Feasible Banzhaf Power Distributions for Five-Player Weighted Voting Systems[1]

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In 1954, Lloyd Shapley and Martin Shubik introduced **[SS]** a method of measuring the voting efficacy of a single voting party (whether an individual or a bloc), as a percentage of the total amount of "power" present in a weighted voting system. Throughout this paper we shall use the term *player* to describe a voting party who must cast all of its votes in favor of a measure, or against the measure. We may briefly describe the Shapley-Shubik approach as considering all permutations of the players and determining the unique player in each which "swings" the vote from one outcome to another, provided all players up to and including the "swing player" vote together. If we equate *power* with the potential for casting deciding votes, then the power of a player is proportional to the number of permutations for which that player is the swing player.

In many applications, the order in which the players vote is not a factor or is simply not applicable, e.g., if all players vote at once by secret ballot. In these contexts, the Banzhaf model **[B]** provides an alternative for describing how the power is distributed.

We assume a finite number of players $P_1, P_2, \ldots, P_n$. Player P_i casts a positive integer number of votes, v_i. The number q of votes needed to pass a motion shall be called the *quota*, and we assume

$$\frac{v_1 + \cdots + v_n}{2} < q \leq v_1 + \cdots + v_n$$

and that q is an integer. We represent a *weighted voting system* (WVS) by

$$[q; v_1, v_2, \ldots, v_n]$$

and assume $v_1 \geq v_2 \geq \ldots \geq v_n$.

We shall also stipulate that $v_i < q$ holds for each i; otherwise it would be possible for P_i to vote alone and pass a motion. But in addition, we assume that for all i,

$$\sum_{j \neq i} v_j \geq q.$$

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If this were not true for some i, then P_i would have the power to prevent any motion from passing; the votes of all the other players would not meet the quota. We call such a condition *veto power*.

A *coalition* in a WVS is a subset of the players. A *winning coalition* is a coalition for which the combined votes of the players meet or exceed the quota. Otherwise, we have a *losing coalition*. We call a player in a winning coalition *critical* if the removal of that player results in a losing coalition. The presence of a critical player in a winning coalition will be called a *critical instance*; if a winning coalition has k critical players, we shall say that there are k critical instances corresponding to that coalition.

We are now ready to define the *Banzhaf power index* for a player P_i (or we may say the *Banzhaf power of* P_i) as the ratio of the number of instances in which P_i is critical to the total number of critical instances. We shall denote this ratio by $B(P_i)$. Note that since each $B(P_i)$ is a percentage, we have

$$B(P_1) + \cdots + B(P_n) = 1.$$

The *Banzhaf power distribution* associated with a WVS shall be denoted by

$$\beta = (B(P_1), \ldots, B(P_n)).$$

We shall call two weighted voting systems *equivalent* if β is the same for each.

We note that $i \geq j$ implies $v_i \geq v_j$, and in turn $B(P_i) \geq B(P_j)$. Hence

$$B(P_1) \geq \cdots \geq B(P_n).$$

We also have that $v_i = v_j$ implies $B(P_i) = B(P_j)$; however, the converse is not true, and in some sense this is what makes the whole endeavor to model power interesting. We may (and often do) discover that two players have the same power index even though one player casts many more votes than the other. The point is that the ratios v_i/v_j are not reliable indicators of comparative power.

Practical considerations make the absence of veto power a reasonable stipulation, but as a mathematical notion, Banzhaf power is still an applicable idea if veto power is present. One can also lift the assumption that $v_1, \ldots, v_n$ and q are integers, and one may even consider *low quota weighted voting systems*, i.e., those for which

$$0 \leq q \leq \frac{v_1 + \cdots + v_n}{2},$$

and extend the notion of the Banzhaf power distribution to this setting. However, by lifting the requirement

$$q > \frac{v_1 + \cdots + v_n}{2},$$

we are no longer assured that the complement of a winning coalition is a losing coalition, and so the terms *winning* and *losing* serve merely to sort the power set of $\{P_1, \ldots, P_n\}$ into two subcollections. As a game-theoretic notion, the low quota WVS is meaningful (and might better be called simply a *voting game*, as in [**DS**] and [**TZ**]), but for most applications, a low quota is unacceptable.

In short, our results apply if the voters' weights and the quota are merely nonnegative real numbers, but do not apply in the case of a low quota or in the case that veto power is present.

Now let us observe that if we have $v_i \geq q$ for any i, then P_i has veto power, since $\sum_{j \neq i} v_j < q$. So the requirement that veto power be absent automatically guarantees that every player's weight is below the quota. Next we note that it is enough to ask that P_1 not have veto power, for then we have

$$v_2 + v_3 + \cdots + v_n \geq q,$$

which expresses the fact that $\{P_2, P_3, \ldots, P_n\}$ is a winning coalition. It follows that if we replace any player P_i, $i \neq 1$, in this coalition with P_1, then we still have a winning coalition, since $v_1 \geq v_i$, so that

$$\sum_{j \neq i} v_j \geq q. \tag{1}$$

Hence (1) holds for all i if it holds for $i = 1$. We observe, then, that veto power is absent from a WVS if and only if

$$\{P_2, P_3, \ldots, P_n\}$$

is a winning coalition.

In view of the above observations, we can present a fairly simple proof of the following result.

Theorem 1: In any WVS with no veto power, we must have $B(P_3) > 0$.

