+\min\{\lambda,\lambda^{-1}\}} \\ 1 - \frac{(1+\lambda)(1+\lambda^{-1})}{2}(1-p)^2, & \frac{1}{1+\min\{\lambda,\lambda^{-1}\}} < p \le 1 \end{cases} \tag{21}$$

Now we can proceed with evaluating $\mathbf{E}L_n$ given by (20). First observe that we have

$$\mathbf{E}\kappa(G_{n,q}) = 1 + o(n^{-200}), \qquad \text{when } \ q \ge \frac{1000\log n}{n}. \tag{22}$$

This is because

$$\begin{aligned} 1 \le \mathbf{E}\kappa(G_{n,q}) &\le 1 + n\mathbf{Pr}(G_{n,q} \text{ is not connected}) \\ &\le 1 + n\sum_{k=1}^{n/2}\binom{n}{k}k^{k-2}q^{k-1}(1-q)^{k(n-k)} \\ &\le 1 + \frac{n}{q}\sum_{k=1}^{n/2}\left(\frac{en}{k}\right)^k k^k e^{-qk(n-k)} \\ &\le 1 + \frac{n^2}{1000\log n}\sum_{k=1}^{n/2}\left(ene^{-\frac{1000\log n}{n}\frac{n}{2}}\right)^k \\ &\le 1 + \frac{n^3}{1000\log n}\frac{e}{n^{499}} \\ &= 1 + o(n^{-200}). \end{aligned} \tag{23}$$

Therefore we can distinguish the following cases depending on the value of λ.

Case 1. $\frac{2000\log n}{n} \le \lambda \le \frac{n}{2000\log n}$. Note that then

$$\hat{p}\left(\frac{1}{1+\max\{\lambda,\lambda^{-1}\}}\right) = \frac{1}{2}\frac{1+\min\{\lambda,\lambda^{-1}\}}{1+\max\{\lambda,\lambda^{-1}\}} = \frac{1}{2}\min\{\lambda,\lambda^{-1}\} \ge \frac{1000\log n}{n},$$

so by (22), the integration over the second and third range from (21) gives the contribution $(1+\lambda)o(n^{-100})$ in (20). Consequently,

$$\mathbf{E}L_n = (1+\lambda)\int_0^{\frac{1}{1+\max\{\lambda,\lambda^{-1}\}}}\mathbf{E}\big(\kappa(G_{n,\frac{(1+\lambda)(1+\lambda^{-1})}{2}p^2}) - 1\big)\mathrm{d}p + (1+\lambda)o(n^{-200}).$$

By the same reason, we also have

$$(1+\lambda)\int_{\frac{1}{1+\max\{\lambda,\lambda^{-1}\}}}^{\sqrt{\frac{2}{(1+\lambda)(1+\lambda^{-1})}}}\mathbf{E}\big(\kappa(G_{n,\frac{(1+\lambda)(1+\lambda^{-1})}{2}p^2}) - 1\big)\mathrm{d}p = \sqrt{2\lambda}o(n^{-200}).$$

Thus

$$\begin{aligned} \mathbf{E}L_n &= (1+\lambda)\int_0^{\sqrt{\frac{2}{(1+\lambda)(1+\lambda^{-1})}}}\mathbf{E}\big(\kappa(G_{n,\frac{(1+\lambda)(1+\lambda^{-1})}{2}p^2}) - 1\big)\mathrm{d}p + (1+\sqrt{2\lambda}+\lambda)o(n^{-200}) \\ &= (1+\lambda)\int_0^{\sqrt{\frac{2}{(1+\lambda)(1+\lambda^{-1})}}}\mathbf{E}\big(\kappa(G_{n,\frac{(1+\lambda)(1+\lambda^{-1})}{2}p^2}) - 1\big)\mathrm{d}p + o(n^{-100}). \end{aligned}$$

Changing the variables yields

$$\mathbf{E}L_n = \sqrt{\frac{\lambda}{2}}\int_0^1 \mathbf{E}\big(\kappa(G_{n,q})-1\big)\frac{\mathrm{d}q}{\sqrt{q}} + o(n^{-100}). \tag{24}$$

It remains to deal with the integral $\int_0^1 \mathbf{E}\big(\kappa(G_{n,q})-1\big)\frac{\mathrm{d}q}{\sqrt{q}}$. As before, thanks to (22), we have

$$\int_0^1 \mathbf{E}\big(\kappa(G_{n,q})-1\big)\frac{\mathrm{d}q}{\sqrt{q}} = \int_0^{\frac{1000\log n}{n}} \mathbf{E}\big(\kappa(G_{n,q})-1\big)\frac{\mathrm{d}q}{\sqrt{q}} + o(n^{-100}). \tag{25}$$

Decompose

$$\kappa(G_{n,q}) = \sum_{k=1}^{k_0} A_k + \sum_{k=3}^{k_0} B_k + R, \tag{26}$$

where A_k is the number of components which are k vertex trees, B_k is the number of non-tree components on k vertices and R is the number of components on at least k_0 vertices. Here we set $k_0 = \log n$.

For the tree components, we have

$$\mathbf{E}A_k = \binom{n}{k}k^{k-2}q^{k-1}(1-q)^{k(n-k)+\binom{k}{2}-k+1}. \tag{27}$$

For $q \le \frac{1000\log n}{n}$ and $k \le \log n$, we have $(1-q)^{-k^2+\binom{k}{2}-k+1} \le e^{qk^2} \le e^{\frac{1000(\log n)^3}{n}} = 1+o(1)$ and $\binom{n}{k} = (1+o(1))\frac{n^k}{k!}$, hence

$$\mathbf{E}A_k = (1+o(1))\frac{n^k}{k!}k^{k-2}q^{k-1}(1-q)^{kn}.$$

Thus

$$\int_0^{\frac{1000\log n}{n}} \mathbf{E}\Big(\sum_{k=1}^{\log n} A_k - 1\Big)\frac{\mathrm{d}q}{\sqrt{q}} = (1+o(1))\sum_{k=1}^{\log n}\int_0^{\frac{1000\log n}{n}} \frac{n^k}{k!}k^{k-2}q^{k-1}(1-q)^{kn}\frac{\mathrm{d}q}{\sqrt{q}} + O\left(\sqrt{\frac{\log n}{n}}\right).$$