Proof: Begin by considering the coalition $\{P_2, P_3\}$. If this is a winning coalition for the WVS, then both P_2 and P_3 are critical, so that we immediately have $B(P_3) > 0$. If, on the other hand, $\{P_2, P_3\}$ is a losing coalition, then so is $\{P_2, P_n\}$, since $v_n \leq v_3$.

Now consider $\{P_2, P_3, P_n\}$. If this coalition is a winner, it follows that P_3 is a critical player, since $\{P_2, P_n\}$ loses. Therefore $B(P_3) > 0$. But if $\{P_2, P_3, P_n\}$ loses, then by replacing P_3 with P_{n-1}, we obtain another losing coalition $\{P_2, P_{n-1}, P_n\}$.

Now consider $\{P_2, P_3, P_{n-1}, P_n\}$. If this coalition wins, then P_3 is critical, and $B(P_3) > 0$. Otherwise, the coalition loses, and it follows that

$$\{P_2, P_{n-2}, P_{n-1}, P_n\}$$

loses as well, as we have replaced P_3 with the weaker player P_{n-2}. If we add P_3 to this coalition and find that

$$\{P_2, P_3, P_{n-2}, P_{n-1}, P_n\}$$

wins, then P_3 is critical for this coalition, so that $B(P_3) > 0$.

The idea is to continue examining coalitions of the form

$$\{P_2, P_3, P_{n-k}, \dots, P_{n-1}, P_n\}$$

until we find one in which P_3 is critical. At worst, one finally arrives at the losing coalition

$$\{P_2, P_4, P_5, \dots, P_n\},$$

and observes that if this coalition loses, then P_3 is critical for the coalition

$$\{P_2, P_3, P_4, P_5, \dots, P_n\},$$

which is guaranteed to be a winner due to the absence of veto power. The result is proved.

A natural question to consider at this point is whether P_4 is guaranteed some positive percentage of Banzhaf power, if $n \geq 4$. The answer is no, for given n, we can construct the WVS

$$[2(n-2); n-2, n-2, n-2, 1, 1, \dots, 1], \tag{2}$$

which meets all of our requirements. For example,

$$\frac{v_1 + \cdots + v_n}{2} = \frac{3(n-2) + n - 3}{2} = \frac{4n-9}{2}$$
$$= 2n - \frac{9}{2} < 2n - 4 = q < 4n - 9 = v_1 + \cdots + v_n.$$

Veto power is absent since $v_2 + v_3 + \cdots + v_n = 3n - 7 > q$. Players P_1, P_2 and P_3 have positive power, since each two-player coalition involving these players wins, with each player involved critical. But noting that every winning coalition contains at least two of P_1, P_2, P_3, we see that P_4 cannot be critical in any winning coalition. The same is true for $P_5, \dots, P_n$. Hence $B(P_1) = B(P_2) = B(P_3) = 1/3$ and $B(P_4) = \cdots = B(P_n) = 0$.

Now we turn our attention to weighted voting systems of given sizes; by the "size" of a WVS we mean the number of players, n. We wish to explore how many distinct Banzhaf power distributions are feasible for a given value of n, under the restriction that veto power is absent.

We note first that if $n = 3$, then we must have $\beta = (1/3, 1/3, 1/3)$; this is the only power distribution possible. So all weighted voting systems of size 3 (with no veto power) are equivalent, since the winning coalitions must be precisely

$$\{P_1, P_2\}, \ \{P_1, P_3\}, \ \{P_2, P_3\}, \ \text{and} \ \{P_1, P_2, P_3\}.$$

In each of the two-player coalitions, both players are critical, but in the three-player coalition, no player is critical. So we have six critical instances; each player is critical twice and therefore has Banzhaf power $2/6 = 1/3$. Examples are $[13; 12, 11, 2]$, $[66; 64, 55, 11]$, and $[2000; 1999, 1999, 1]$. Despite the very different appearances of these systems and the disparity between v_1 and v_3, each player has precisely the same potential for criticality, so that all three systems are equivalent to $[2; 1, 1, 1]$.

In the case $n = 4$, we have the following result, proved in **[To]**.

Theorem 2: In any four-player WVS with no veto power, there are only five feasible power distributions:

(a) $\beta = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

(b) $\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$

(c) $\beta = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

(d) $\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$

(e) $\beta = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}\right)$

The proof is merely an exercise in considering the possible compositions for the collection of winning coalitions, and there are but five cases. So each case corresponds to a distinct power distribution. In every case, the absence of veto power guarantees that all four 3-player coalitions win (and of course the 4-player coalition wins as well but contributes no critical instances); the cases are thus distinguished by which 2-player coalitions are winners. All cases result in a total of 12 critical instances, but this is a special feature of 4-player systems; as we shall see, the total number of critical instances is a characteristic of the WVS itself and is not determined solely by n.

Specific "canonical"[5] examples of 4-player weighted voting systems are given below, along with more complex equivalent systems. The point is that in practice, any 4-player WVS with no veto power is equivalent to and may be replaced by one of the canonical forms to make the power structure more evident, since the canonical examples have the property that $v_i = v_j$ implies $B(P_i) = B(P_j)$.

[5] We hesitate to provide a precise definition of *canonical* and use it here as an informal notion. We are not claiming any form we are calling canonical is in every case the *unique* WVS with no veto power which corresponds to a given power distribution and minimizes q and all v_i; though in many cases this is undoubtedly true. We only guarantee that $v_i = v_j$ implies $B(P_i) = B(P_j)$ for these forms.

$[3;1,1,1,1]$ yields case (a), as does $[162;94,67,57,38]$
$[4;2,2,2,1]$ yields case (b), as does $[46;38,25,23,4]$
$[3;2,1,1,1]$ yields case (c), as does $[112;76,51,42,36]$
$[4;2,2,1,1]$ yields case (d), as does $[125;88,84,35,25]$
$[5;3,2,2,1]$ yields case (e), as does $[82;53,49,32,28]$

Note that every power distribution feasible for the 4-player case results in equal power for at least two of the players; that is, a strict hierarchy of Banzhaf power is not possible for $n = 4$.

Before leaving 4-player systems, we observe that when $n = 4$, the maximum percentage of Banzhaf power possible is 1/2. Is this true for any value of n? The answer is no. Of course, given n, only finitely many power distributions are feasible, so that there is a maximum value for $B(P_1)$ corresponding to that value of n. Call this value M_n. We shall see that $M_5 = 7/11$; we conjecture that the proof of the following theorem gives a general formula for M_n. We can at least conclude the following:

Theorem 3: $\lim_{n\to\infty} M_n = 1$

Proof: Given n, consider the WVS $[n-1; n-2, 1, 1, \ldots, 1]$. There are a total of $2n - 3$ votes, and the quota is greater than half of this total. We note that P_1 belongs to every winning coalition except

$$\{P_2, P_3, \ldots, P_n\}$$

and moreoever, P_1 is critical in every winning coalition to which it belongs except the n-player coalition. We now undertake to count these coalitions: There are $n - 1$ two-player winning coalitions; they are

$$\{P_1, P_2\}, \{P_1, P_3\}, \ldots, \{P_1, P_n\}.$$

Noting that each of these contains P_1 and one other player, we have $\binom{n-1}{1}$ such coalitions.

The three-player winning coalitions number $\binom{n-1}{2}$, since each contains P_1 and two other players. Similarly, each four-player winning coalition contains P_1 and three other players, so that these number $\binom{n-1}{3}$. We continue in this manner, finally considering the $(n-1)$-player coalitions, which must all win; there are n of these, and P_1 belongs to $n - 1$ of them. Each contains P_1 and $n - 2$ other players, so that we can describe the number of such coalitions by $\binom{n-1}{n-2}$. So the total number of coalitions in which P_1 is critical is

$$\alpha = \sum_{k=1}^{n-2} \binom{n-1}{k}.$$

Now we must compute the total number of critical instances; for this, we simply determine how players other than P_1 may be critical. Choose P_i with $i \neq 1$, and note that $\{P_1, P_i\}$ is a winning coalition with both players critical. So this contributes one critical instance for P_i. Next consider the coalition $C = \{P_2, P_3, \ldots, P_n\}$. This coalition wins, and P_i is critical, since C is the only winning coalition not containing P_1. (So if P_i is removed from C, the result is a losing coalition by virtue of $P_1 \notin C - \{P_i\}$.)

So far we have 2 critical instances for P_i. But there can be no more, because if $3 \leq k \leq n-2$, and P_i belongs to a k-player winning coalition, then P_1 also belongs to this coalition, so removing P_i results in a $(k-1)$-player coalition to which P_1 also belongs, yielding another winning coalition. Hence P_i cannot be critical in the k-player coalition.

So P_1 is critical in α instances, and each other player is critical in 2 instances. The total number of critical instances is therefore $\alpha + 2(n-1)$, so that

$$B(P_1) = \frac{\alpha}{\alpha + 2(n-1)}.$$

We note that α is polynomial in n, with degree greater than or equal to 2 whenever $n \geq 5$. Since

$$\lim_{n\to\infty} \frac{2(n-1)}{\alpha} = 0,$$

and since $M_n \geq B(P_1)$, we obtain the result.

Now we consider power distribution for weighted voting systems of size 5. As in the 4-player case, we consider each feasible composition for the collection of winning coalitions, but here the situation is considerably more complicated. The winning coalitions must include the 5-player coalition and all five 4-player coalitions, so the cases are distinguished by which 2- and 3-player coaltions win. The total number of critical instances is not the same for all cases, an issue which we discuss shortly.

We shall close this paper with the proof (by exhaustion of cases) that a total of 35 power distributions are possible for weighted voting systems of size 5. (This is Theorem 5.) The proof is not difficult, but it is tedious, so for now we merely preview the results. Some of the power distributions in the 5-player case are obviously feasible, such as

$$\beta = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \tag{3}$$

and

$$\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right). \tag{4}$$

Note that to obtain (4), we simply construct the WVS given by (2), which is $[6; 3, 3, 3, 1, 1]$.

The proof of Theorem 5 reveals the interesting fact that all of the feasible power distributions except (3) are attainable in only one way, which is to say that if two weighted voting systems of size 5 are equivalent, then the collection of winning coalitions must be precisely the same for both, unless the power is evenly distributed. So there are in total 36 possibilities for how the collection of winning coalitions is composed; two of them give rise to (3), and we shall discuss these two later.

Our results for $n = 5$ raise the interesting open question of whether it is possible to find two equivalent n-player weighted voting systems (with no veto power) for which

$$\beta \neq \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$$

and the collection of winning coalitions is different for each. If this is possible, what is the minimum value of n for which this can occur? And if there is such a minimum value N, can such a pair of examples always be found for any $n \geq N$?