Setting $q = \frac{x}{n}$ gives

$$\int_0^{\frac{1000\log n}{n}} \frac{n^k}{k!}k^{k-2}q^{k-1}(1-q)^{kn}\frac{\mathrm{d}q}{\sqrt{q}} = \sqrt{n}\frac{k^{k-2}}{k!}\int_0^{2000\log n} x^{k-1}\left(1-\frac{x}{n}\right)^{kn}\frac{\mathrm{d}x}{\sqrt{x}}.$$

Using $1-t = e^{-t+O(t^2)}$ as $t \to 0$, for $x \le 1000\log n$ and $k \le \log n$, we have $\left(1-\frac{x}{n}\right)^{kn} = e^{-kx+O(\frac{(\log n)^3}{n})} = (1+o(1))e^{-kx}$. Therefore

$$\int_0^{\frac{1000\log n}{n}} \mathbf{E}\Big(\sum_{k=1}^{\log n} A_k - 1\Big)\frac{\mathrm{d}q}{\sqrt{q}} = (1+o(1))\sqrt{n}\sum_{k=1}^{\log n}\frac{k^{k-2}}{k!}\int_0^{1000\log n} x^{k-1}e^{-kx}\frac{\mathrm{d}x}{\sqrt{x}} + O\left(\sqrt{\frac{\log n}{n}}\right).$$

If the integral was from 0 to ∞, we could express it using the gamma function. Since

$$\sqrt{n}\sum_{k=1}^{\log n}\frac{k^{k-2}}{k!}\int_{1000\log n}^{\infty} x^{k-1}e^{-kx}\frac{\mathrm{d}x}{\sqrt{x}} \le \sqrt{1000n\log n}\sum_{k=1}^{\log n}\frac{k^{k-2}}{k!}\int_{1000\log n}^{\infty} x^{k-1}e^{-kx}\mathrm{d}x$$

and for $k=1$ on the right hand side we get $\sqrt{1000n\log n}e^{-1000\log n}=o(n^{-900})$, whereas for $k\ge 2$ we get

$$\begin{aligned}\sqrt{1000n\log n}\sum_{k=2}^{\log n}\frac{k^{k-2}}{k!}\int_{1000\log n}^{\infty}&x^{k-1}e^{-x}e^{-(k-1)\cdot 1000\log n}\mathrm{d}x\\ &\le O(n^{1001})\sum_{k=2}^{\log n}\frac{k^{k-2}}{k!}(k-1)!n^{-1000k}\\ &\le O(n^{1001})\sum_{k=2}^{\log n}\left(\frac{k}{n^{1000}}\right)^{k}=O(n^{-500}).\end{aligned}$$

We can conclude that

$$\int_0^{\frac{1000\log n}{n}}\mathbf{E}\Big(\sum_{k=1}^{\log n}A_k-1\Big)\frac{\mathrm{d}q}{\sqrt{q}}=(1+o(1))\sqrt{n}\sum_{k=1}^{\log n}\frac{k^{k-2}}{k!}\int_0^{\infty}x^{k-3/2}e^{-kx}\mathrm{d}x\\ +O\left(\sqrt{\frac{\log n}{n}}\right).$$

It remains to compute the sum over k. We have

$$\sum_{k=1}^{\log n}\frac{k^{k-2}}{k!}\int_0^{\infty}x^{k-3/2}e^{-kx}\mathrm{d}x=\sum_{k=1}^{\log n}\frac{k^{k-2}}{k!}\frac{\sqrt{k}}{k^k}\Gamma\left(k-\frac{1}{2}\right)=\sum_{k=1}^{\log n}\frac{1}{k^{3/2}}\frac{\Gamma\left(k-\frac{1}{2}\right)}{k!}. \tag{28}$$

Since for $k\ge 3$, $\Gamma(k-1/2)\le\Gamma(k)=(k-1)!$, the series converges and we have

$$\int_0^{\frac{100\log n}{n}}\mathbf{E}\Big(\sum_{k=1}^{\log n}A_k-1\Big)\frac{\mathrm{d}q}{\sqrt{q}}=(1+o(1))c_0\sqrt{n}, \tag{29}$$

where

$$c_0=\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\frac{\Gamma\left(k-\frac{1}{2}\right)}{k!}. \tag{30}$$

To bound the contribution form non-tree components, note that

$$\mathbf{E}B_k\le\binom{n}{k}k^kq^k(1-q)^{k(n-k)}\le\left[enqe^{-qn}\right]^ke^{qk^2}. \tag{31}$$

Thus

$$\begin{aligned}\int_0^{\frac{1000\log n}{n}}\mathbf{E}\Big(\sum_{k=3}^{\log n}B_k\Big)\frac{\mathrm{d}q}{\sqrt{q}}&\le e^{\frac{1000\log n}{n}(\log n)^2}\sum_{k=3}^{\log n}\int_0^{\frac{1000\log n}{n}}\left[enqe^{-qn}\right]^k\frac{\mathrm{d}q}{\sqrt{q}}\\ &\le(1+o(1))(\log n)\int_0^{\frac{1000\log n}{n}}\left[enqe^{-qn}\right]^3\frac{\mathrm{d}q}{\sqrt{q}}\\ &=O(\log n)\frac{1}{\sqrt{n}}\int_0^{1000\log n}x^{5/2}e^{-3x}\mathrm{d}x,\end{aligned}$$

so

$$\int_0^{\frac{1000\log n}{n}}\mathbf{E}\Big(\sum_{k=3}^{\log n}B_k\Big)\frac{\mathrm{d}q}{\sqrt{q}}=O\left(\frac{\log n}{\sqrt{n}}\right). \tag{32}$$

Finally, for the large components, since

$$R \leq \frac{n}{k_0}, \tag{33}$$

we get $R \leq \frac{n}{\log n}$, so we have

$$\int_0^{\frac{1000 \log n}{n}} \mathbf{E}(R) \frac{\mathrm{d}q}{\sqrt{q}} \leq 2\sqrt{\frac{1000 \log n}{n}} \frac{n}{\log n} = O\left(\frac{\sqrt{n}}{\sqrt{\log n}}\right). \tag{34}$$

Combing (29), (32), (34) with (26) and plugging into (25), we obtain

$$\int_0^1 \mathbf{E}\big(\kappa(G_{n,q}) - 1\big) \frac{\mathrm{d}q}{\sqrt{q}} = (1 + o(1))c_0\sqrt{n}.$$

In view of (24) this gives (16).