The power distribution which maximizes $B(P_1)$ in the 5-player case is

$$\beta = \left(\frac{7}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right),$$

and we note that this is the outcome predicted by the proof of Theorem 3, corresponding to weighted voting systems which are equivalent to $[4; 3, 1, 1, 1, 1]$. There is one other feasible power distribution which gives P_1 over half of the power; it is

$$\beta = \left(\frac{13}{23}, \frac{3}{23}, \frac{3}{23}, \frac{3}{23}, \frac{1}{23}\right).$$

Only two more power distributions on the list give P_1 at least half of the total power, and they are

$$\beta = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0\right) \tag{5}$$

and

$$\beta = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right).$$

If the reader is interested in the issue of powerlessness, we can preview how this occurs when $n = 5$; in addition to (4) and (5), there are three other ways in which we can have a powerless player. The corresponding power distributions are

$$\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0\right),$$

$$\beta = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right),$$

and

$$\beta = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, 0\right).$$

Comparing these five power distributions involving a powerless player, with $n = 5$, to the five power distributions in Theorem 2, with $n = 4$, we note that whenever P_5 is powerless, the WVS is in some sense reduced to a *de facto* 4-player system, and the distribution of power among the other players obeys the restrictions of Theorem 2. We wonder, then, if there are exactly 35 ways for P_6 to be powerless in a 6-player WVS, with the power distribution for the other five players conforming to one of the possibilities given in Theorem 5.

In contrast to the 4-player case, a strict hierarchy of Banzhaf power is possible when $n = 5$. But surprisingly, there is only *one* way for this to occur, and in that case we have

$$\beta = \left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{3}{25}, \frac{1}{25}\right). \tag{6}$$

A canonical example is $[9; 5, 4, 3, 2, 1]$. (See Case 21 in the proof of Theorem 5 to see why the WVS $[8; 5, 4, 3, 2, 1]$ does not induce (6).) The almost irresistible pattern in (6), viz.,

$$\left(\frac{2n-1}{n^2}, \frac{2n-3}{n^2}, \ldots, \frac{1}{n^2}\right),$$

is probably a red herring[6], for though we have not undertaken an exhaustive analysis of 6-player systems, we do know that strict hierarchy of power for $n = 6$ need not induce

$$\beta = \left(\frac{11}{36}, \frac{1}{4}, \frac{7}{36}, \frac{5}{36}, \frac{1}{12}, \frac{1}{36}\right). \tag{7}$$

In fact, we have not found an example yielding (7). However, we have found the example $[15; 9, 7, 4, 3, 2, 1]$, which induces the following strict hierarchy of power:

$$\beta = \left(\frac{5}{12}, \frac{3}{16}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \frac{1}{24}\right).$$

The most naive attempt to effect strict hierarchy of power with $n = 6$, the WVS $[11; 6, 5, 4, 3, 2, 1]$, fails to deliver, for this WVS yields

$$\beta = \left(\frac{9}{28}, \frac{1}{4}, \frac{5}{28}, \frac{3}{28}, \frac{3}{28}, \frac{1}{28}\right).$$

We observe, then, that it is far from obvious how to effect strict hierarchy of power when such an outcome is appropriate and/or desired. Merely constructing the WVS

$$[q; n, n-1, n-2, \ldots, 2, 1],$$

[6]In fact, we cannot say with certainty that n^2 critical instances is always possible for an n-player WVS, though Theorem 4 does allow this possibility.

with q just large enough to avoid low quota, i.e.,

$$q = 1 + \left[\frac{n(n+1)}{4}\right],$$

may not work. (Here $[r]$ denotes the integer round-down of r.) Of course, by Theorem 2, the strategy cannot work for $n = 4$; we obtain $[6; 4, 3, 2, 1]$, which is alternative (e). For $n = 5$, the strategy "almost" works ($[8; 5, 4, 3, 2, 1]$) but results in equal power for P_2 and P_3. Raising the quota from 8 to 9 achieves the desired effect. But for $n = 6$, no appropriate[7] value of q yields strict hierarchy of power for $[q; 6, 5, 4, 3, 2, 1]$.

A "real-life" 6-player WVS was among the original motivations for attorney John Banzhaf to devise his model of power. The Nassau County, New York, Board of Supervisors consisted of six voting members, each representing one municipality and casting a number of votes in proportion to the population of that municipality[8]. As of 1958 the operative WVS was $[16; 9, 9, 7, 3, 1, 1]$, and by 1964 updated population data resulted in the adoption of the WVS $[58; 31, 31, 28, 21, 2, 2]$. Unfortunately, the two systems are equivalent and have

$$\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\right),$$

so that they are in fact equivalent to $[8; 4, 4, 4, 1, 1, 1]$, given by (2). How long would the power structure have gone unnoticed if this canonical WVS had been in use instead?

However, in view of the fact that five players are just enough to make strict hierarchy of power possible and the likelihood that the number of feasible power distributions is significantly larger than 35 when $n = 6$, it is conceivable that the variety of feasible power distributions for 6-player systems is quite great. Perhaps Nassau County's predicament could have been remedied by *some* WVS for which β more closely reflected the population distribution, which was clearly the intent[9].

Our proof of Theorem 5, below, unfortunately sheds no light on how one might obtain a formula, in terms of n, for the number of power distributions feasible for weighted voting systems of size n. A more efficient method of counting cases might produce such a formula as a byproduct. But from an applied

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[7]For the WVS $[q; 6, 5, 4, 3, 2, 1]$, we must have $q \geq 11$ to avoid low quota, but we must also have $q \leq 15$ to avoid veto power.

[8]More precisely, five municipalities were represented, the largest by two voting members.

[9]The population figures given in **[B]**, when compared with the 1958 WVS, do reveal one anomaly: $v_3 = 7$ and $v_4 = 3$, even though the two municipalities in the roles of P_3 and P_4 are fairly close in population. However, the narrative in **[B]** strongly suggest that representation in proportion to population was the aim of the WVS. We do not know if a misprint may be responsible or if there were unrelated political reasons for the underallocation of votes to P_4. We have inferred the quotas for each WVS; they are not explicitly stated. If the 1958 vote allocations were given instead as $[q; 9, 9, 7, 6, 1, 1]$, then we would infer $q = 17$. But then the power distribution is $\beta = \left(\frac{2}{7}, \frac{2}{7}, \frac{3}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14}\right)$.

standpoint, an even more desirable tool would be a way to determine, given n and a prescribed power distribution β^*, how to construct a WVS so that the actual power distribution β comes as close as possible (by some measure) to β^*. Merely knowing the *number* of feasible power distributions would not give enough information to construct the desired WVS. But surely the larger the value of n, the more realistic the stated goal becomes. In **[T]**, β is computed for the US Electoral College, which one can view as a 51-player WVS involving the 50 states and the District of Columbia. If β^* is given by computing each player's percentage of the total number of electoral votes, then one observes that β approximates β^* reasonably well. Even Banzhaf's paper, trumpeting (in its very title) the claim that *Weighted Voting Doesn't Work* **[B]**, contains the computation of β for the weighted voting plan adopted by the New Jersey Senate[10], amounting to a 21-player WVS, with each county casting a (noninteger) number of votes exactly equal to that county's percentage of the state's population. A more naive approach is hard to imagine; yet β is again close to the ideal. In fairness, however, we should mention that the Banzhaf power of the three largest counties was slightly above the ideal, while the power of the other 18 counties was slightly below[11]. The same was true (prior to the 2001 reapportionment) for the US Electoral College; the three largest states[12] held slightly more Banzhaf power than β^* would give, and the 48 other players held slightly less[13].

We now return to the question of the total number of critical instances one may expect, given n. In general, n alone does not determine this number; the structure of the WVS itself contributes as well. Dubey and Shapley considered this question, and one may refer to their result as a partial check on one's work in calculating Banzhaf power for a WVS. Theorems 2 and 3 in **[DS]**, combined and adapted for our assumptions, become the following result.

Theorem 4: Given a WVS with no veto power, let n be the number of players, and let $m = 1 + [n/2]$. Then the number c of critical instances satisfies

$$n(n-1) \leq c \leq m \binom{n}{m}.$$

[10] According to Banzhaf, considerations of New Jersey state law resulted in the courts quickly squelching the plan.

[11] Also in fairness to Banzhaf, we should mention his statement in **[B]**: "The purpose in this paper is neither to attack nor defend weighted voting *per se*", nor is this the legacy of Banzhaf's work, which is rather the astute observation that weighted voting "does not even theoretically produce the effects which have been claimed to justify it."

[12] California, New York, and Texas

[13] Interestingly, the Shapley-Shubik model of power also yields a power distribution for the Electoral College in which the large states hold more than the ideal share of power. The seven largest states have power greater than ideal, and the other 44 players have power below the ideal. See **[T]**.

Moreover, c is exactly equal to $m \binom{n}{m}$ if and only if every coalition containing more than $n/2$ players wins and every coalition containing less than $n/2$ players loses.

Note that in case $n = 4$, the absence of veto power ensures that every coalition with less than 2 players loses and every coalition with more than 2 players wins; this explains why all five cases have $c = 12$. But for $n = 5$, the number of critical instances does not always meet the upper bound of 30. In fact, there is only one of the 36 cases in which we do have $c = 30$, and this is the case in which we have no 2-player winning coalitions and in which every 3-player coalition wins. As intuition would suggest, this scenario gives all five players equal power, so that we have (2). A canonical example is $[3; 1, 1, 1, 1, 1]$. At the other end of the spectrum is the unique case in which $c = 20$, which occurs when the winning coalitions are exactly (and only) those which are guaranteed – namely, the 4-player coalitions and the 5-player coalition. This scenario also induces (2), and a canonical example is $[4; 1, 1, 1, 1, 1]$.

We are now ready to state and prove our main result.

Theorem 5: For weighted voting systems of size 5 with no veto power, there are 35 feasible Banzhaf power distributions. They are as follows.

1. $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$
2. $\left(\frac{7}{29}, \frac{7}{29}, \frac{5}{29}, \frac{5}{29}, \frac{5}{29}\right)$
3. $\left(\frac{2}{7}, \frac{3}{14}, \frac{3}{14}, \frac{1}{7}, \frac{1}{7}\right)$
4. $\left(\frac{1}{3}, \frac{5}{27}, \frac{5}{27}, \frac{5}{27}, \frac{1}{9}\right)$
5. $\left(\frac{7}{27}, \frac{7}{27}, \frac{7}{27}, \frac{1}{9}, \frac{1}{9}\right)$
6. $\left(\frac{5}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}\right)$
7. $\left(\frac{4}{13}, \frac{3}{13}, \frac{3}{13}, \frac{2}{13}, \frac{1}{13}\right)$
8. $\left(\frac{9}{25}, \frac{1}{5}, \frac{1}{5}, \frac{3}{25}, \frac{3}{25}\right)$

9. $\left(\frac{7}{25}, \frac{7}{25}, \frac{1}{5}, \frac{1}{5}, \frac{1}{25}\right)$

10. $\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}\right)$

11. $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right)$

12. $\left(\frac{7}{23}, \frac{7}{23}, \frac{3}{23}, \frac{3}{23}, \frac{3}{23}\right)$