Case 2. $\lambda < \frac{2000 \log n}{n}$. Then plainly $\min\{\lambda, \lambda^{-1}\} = \lambda$ and $\max\{\lambda, \lambda^{-1}\} = \lambda^{-1}$. Since $\hat{p}(p) \geq \hat{p}(\frac{1}{1+\lambda}) = 1 - \frac{\lambda}{2}$, for $p \geq \frac{1}{1+\lambda}$, in view of (22), the third range in (21), that is $\frac{1}{1+\lambda} < p \leq 1$, gives the contribution $(1 + \lambda)o(n^{-200}) = o(n^{-200})$ in (20). For the remaining two ranges, changing the variables $q = \hat{p}(p)$ in (20) gives

$$\mathbf{E}L_n = \sqrt{\frac{\lambda}{2}} \int_0^{\lambda/2} \mathbf{E}\Big[\kappa(G_{n,q}) - 1\Big] \frac{\mathrm{d}q}{\sqrt{q}} + \int_{\lambda/2}^{1-\lambda/2} \mathbf{E}\Big[\kappa(G_{n,q}) - 1\Big] \mathrm{d}q + o(n^{-100}).$$

By (22), for the second integral we get

$$\int_{\lambda/2}^{1-\lambda/2} \mathbf{E}\Big[\kappa(G_{n,q}) - 1\Big] \mathrm{d}q = \int_{\lambda/2}^{\frac{1000 \log n}{n}} \mathbf{E}\Big[\kappa(G_{n,q}) - 1\Big] \mathrm{d}q + o(n^{-200}),$$

so

$$\mathbf{E}L_n = \sqrt{\frac{\lambda}{2}} \int_0^{\lambda/2} \mathbf{E}\Big[\kappa(G_{n,q}) - 1\Big] \frac{\mathrm{d}q}{\sqrt{q}} + \int_{\lambda/2}^{\frac{1000 \log n}{n}} \mathbf{E}\Big[\kappa(G_{n,q}) - 1\Big] \mathrm{d}q + o(n^{-100}). \tag{35}$$

We again decompose $\kappa(G_{n,q})$ as in (26). Here we set $k_0 = (\log n)^2$. First we show that the B_k and R have small contribution in the integrals above. By (31),

$$\begin{aligned}
\sqrt{\frac{\lambda}{2}} \int_0^{\lambda/2} \mathbf{E}\Big[\sum_{k=3}^{k_0} B_k\Big] \frac{\mathrm{d}q}{\sqrt{q}} &\leq \sum_{k=3}^{k_0} \sqrt{\frac{\lambda}{2}} \int_0^{\lambda/2} \big[enqe^{-qn}\big]^k e^{qk^2} \frac{\mathrm{d}q}{\sqrt{q}} \\
&\leq e^{\frac{\lambda}{2}k_0^2} \sum_{k=3}^{k_0} \sqrt{\frac{\lambda}{2}} \int_0^{\infty} \big[enqe^{-qn}\big]^k \frac{\mathrm{d}q}{\sqrt{q}} \\
&\leq e^{\frac{1000(\log n)k_0^2}{n}} \sqrt{\frac{\lambda}{2}} \sum_{k=3}^{k_0} \frac{1}{\sqrt{n}} \int_0^{\infty} \big[exe^{-x}\big]^k \frac{\mathrm{d}x}{\sqrt{x}} \\
&\leq e^{\frac{1000(\log n)k_0^2}{n}} \sqrt{\frac{1000 \log n}{n}} \frac{k_0}{\sqrt{n}} \int_0^{\infty} \big[exe^{-x}\big]^3 \frac{\mathrm{d}x}{\sqrt{x}} \\
&= O\left(\frac{(\log n)^{5/2}}{n}\right).
\end{aligned}$$

and similarly

$$\begin{aligned}\int_{\lambda/2}^{\frac{1000\log n}{n}} \mathbf{E}\Big[\sum_{k=3}^{k_0} B_k\Big]\mathrm{d}q &\le \sum_{k=3}^{k_0}\int_{\lambda/2}^{\frac{1000\log n}{n}} \big[enqe^{-qn}\big]^k e^{qk^2}\mathrm{d}q\\ &\le e^{\frac{1000(\log n)k_0^2}{n}}\frac{k_0}{n}\int_0^\infty \big[exe^{-x}\big]^3\mathrm{d}x\\ &= O\left(\frac{(\log n)^2}{n}\right).\end{aligned}$$

By (33),

$$\begin{aligned}\sqrt{\frac{\lambda}{2}}\int_0^{\lambda/2}\mathbf{E}R\frac{\mathrm{d}q}{\sqrt{q}}+\int_{\lambda/2}^{\frac{1000\log n}{n}}\mathbf{E}R\,\mathrm{d}q &\le \frac{n}{k_0}\left(\sqrt{\frac{\lambda}{2}}\int_0^{\lambda/2}\frac{\mathrm{d}q}{\sqrt{q}}+\int_{\lambda/2}^{\frac{1000\log n}{n}}\mathrm{d}q\right)\\ &\le \frac{n}{k_0}\left(\lambda+\frac{1000\log n}{n}\right)\\ &= O\left(\frac{1}{\log n}\right).\end{aligned}$$

Putting the last three estimates together with (35) yields

$$\begin{aligned}\mathbf{E}L_n &= \sqrt{\frac{\lambda}{2}}\int_0^{\lambda/2}\mathbf{E}\Big[\sum_{k=1}^{k_0}A_k-1\Big]\frac{\mathrm{d}q}{\sqrt{q}}+\int_{\lambda/2}^{\frac{1000\log n}{n}}\mathbf{E}\Big[\sum_{k=1}^{k_0}A_k-1\Big]\mathrm{d}q+O\left(\frac{1}{\log n}\right)\\ &= \sqrt{\frac{\lambda}{2}}\int_0^{\lambda/2}\mathbf{E}\Big[\sum_{k=1}^{k_0}A_k\Big]\frac{\mathrm{d}q}{\sqrt{q}}+\int_{\lambda/2}^{\frac{1000\log n}{n}}\mathbf{E}\Big[\sum_{k=1}^{k_0}A_k\Big]\mathrm{d}q+O\left(\frac{1}{\log n}\right).\end{aligned}\tag{36}$$