13. $\left(\frac{7}{23}, \frac{5}{23}, \frac{5}{23}, \frac{5}{23}, \frac{1}{23}\right)$

14. $\left(\frac{3}{11}, \frac{3}{11}, \frac{2}{11}, \frac{2}{11}, \frac{1}{11}\right)$

15. $\left(\frac{5}{21}, \frac{5}{21}, \frac{5}{21}, \frac{1}{7}, \frac{1}{7}\right)$

16. $\left(\frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$

17. $\left(\frac{1}{3}, \frac{7}{27}, \frac{5}{27}, \frac{1}{9}, \frac{1}{9}\right)$

18. $\left(\frac{5}{13}, \frac{3}{13}, \frac{2}{13}, \frac{2}{13}, \frac{1}{13}\right)$

19. $\left(\frac{8}{25}, \frac{8}{25}, \frac{1}{5}, \frac{2}{25}, \frac{2}{25}\right)$

20. $\left(\frac{11}{25}, \frac{1}{5}, \frac{3}{25}, \frac{3}{25}, \frac{3}{25}\right)$

21. $\left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{3}{25}, \frac{1}{25}\right)$

22. $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0\right)$

23. $\left(\frac{5}{12}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right)$

24. $\left(\frac{9}{23}, \frac{7}{23}, \frac{3}{23}, \frac{3}{23}, \frac{1}{23}\right)$

25. $\left(\frac{4}{11}, \frac{4}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right)$

26. $\left(\frac{5}{13}, \frac{3}{13}, \frac{3}{13}, \frac{1}{13}, \frac{1}{13}\right)$

27. $\left(\frac{11}{25}, \frac{1}{5}, \frac{1}{5}, \frac{3}{25}, \frac{1}{25}\right)$

28. $\left(\frac{9}{25}, \frac{7}{25}, \frac{7}{25}, \frac{1}{25}, \frac{1}{25}\right)$

29. $\left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, 0\right)$

30. $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right)$

31. $\left(\frac{11}{23}, \frac{5}{23}, \frac{5}{23}, \frac{1}{23}, \frac{1}{23}\right)$

32. $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0\right)$

33. $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right)$

34. $\left(\frac{13}{23}, \frac{3}{23}, \frac{3}{23}, \frac{3}{23}, \frac{1}{23}\right)$

35. $\left(\frac{7}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right)$

Proof: The feasible power distributions above are listed in the order in which we consider the various cases. (The only exception arises from the two distinct cases inducing power distribution number 1.) We repeat here that all five 4-player coalitions are guaranteed to win. However, the critical instances corresponding to these coalitions depend on which 3-player coalitions are winners. Hence for every case, we must list all 2-, 3-, and 4-player winning coalitions to find all instances of criticality.

Our work is slightly simplified by observing that there can be no more than four 2-player winning coalitions, for note $\{P_4, P_5\}$ can never be a winner: $v_4 + v_5 \leq v_1 + v_2 + v_3$. Similarly we can eliminate $\{P_3, P_5\}$, $\{P_3, P_4\}$, $\{P_2, P_4\}$, and $\{P_2, P_5\}$, leaving only five potential 2-player winning coalitions. But then we note that $\{P_1, P_4\}$ and $\{P_2, P_3\}$ cannot win simultaneously; nor can $\{P_1, P_5\}$

and $\{P_2, P_3\}$. It follows that the only way we can have as many as four 2-player winning coalitions is if they are exactly $\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}$, and $\{P_1, P_5\}$. We must also keep in mind that whenever a 2- or 3-player coalition wins, its complement loses. However, the converse is not true; if a coalition loses, it does not follow that the complement wins.

We now organize the proof into 5 parts, according to the number of 2-player coalitions which win. The cases are numbered 1 through 35, so that Case i induces power distribution number i on our list. An exception to the numbering scheme is Case 1(a), presented first, and Case 1(b), presented between Case 15 and Case 16. Cases 1(a) and 1(b) each induce power distribution number 1.

For each case, we have included a canonical example of a particular WVS, so that the reader may see more easily that the case we describe is indeed feasible.

Part I: Assume there are no winning 2-player coalitions. If this is the case, then in every winning 3-player coalition, all three players will be critical.

Case 1(a): All ten 3-player coalitions win. An example is $[3; 1, 1, 1, 1, 1]$. In this case, no player can be critical in a 4-player winning coalition, so all instances of criticality come from the 3-player coalitions. We have a total of 30 such instances, 6 for each player. Hence $B(P_i) = 6/30 = 1/5$ for each i. So we have

$$\beta = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

Case 2: Nine 3-player coalitions win. Then the only loser is $\{P_3, P_4, P_5\}$. The situation is then summarized in the following table. The left-hand column gives an abbreviated description of each winning coalition; for example, 235 means that $\{P_2, P_3, P_5\}$ is a winning coalition. The right-hand column indicates which players are critical for the coalition to the left; for example, 2 means that only P_2 is critical, while 134 means that P_1, P_3, and P_4 are critical. Here and in the sequel we omit 12345, which represents the 5-player coalition, for though it wins, it never contributes critical instances.

Winners	Critical
123	123
124	124
125	125
134	134
135	135
145	145
234	234
235	235
245	245
2345	2
1345	1
1245	none
1235	none
1234	none

An example is $[7; 3, 3, 2, 2, 2]$, and we have

$$\beta = \left(\frac{7}{29}, \frac{7}{29}, \frac{5}{29}, \frac{5}{29}, \frac{5}{29} \right).$$

Case 3: Eight 3-player coalitions win. Then the losers must be $\{P_3, P_4, P_5\}$ and $\{P_2, P_4, P_5\}$. The situation is depicted below.