Using (27) and repeating verbatim the arguments following it to bound $1-q$, to change the variables $q=\frac{x}{n}$ and to replace $\left(1-\frac{x}{n}\right)^{kn}$ with e^{-kx}, we obtain

$$\begin{aligned}\mathbf{E}L_n = (1+o(1))\sum_{k=1}^{k_0}\frac{k^{k-2}}{k!}\Big[\sqrt{\frac{\lambda n}{2}}\int_0^{\frac{\lambda n}{2}}x^{k-3/2}e^{-kx}\mathrm{d}x+\int_{\frac{\lambda n}{2}}^{1000\log n}x^{k-1}e^{-kx}\mathrm{d}x\Big]&\\ +O\left(\frac{1}{\log n}\right)&.\end{aligned}$$

As in Case 1, $\sum_{k=1}^{k_0}\frac{k^{k-2}}{k!}\int_{1000\log n}^{\infty}x^{k-1}e^{-kx}\mathrm{d}x=O(n^{-100})$, so we can replace the integral $\int_{\frac{\lambda n}{2}}^{1000\log n}x^{k-1}e^{-kx}\mathrm{d}x$ with$\int_{\frac{\lambda n}{2}}^{\infty}x^{k-1}e^{-kx}\mathrm{d}x$. Moreover, crude estimates show that

$$\begin{aligned}&\sum_{k=k_0}^{\infty}\frac{k^{k-2}}{k!}\Big[\sqrt{\frac{\lambda n}{2}}\int_0^{\frac{\lambda n}{2}}x^{k-3/2}e^{-kx}\mathrm{d}x+\int_{\frac{\lambda n}{2}}^{\infty}x^{k-1}e^{-kx}\mathrm{d}x\Big]\\ &\le \sum_{k=k_0}^{\infty}\frac{k^{k-2}}{k!}\Big[\sqrt{1000\log n}\int_0^\infty x^{k-3/2}e^{-kx}\mathrm{d}x+\int_0^\infty x^{k-1}e^{-kx}\mathrm{d}x\Big]\\ &= \sum_{k=k_0}^{\infty}\frac{k^{k-2}}{k!}\Big[\sqrt{1000\log n}\frac{\Gamma(k-\frac{1}{2})}{k^{k-1/2}}+\frac{\Gamma(k)}{k^k}\Big]\\ &\le \sqrt{1000\log n}\sum_{k=k_0}^{\infty}k^{-5/2}+\sum_{k=k_0}^{\infty}k^{-3}=O\left(\frac{1}{(\log n)^{5/2}}\right).\end{aligned}$$

Thus finally

$$\mathbf{E}L_n = (1+o(1))\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}\Big[\sqrt{\frac{\lambda n}{2}}\int_0^{\frac{\lambda n}{2}} x^{k-3/2}e^{-kx}\mathrm{d}x + \int_{\frac{\lambda n}{2}}^{\infty} x^{k-1}e^{-kx}\mathrm{d}x\Big] + O\left(\frac{1}{\log n}\right).$$

Note that in the first integral, we have $\sqrt{\frac{\lambda n}{2}}\frac{1}{\sqrt{x}} \geq 1$, hence the main term (the sum over k) is lower-bounded by $\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}\int_0^{\infty} x^{k-1}e^{-kx}\mathrm{d}x = \zeta(3)$ and consequently, the $O\left(\frac{1}{\log n}\right)$ term can be incorporated into the $o(1)$ term, which gives (17).

Case 3. $\lambda > \frac{n}{2000\log n}$. Then plainly $\min\{\lambda,\lambda^{-1}\} = \lambda^{-1}$ and $\max\{\lambda,\lambda^{-1}\} = \lambda$. Changing the variables $q = \hat{p}(p)$ in (20) yields

$$\begin{aligned}\mathbf{E}L_n &= \sqrt{\frac{\lambda}{2}}\int_0^{\frac{1}{2\lambda}} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\frac{\mathrm{d}q}{\sqrt{q}} \\ &\quad + \lambda\int_{\frac{1}{2\lambda}}^{1-\frac{1}{2\lambda}} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\mathrm{d}q \\ &\quad + \sqrt{\frac{\lambda}{2}}\int_{1-\frac{1}{2\lambda}}^{1} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\frac{\mathrm{d}q}{\sqrt{1-q}}.\end{aligned}$$

Since $1-\frac{1}{2\lambda} \geq \frac{1000\log n}{n}$, in view of (22), the third integral gives

$$\sqrt{\frac{\lambda}{2}}\int_{1-\frac{1}{2\lambda}}^{1} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\frac{\mathrm{d}q}{\sqrt{1-q}} = o(n^{-200})\sqrt{\frac{\lambda}{2}}\int_{1-\frac{1}{2\lambda}}^{1}\frac{\mathrm{d}q}{\sqrt{1-q}} = o(n^{-200}).$$

Similarly, for the second integral we have

$$\lambda\int_{\frac{1000\log n}{n}}^{1-\frac{1}{2\lambda}} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\mathrm{d}q = \lambda o(n^{-100})\int_{\frac{1000\log n}{n}}^{1-\frac{1}{2\lambda}}\mathrm{d}q = \lambda o(n^{-100}).$$

Thus we can write (we incorporate the term $o(n^{-200})$ in $\lambda o(n^{-200})$)

$$\mathbf{E}L_n = \lambda\left(\sqrt{\frac{1}{2\lambda}}\int_0^{\frac{1}{2\lambda}} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\frac{\mathrm{d}q}{\sqrt{q}} + \int_{\frac{1}{2\lambda}}^{\frac{1000\log n}{n}} \mathbf{E}\Big[\kappa(G_{n,q})-1\Big]\mathrm{d}q + o(n^{-200})\right).$$

The expression in the bracket is exactly (35) with λ being replaced by λ^{-1}. Therefore, from (17), we obtain (18). □

Lemma 3. *With the notation of Lemma 2, if $\lambda = O(n)$, we have*

$$\mathbf{E}L_n = \begin{cases}\Theta((\lambda n)^{1/2}) & \textit{Case 1.}\\ \Theta(\max\left\{1,(\lambda n)^{1/2}\right\}) & \textit{Case 2.}\\ \Theta((\lambda n)^{1/2}) & \textit{Case 3.}\end{cases} \tag{37}$$

and with probability $1-o(n^{-200})$,

$$Z_{\max} = \begin{cases}O\left(\left(\frac{\lambda\log n}{n}\right)^{1/2}\right) & \textit{Case 1.}\\ O\left(\frac{\log n}{n}\right) & \textit{Case 2.}\\ O(\log n) & \textit{Case 3.}\end{cases} \tag{38}$$

where $Z_{\max} = \max\{Z_e : e \in T^*\}$ *and* T^* *is the minimum spanning tree with weights* Z_e*.*

Also in Case 3 we have

$$C_{\max} = O\left(\frac{(\log n)^{1/2}}{n^{1/2}}\right), \tag{39}$$

where $C_{\max} = \max\{C_e : e \in T^*\}$*.*

Proof. The claims concerning $\mathbf{E}L_n$ follow directly from (16), (17), (18).

To justify (38), fix p_0 and let $X = |\{e \in T^*,\ Z_e > (1+\lambda)p_0\}|$ be the number of edges on the minimum spanning tree having weights Z_e above $(1+\lambda)p_0$. By Janson's formula from [11], $X = \kappa(G_{n,\hat{p}(p_0)})-1$ with $\hat{p}$ given by (21). By the first moment, $\mathbf{Pr}(X > 0) \leq \mathbf{E}X = \mathbf{E}[\kappa(G_{n,\hat{p}(p_0)}) - 1]$. By (23), choosing p_0 such that $\hat{p}(p_0) = \frac{1000 \log n}{n}$ gives $X = 0$, equivalently $Z_e \leq (1+\lambda)p_0$, with probability $1 - o(n^{-200})$. It remains to bound $(1+\lambda)p_0$. In Case 1, we see from (21) that $\frac{(1+\lambda)(1+\lambda^{-1})}{2}p_0^2 = \frac{1000 \log n}{n}$, so $(1+\lambda)p_0 = \sqrt{2\lambda}\sqrt{\frac{1000 \log n}{n}} = O(\sqrt{\frac{\lambda \log n}{n}})$. In Case 2 we see that we have to use the second formula in (21) and $p_0(1+\lambda) = \frac{1000 \log n}{n} + \frac{1}{2}\lambda = O(\frac{\log n}{n})$. Similarly in Case 3, $p_0(1+\lambda^{-1}) = \frac{1000 \log n}{n} + \frac{1}{2\lambda}$, hence $p_0(1+\lambda) = O(\log n)$.

For (39), we note that $\mathbf{Pr}(W_e \leq q, C_e \leq q) = q^2$. Putting $q = (1000 \log n/n)^{1/2}$ we see that with the required probability, the random graph G_{n,q^2} is connected. This implies that with the same probability there is a spanning tree T with $Z_e \leq (1+\lambda)q\ \forall e \in T$. It follows that a spanning tree that minimises Z will have $Z_{\max} \leq (1+\lambda)q$. (Applying the greedy algorithm will finish before needing an edge with $Z_e > (1+\lambda)q$.) So $Z_{\max} \leq (1+\lambda)q$ and consequently $C_{\max} \leq 2q$. $\square$

3.2 Concentration

The goal of this section is to prove that for any $\epsilon = \frac{1}{\log n}$,

$$\mathbf{Pr}(|L_n - \mathbf{E}(L_n)| \geq \epsilon \mathbf{E}(L_n)) = o(n^{-100}).$$

And this immediately implies that

$$\mathbf{Pr}(|\phi(\lambda) - \mathbf{E}(\phi(\lambda))| \geq \epsilon(\mathbf{E}(L_n))) = o(n^{-200}) \tag{40}$$

In our analysis we consider separately the contribution of long and short edges. Let $L = n^{1/10}\mathbf{E}(L_n)/n$ and let Y_L denote the total cost of the edges used on the minimum spanning tree with $Z_e \leq L$. Let $N = \binom{n}{2}$ and note that Y_L is a function of N i.i.d. random variables $X_1, \ldots, X_N$.

We will show Y_L is concentrated using a variant of the Symmetric Logarithmic Sobolev Inequality from [3]. Let $Y'_{L,i}$ denote the same quantity as Y_L, but with the variable X_i replaced by an independent copy X'_i. Then a simplified form of the Symmetric Logarithmic Sobolev Inequality [3, Corollary 3] says that if

$$\mathbf{E}\left(\sum_{i=1}^{N}(Y_L - Y'_{L,i})^2 \mathbf{1}_{Y_L > Y'_{L,i}} | X_1, \ldots, X_N\right) \leq c$$

then for all $t > 0$,

$$\mathbf{Pr}[Y_L > \mathbf{E}Y_L + t] \leq e^{-t^2/4c},$$

and if

$$\mathbf{E}\left(\sum_{i=1}^{N}(Y'_{L,i}-Y_L)^2\mathbf{1}_{Y'_{L,i}>Y_L}\big|X_1,\ldots,X_N\right)\leq c$$

then for all $t>0$,

$$\mathbf{Pr}(Y_L<\mathbf{E}Y_L-t)\leq e^{-t^2/4c}.$$

Changing the value of one edge can change the value of Y_L by at most L, so $(Y_L-Y'_{L,i})^2<L^2$. Let I denote the indices of the edges which contribute to Y_L. If $i\notin I$ then $Y'_{L,i}<Y_L$ implies $X'_i\leq L$. So

$$\sum_{i=1}^{N}(Y_L-Y'_{L,i})^2\mathbf{1}_{Y_L>Y'_{L,i}}\leq\sum_{i\in I}L^2+\sum_{i\notin I}L^2\mathbf{1}_{X'_i\leq L}.$$

Now $\mathbf{Pr}(X'_i<L)\leq\mathbf{Pr}(W_e\leq L,\ \lambda C_e\leq L)\leq L/\lambda^*$ where $\lambda^*=\max\{\lambda,1\}$. Then, since there are less than n terms in the first sum and less than n^2 terms in the second sum, we have

$$\mathbf{E}\left(\sum_{i=1}^{N}(Y_L-Y'_{L,i})^2\mathbf{1}_{Y_L>Y'_{L,i}}\big|X_1,\ldots,X_N\right)\leq L^2n+L^3n^2/\lambda^*.$$

If $i\notin I$ then we also have that $Y'_{L,i}>Y_L$ implies $X'_i\leq L$. So we also have

$$\mathbf{E}\left(\sum_{i=1}^{N}(Y'_{L,i}-Y_L)^2\mathbf{1}_{Y'_{L,i}<Y_L}\big|X_1,\ldots,X_N\right)\leq L^2n+L^3n^2/\lambda^*.$$