Winners	Critical
123	123
124	124
125	125
134	134
135	135
145	145
234	234
235	235
2345	23
1345	1
1245	1
1235	none
1234	none

An example is $[8; 4, 3, 3, 2, 2]$, and we have

$$\beta = \left(\frac{2}{7}, \frac{3}{14}, \frac{3}{14}, \frac{1}{7}, \frac{1}{7} \right).$$

Case 4: Seven 3-player coalitions win, and the losers are $\{P_3, P_4, P_5\}$, $\{P_2, P_4, P_5\}$, and $\{P_2, P_3, P_5\}$. (There is one other possibility, namely that $\{P_1, P_4, P_5\}$ loses instead of $\{P_2, P_3, P_5\}$. This is Case 5.)

Winners	Critical
123	123
124	124
125	125
134	134
135	135
145	145
234	234
2345	234
1345	1
1245	1
1235	1
1234	none

An example is $[6; 3, 2, 2, 2, 1]$, and we have

$$\beta = \left(\frac{1}{3}, \frac{5}{27}, \frac{5}{27}, \frac{5}{27}, \frac{1}{9} \right).$$

Case 5: Seven 3-player coalitions win as indicated below.

Winners	Critical
123	123
124	124
125	125
134	134
135	135
234	234
235	235
2345	23
1345	13
1245	12
1235	none
1234	none

An example is $[5; 2, 2, 2, 1, 1]$, and we have

$$\beta = \left(\frac{7}{27}, \frac{7}{27}, \frac{7}{27}, \frac{1}{9}, \frac{1}{9} \right).$$

Case 6: Six 3-player coalitions are winners. One way this can occur is if all six of these contain P_1.

Winners	Critical
123	123
124	124
125	125
134	134
135	135
145	145
2345	2345
1345	1
1245	1
1235	1
1234	1

An example is $[4; 2, 1, 1, 1, 1]$, and we have

$$\beta = \left(\frac{5}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13} \right).$$

Case 7: The six 3-player winning coalitions are as above but with $\{P_1, P_4, P_5\}$ replaced by $\{P_2, P_3, P_4\}$.

Winners	Critical
123	123
124	124
125	125
134	134
135	135
234	234
2345	234
1345	13
1245	12
1235	1
1234	none

An example is $[8; 4, 3, 3, 2, 1]$, and we have

$$\beta = \left(\frac{4}{13}, \frac{3}{13}, \frac{3}{13}, \frac{2}{13}, \frac{1}{13} \right).$$

There is no other way to have six 3-player winning coalitions.

Case 8: Five 3-player coalitions are winners. There are two ways for this to occur. One way is as follows.

Winners	Critical
123	123
124	124
125	125
134	134
135	135
2345	2345
1345	13
1245	12
1235	1
1234	1

An example is $[6; 3, 2, 2, 1, 1]$, and we have

$$\beta = \left(\frac{9}{25}, \frac{1}{5}, \frac{1}{5}, \frac{3}{25}, \frac{3}{25} \right).$$

Case 9: Another way in which we may have five 3-player winning coalitions is by replacing $\{P_1, P_3, P_5\}$ with $\{P_2, P_3, P_4\}$.

Winners	Critical
123	123
124	124
125	125
134	134
234	234
2345	234
1345	134
1245	12
1235	12
1234	none

An example is $[7; 3, 3, 2, 2, 1]$, and we have

$$\beta = \left(\frac{7}{25}, \frac{7}{25}, \frac{1}{5}, \frac{1}{5}, \frac{1}{25} \right).$$

Case 10: There are two ways in which we may have four 3-player winning coalitions. One is as follows.

Winners	Critical
123	123
124	124
125	125
134	134
2345	2345
1345	134
1245	12
1235	12
1234	1

An example is [8; 4, 3, 2, 2, 1], and we have

$$\beta = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}\right).$$

Case 11: By replacing $\{P_1, P_2, P_5\}$ in the preceding case by $\{P_2, P_3, P_4\}$, we obtain the following.

Winners	Critical
123	123
124	124
134	134
234	234
2345	234
1345	134
1245	124
1235	123
1234	none

An example is [6; 2, 2, 2, 2, 1], and we have

$$\beta = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\right).$$

Case 12: There are three 3-player winning coalitions; this may occur two ways. One is the following.

Winners	Critical
123	123
124	124
125	125
2345	2345
1345	1345
1245	12
1235	12
1234	none

An example is $[5; 2, 2, 1, 1, 1]$, and we have

$$\beta = \left(\frac{7}{23}, \frac{7}{23}, \frac{3}{23}, \frac{3}{23}, \frac{3}{23}\right).$$

Case 13: By replacing $\{P_1, P_2, P_5\}$ in the preceding case with $\{P_1, P_3, P_4\}$, we have the following.

Winners	Critical
123	123
124	124
134	134
2345	2345
1345	134
1245	124
1235	123
1234	1

An example is $[7; 3, 2, 2, 2, 1]$, and we have

$$\beta = \left(\frac{7}{23}, \frac{5}{23}, \frac{5}{23}, \frac{5}{23}, \frac{1}{23}\right).$$

Case 14: If there are but two 3-player winning coalitions, then they must be $\{P_1, P_2, P_3\}$ and $\{P_1, P_2, P_4\}$.