Therefore,

$$\begin{aligned}\mathbf{Pr}\left[|Y_L-\mathbf{E}Y_L|\geq\epsilon\mathbf{E}(L_n)\right]&\leq 2\exp\left\{-\frac{\epsilon^2\mathbf{E}(L_n)^2}{4(L^2n+L^3n^2/\lambda^*)}\right\}=2\exp\left\{-\frac{\epsilon^2}{4(n^{2/10}/n+n^{3/10}\mathbf{E}(L_n)/(n\lambda^*)})\right\}\\&\leq 2\exp\left\{-\frac{\epsilon^2}{4(n^{-4/5}+A\frac{n^{-7/10}\max\{1,\sqrt{\lambda n}\}}{\max\{1,\lambda\}})}\right\}\leq 2\exp\left\{-\frac{\epsilon^2 n^{1/5}}{A'}\right\}=o(n^{-200}),\end{aligned}\tag{41}$$

where we have used $\mathbf{E}(L_n)\leq A\max\{1,(\lambda n)^{1/2}\}$, see Lemma 3 and A,A' are universal constants.

Let Y'_L denote the total cost of the edges used with edge cost at least L. We have from Lemma 3 that for some $B>0$, with probability $1-o(n^{-200})$,

$$Z_{\max}\leq\begin{cases}B\left(\frac{\lambda\log n}{n}\right)^{1/2}\leq L=\Theta\left(\frac{n^{1/10}}{n}\sqrt{\lambda n}\right) & \text{Case 1.}\\[2ex] \frac{B\log n}{n}\leq L=\Omega\left(\frac{n^{1/10}}{n}\right) & \text{Case 2.}\\[2ex] B\log n\leq L=\Theta\left(\frac{n^{1/10}}{n}\sqrt{\lambda n}\right)=\Omega(n^{1/20}) & \text{Case 3.}\end{cases}\tag{42}$$

And so $Y'_L=0$ with probability $1-o(n^{-200})$.

3.3 Optimising over λ

The first thing to observe is that ϕ is a concave function of λ, see for example Boyd and Vandenberghe [4].This is because it is the minimum of a collection of linear functions. Ignoring the $(1+o(1))$ factor, it will be differentiable. It follows then that we can maximise $\phi(\lambda)$ by setting its (asymptotic) derivative to zero.

Case 1: (8) holds.
Suppose now that we divide the interval $I = \left[\frac{2000\log n}{n}, \frac{n}{2000\log n}\right]$ into n^4 sub-intervals of equal length less than n^{-3}. Suppose that the ith interval is $[\lambda_i, \lambda_{i+1}]$. We observe that for any spanning tree T we have that for $\lambda \in [\lambda_i, \lambda_{i+1}]$,

$$|(W(T) + \lambda_i C(T)) - (W(T) + \lambda C(T))| \leq \frac{1}{n^3}$$

and so

$$|\phi(\lambda_i) - \phi(\lambda)| \leq \frac{1}{n^3} + c_0|\lambda_i - \lambda| \leq \frac{2}{n^3}. \tag{43}$$

So, maximising ϕ over $\lambda_1, \lambda_2, \ldots, \lambda_{n^3}$ makes an error in maximising $\phi(\lambda)$ over I of at most $2n^{-3}$.

If $\lambda \in I$, then using the concentration result (40) of Section 3.2, we see that for a fixed $\lambda = \lambda_i$ we have

$$\phi(\lambda) = \mathbf{E}\phi(\lambda) + \epsilon\mathbf{E}L_n = (1+\epsilon)\mathbf{E}L_n - \lambda c_0 = (1+o(1))c_1\sqrt{\lambda n} - \lambda c_0 \text{ with probability } 1 - o(n^{-200}). \tag{44}$$

We see therefore that w.h.p. the expression for $\lambda = \lambda_i$ in (44) holds simultaneously for all $i = 1, 2, \ldots, n^4$. Differentiating the expression $\phi(\lambda)$ and setting it to zero we see that ϕ is maximised at

$$\lambda^* = (1+o(1))\frac{c_1^2 n}{4c_0^2} \tag{45}$$

and that $\phi(\lambda^*) = (1+o(1))\frac{c_1^2 n}{4c_0}$. We then note that $\lambda^* \in I$ for c_0 as in (a) and then the fact that ϕ is concave and is maximised where $\phi'(\lambda) = 0$ now implies that in this case w.h.p.

$$\max_\lambda \phi(\lambda) = (1+o(1))\frac{c_1^2 n}{4c_0} \tag{46}$$

Case 2: $c_0 = \alpha n$ where $0 < \alpha < 1/2$.
We proceed as in Case 1. We argue that if $c_0 = \alpha n$ and we minimise the expression for $\mathbf{E}L_n$ in (17) less λc_0 then this is optimised at $\lambda^* = o(\log n/n)$ and this is sufficient to imply that in this case w.h.p.

$$\max_\lambda \phi(\lambda) = (1+o(1))(f(\beta^*) - 2\alpha\beta^*), \tag{47}$$

where β^* is the unique solution to $f'(\beta) = 2\alpha$, see (10), (11).

Putting $\beta = \lambda n/2$ and $c_0 = \alpha n$ into the expression in (17) we get

$$\phi(\beta) = (1+o(1))\sum_{k=1}^\infty \frac{k^{k-2}}{k!}\beta^{1/2}\int_0^\beta x^{k-3/2}e^{-kx}\mathrm{d}x + \int_\beta^\infty x^{k-1}e^{-kx}\mathrm{d}x - 2\alpha\beta = (1+o(1))f(\beta) - 2\alpha\beta.$$

Differentiating w.r.t. β we get

$$\phi'(\beta) = (1+o(1))f'(\beta) - 2\alpha \tag{48}$$

and hence the solution β^* to $\phi'(\beta) = 0$ asymptotically satisfies $f'(\beta) = 2\alpha$. Clearly $\beta^* = \Theta(1)$ which implies that $\lambda^* = O(1/n)$ and so $\lambda^* = o(\log n/n)$ as claimed. It then follows that w.h.p.

$$\max_{\lambda} \phi(\lambda) = (1 + o(1))(f(\beta^*) - 2\alpha\beta^*). \tag{49}$$

Case 3: $c_0 = \alpha$ where $\alpha > \zeta(3)$.
In this case we put $\beta = n/2\lambda$ and proceed as in Case 2. Putting $\beta = \lambda n/2$ and $c_0 = \alpha$ into the expression in (18) we get

$$\phi(\beta) = (n + o(n)) \left(\frac{1}{2} \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \beta^{-1/2} \int_0^{\beta} x^{k-3/2} e^{-kx} \mathrm{d}x + \beta^{-1} \int_{\beta}^{\infty} x^{k-1} e^{-kx} \mathrm{d}x \right) - \frac{\alpha}{2\beta} = (n + o(n)) \frac{f(\beta)}{2\beta} - \frac{\alpha n}{2\beta}.$$

Differentiating w.r.t. β we get

$$\phi'(\beta) = (n + o(n)) \left(\frac{f'(\beta)}{2\beta} - \frac{f(\beta)}{2\beta^2} \right) - \frac{\alpha n}{2\beta^2}$$

and hence the solution to $\phi'(\beta) = 0$ asymptotically satisfies $f(\beta) - \beta f'(\beta) = \alpha$. Clearly $\beta^* = \Theta(1)$ which implies that $\lambda^* = \Omega(n)$. It then follows that w.h.p.

$$\max_{\lambda} \phi(\lambda) = (n + o(n)) \frac{f(\beta^*) - \alpha}{2\beta^*}. \tag{50}$$

Note that in all cases, $\lambda^* = O(n)$ and so Lemma 3 applies to λ^* in Section 3.2.

4 Proof of Theorem 1

We will use the following theorem from Goemans and Ravi [10]:

Theorem 4. *There exists a spanning tree $\tilde{T}$ such that $W(\tilde{T}) \leq \phi(\lambda^*) \leq W^*$ and $C(\tilde{T}) \leq c^* + C_{\max}(\tilde{T})$, where $C_{\max}(\tilde{T})$ is the maximum cost of an edge of $\tilde{T}$.*

For Cases 1 and 2 from Lemma 2 we let $\hat{c}_0 = c_0 - \delta$ where $\delta = \frac{2}{\lambda^*} B R_{38}$ where B is a suitable hidden constant for (38) and R_{38} is the RHS of (38). Suppose now that we replace c_0 by $\hat{c}_0$ and let $\hat{W}$ denote the minimum weight of a tree with cost at most $\hat{c}_0$. Applying Theorem 4 we obtain a spanning tree $\hat{T}$ such that $W(\hat{T}) \leq \phi(\hat{\lambda}) \leq \hat{W}$ and $c(\hat{T}) \leq \hat{c}_0 + \frac{1}{\lambda^*} B R_{38} \leq c_0$. It only remains to show that w.h.p. $\phi(\hat{\lambda}) \approx W^*$. This follows from our expressions for $\phi(\lambda^*)$ in Section 3.3 and the fact that $\hat{c}_0 \approx c_0$, which we verify now.

In Case 1 we have from (45) that,

$$\frac{\delta}{c_0} \leq O\left(\sqrt{\frac{\log n}{\lambda^* n c_0^2}} \right) = O\left(\frac{\sqrt{\log n}}{n} \right) = o(1).$$

In Case 2 we have $\delta = O\left(\frac{\log n}{\lambda^* n}\right)$, $c_0 = \Omega(n)$, $\lambda^* = O(\frac{1}{n})$ and so $\delta/c_0 = O\left(\frac{\log n}{n}\right) = o(1)$.

For Case 3 we let $\delta = 1/\log n$ and proceed as above. We find that once again $\phi(\hat{\lambda}) \approx W^*$ because of the expression (50) for $\phi(\lambda^*)$ in Section 3.3 and the fact that $\hat{c}_0 \approx c_0$. We then use Theorem 4 and (39) to show that

$$C(\hat{T}) \leq \hat{c}_0 + O\left(\left(\frac{\log n}{n} \right)^{1/2} \right) = c_0 - \frac{1}{\log n} + O\left(\left(\frac{\log n}{n} \right)^{1/2} \right) \leq c_0.$$

This completes the proof of Theorem 1.

5 Conclusion

We have determined the asymptotic optimum value to problem (1) w.h.p. The proof is constructive in that we can w.h.p. get an asymptotically optimal solution (1) by computing $\hat{T}$ of the previous section. Our theorem covers almost all of the possibilities for c_0, although there are some small gaps between the 3 cases.

The present result assumes that cost and weight are independent. It would be more reasonable to assume some positive correlation. This could be the subject of future research. One could also consider more than one constraint, but then we might lose Theorem 4.

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A Proof of (15)

We want to show that h is strictly decreasing on $(0, +\infty)$, where

$$h(\beta) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \beta^{-1/2} \int_0^{\beta} x^{k-3/2} e^{-kx} \mathrm{d}x. \tag{51}$$

We have

$$-2\beta^{3/2} h'(\beta) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \left[\int_0^{\beta} x^{k-3/2} e^{-kx} \mathrm{d}x - 2\beta^{k-1/2} e^{-k\beta} \right].$$

Call the right hand side $H(\beta)$. We want to show that it is positive for every $\beta > 0$. We have $H(0) = 0$, so it is enough to show that $H'(\beta)$ is positive for every $\beta > 0$. We have

$$H'(\beta) = 2\beta^{-1/2} \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} \left[k\beta^k - (k-1)\beta^{k-1} \right] e^{-k\beta}$$

and want to show that the sum on the right hand side is positive for every $\beta > 0$. Note that for $\beta \geq 1$, we have $k\beta^k - (k-1)\beta^{k-1} > 0$ for every $k \geq 1$, so the sum is positive in this case. Let $0 < \beta < 1$. Separating the first two terms, we rewrite the condition that the sum is positive as

$$\beta e^{-\beta} + \frac{1}{2}(2\beta^2 - \beta) e^{-2\beta} > \sum_{k=3}^{\infty} \frac{k^{k-2}}{k!} \left[k - 1 - k\beta \right] \beta^{k-1} e^{-k\beta}.$$

Equivalently, multiplying by $\beta^{-1} e^{2\beta}$, we want to show that for every $0 < \beta < 1$,

$$e^{\beta} + \beta - \frac{1}{2} > \sum_{k=3}^{\infty} \frac{k^{k-2}}{k!} \left[k - 1 - k\beta \right] \left(\beta e^{-\beta} \right)^{k-2}.$$

Let $0 < \beta \leq \frac{2}{5}$. Estimating crudely $k - 1 - k\beta < k - 1$, using $k! > \sqrt{2\pi} k^{k+1/2} e^{-k}$ and then bounding $\frac{k-1}{k^{5/2}} \leq \frac{2}{3^{5/2}}$ for $k \geq 3$, we get

$$\begin{aligned} \sum_{k=3}^{\infty} \frac{k^{k-2}}{k!} \left[k - 1 - k\beta \right] \left(\beta e^{-\beta} \right)^{k-2} &< \frac{2e^2}{3^{5/2}\sqrt{2\pi}} \sum_{k=3}^{\infty} \left(\beta e^{1-\beta} \right)^{k-2} \\ &= \frac{2e^2}{3^{5/2}\sqrt{2\pi}} \frac{\beta e^{1-\beta}}{1 - \beta e^{1-\beta}}. \end{aligned}$$

Moreover, we have

$$\frac{2e^2}{3^{5/2}\sqrt{2\pi}} \frac{\beta e^{1-\beta}}{1 - \beta e^{1-\beta}} < e^{\beta} + \beta - \frac{1}{2}, \qquad 0 < \beta \leq \frac{2}{5}, \tag{52}$$

(shown below) which finishes the proof in this case.

Let $\frac{2}{5} < \beta < 1$. Estimating crudely $k - 1 - k\beta < k - 1 - \frac{2}{5}k = \frac{3}{5}k - 1$, using $k! > \sqrt{2\pi} k^{k+1/2} e^{-k}$ and then bounding $\left(\beta e^{1-\beta} \right)^{k-2} < \beta e^{1-\beta}$ for $k \geq 3$, we get

$$\begin{aligned} \sum_{k=3}^{\infty} \frac{k^{k-2}}{k!} \left[k - 1 - k\beta \right] \left(\beta e^{-\beta} \right)^{k-2} &< \left(\sum_{k=3}^{\infty} \frac{\frac{3}{5}k - 1}{k^{5/2}} \right) \frac{e^2}{\sqrt{2\pi}} \beta e^{1-\beta} \\ &< \frac{3}{5} \frac{e^2}{\sqrt{2\pi}} \beta e^{1-\beta}, \end{aligned}$$

where it can be checked numerically that $\sum_{k=3}^{\infty} \frac{\frac{3}{5}k-1}{k^{5/2}} < \frac{3}{5}$. Moreover, we have

$$\frac{3e^2}{5\sqrt{2\pi}}\beta e^{1-\beta} < e^\beta + \beta - \frac{1}{2}, \qquad \frac{2}{5} < \beta < 1, \tag{53}$$

(shown below) which finishes the proof in this case.

It remains to prove (52) and (53).

Showing (52) is equivalent to showing that the function

$$u(\beta) = \left(e^\beta + \beta - \frac{1}{2}\right)(1 - \beta e^{1-\beta}) - \frac{2e^3}{3^{5/2}\sqrt{2\pi}}\beta e^{-\beta}$$

is positive on $(0, \frac{2}{5})$. We numerically check that $u(\frac{2}{5}) > 0.1$ and it suffices to show that u is decreasing on $(0, \frac{2}{5})$. We find that

$$e^\beta u'(\beta) = e^{2\beta} + (1-e)e^\beta + e\beta^2 + \left(\frac{2e^3}{3^{5/2}\sqrt{2\pi}} - \frac{5e}{2}\right)\beta + \frac{e}{2} - \frac{2e^3}{3^{5/2}\sqrt{2\pi}}.$$

Call the right hand side $\tilde{u}(\beta)$. We have $\tilde{u}(0) < -0.3$ and for $0 < \beta < \frac{2}{5}$,

$$\begin{aligned}\tilde{u}'(\beta) &= 2e^{2\beta} + (1-e)e^\beta + 2e\beta + \frac{2e^3}{3^{5/2}\sqrt{2\pi}} - \frac{5e}{2} \\ &< 2e^{4/5} + 1 - e + \frac{4e}{5} + \frac{2e^3}{3^{5/2}\sqrt{2\pi}} - \frac{5e}{2} < -0.8\end{aligned}$$

which shows that $\tilde{u}$ decreases, hence $\tilde{u}(\beta)$ is negative, hence $u'(\beta)$ is negative, hence u decreases.

Showing (53) is equivalent to showing that the function

$$v(\beta) = e^\beta + \beta - \frac{1}{2} - \frac{3e^3}{5\sqrt{2\pi}}\beta e^{-\beta}$$

is positive on $(\frac{2}{5}, 1)$. For $\frac{2}{5} < \beta < 1$, we have

$$\begin{aligned}v'(\beta) &= e^\beta + 1 - \frac{3e^3}{5\sqrt{2\pi}}(1-\beta)e^{-\beta} \\ &> e^{2/5} + 1 - \frac{3e^3}{5\sqrt{2\pi}}\frac{3}{5}e^{-2/5} > 0.5\end{aligned}$$

(we used that $(1-\beta)e^{-\beta}$ decreases on $(0, 2)$). This shows that v increases on $(\frac{2}{5}, 1)$, hence $v(\beta) > v(\frac{2}{5}) > 0$ for $\frac{2}{5} < \beta < 1$.

B Proof of (21)

We need to compute the surface area of the subset $\left\{(u, v) \in [0, 1]^2,\ \frac{1}{1+\lambda}u + \frac{1}{1+\lambda^{-1}}v \le p\right\}$ of the unit square $[0, 1]^2$. The line $\frac{1}{1+\lambda}u+\frac{1}{1+\lambda^{-1}}v = p$ intersects the u and v axes respectively at $u_0 = p(1+\lambda)$ and $v_0 = p(1+\lambda^{-1})$. Thus when both u_0 and v_0 are less than 1, the subset is a right triangle whose area is $\frac{1}{2}u_0v_0$. This gives the formula in the first case of (21). When exactly one of u_0 and v_0 is less than 1 and the other one is greater than 1, the subset is a trapezoid and computing its area gives the formula in the second case of (21). Finally, if both u_0 and v_0 are greater than 1, the subset is the complement of a right triangle and the formula in the third case of (21) follows from the first one by changing p to $1 - p$ and taking the complement.