Winners	Critical
123	123
124	124
2345	2345
1345	1345
1245	124
1235	123
1234	12

An example is $[8; 3, 3, 2, 2, 1]$, and we have

$$\beta = \left(\frac{3}{11}, \frac{3}{11}, \frac{2}{11}, \frac{2}{11}, \frac{1}{11}\right).$$

Case 15: Only one 3-player coalition wins, namely $\{P_1, P_2, P_3\}$.

Winners	Critical
123	123
2345	2345
1345	1345
1245	1245
1235	123
1234	123

An example is $[6; 2, 2, 2, 1, 1]$, and we have

$$\beta = \left(\frac{5}{21}, \frac{5}{21}, \frac{5}{21}, \frac{1}{7}, \frac{1}{7} \right).$$

Case 1(b): If no 3-player coalitions win, then we are left with only those coalitions guaranteed to win, such as in $[4; 1, 1, 1, 1, 1]$. The result is that $B(P_i) = 1/5$ for all i.

Part II: For this set of cases (16-25), we consider systems for which the only 2-player winning coalition is $\{P_1, P_2\}$. Then we are guaranteed that $\{P_3, P_4, P_5\}$ is a losing coalition, so we may have at most nine 3-player winning coalitions, as in Case 16. We proceed as in Part I, by decreasing the number of 3-player winners.

Case 16: If all 3-player coalitions except $\{P_3, P_4, P_5\}$ win, then we have the following.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
145	145
234	234
235	235
245	245
2345	2
1345	1
1245	none
1235	none
1234	none

An example is $[4; 2, 2, 1, 1, 1]$, and we have

$$\beta = \left(\frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7} \right).$$

Case 17: If we have eight 3-player winning coalitions, then the losers must be $\{P_3, P_4, P_5\}$ and $\{P_2, P_4, P_5\}$, so we have the following.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
145	145
234	234
235	235
2345	23
1345	1
1245	1
1235	none
1234	none

An example is $[9; 5, 4, 3, 2, 2]$, and we have

$$\beta = \left(\frac{1}{3}, \frac{7}{27}, \frac{5}{27}, \frac{1}{9}, \frac{1}{9}\right).$$

Case 18: If we have seven 3-player winning coalitions, then the new loser may be either $\{P_2, P_3, P_5\}$ or $\{P_1, P_4, P_5\}$. In the former event, we have the following.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
145	145
234	234
2345	234
1345	1
1245	1
1235	1
1234	none

An example is $[8; 5, 4, 2, 2, 1]$, and we have

$$\beta = \left(\frac{5}{13}, \frac{3}{13}, \frac{2}{13}, \frac{2}{13}, \frac{1}{13}\right).$$

Case 19: If $\{P_1, P_4, P_5\}$ is instead struck from Case 17, we have the following.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
234	234
235	235
2345	2
1345	13
1245	12
1235	none
1234	none

An example is $[7; 4, 4, 2, 1, 1]$, and we have

$$\beta = \left(\frac{8}{25}, \frac{8}{25}, \frac{1}{5}, \frac{2}{25}, \frac{2}{25} \right).$$

Case 20: If there are six 3-player winning coalitions, then one way this may occur is as follows.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
145	145
2345	2345
1345	1
1245	1
1235	1
1234	1

An example is $[5; 3, 2, 1, 1, 1]$, and we have

$$\beta = \left(\frac{11}{25}, \frac{1}{5}, \frac{3}{25}, \frac{3}{25}, \frac{3}{25} \right).$$

Case 21: Now we come to the unique case in which strict hierarchy of Banzhaf power is possible for a 5-player WVS. With six 3-player winning coalitions we may have the following.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
234	234
2345	234
1345	13
1245	12
1235	1
1234	none

An example is $[9; 5, 4, 3, 2, 1]$, and we have

$$\beta = \left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{3}{25}, \frac{1}{25}\right).$$

An observation we can make at this point is that giving i votes to P_i seems an obvious, albeit naive, way to effect strict hierarchy of power, but we note that in the case of 5 players, this results in a total of 15 votes, so that we must have $q \geq 8$. But setting q equal to 8 instead of 9 yields a WVS for which $B(P_2) = B(P_3)$, equivalent to $[8; 5, 3, 3, 2, 1]$, which is Case 27.

Case 22: If there are five 3-player winning coalitions, then this may occur as follows.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
234	234
2345	234
1345	134
1245	12
1235	12
1234	none

An example is $[10; 5, 5, 3, 3, 1]$, and we have

$$\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0\right).$$

Case 23: Another way we may have five 3-player winning coalitions is by replacing $\{P_2, P_3, P_4\}$ in the previous case with $\{P_1, P_3, P_5\}$. The we have the following.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
135	135
2345	2345
1345	13
1245	12
1235	1
1234	1

An example is $[7; 4, 3, 2, 1, 1]$, and we have

$$\beta = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12} \right).$$

Case 24: If there are four 3-player winning coalitions, then they must be as follows.

Winners	Critical
12	12
123	12
124	12
125	12
134	134
2345	2345
1345	134
1245	12
1235	12
1234	1

An example is $[9; 5, 4, 2, 2, 1]$, and we have

$$\beta = \left(\frac{9}{23}, \frac{7}{23}, \frac{3}{23}, \frac{3}{23}, \frac{1}{23} \right).$$

Case 25: There can be no fewer than three 3-player winning coalitions in case $\{P_1, P_2\}$ wins, for then the coalitions $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, and $\{P_1, P_2, P_5\}$ automatically win. Hence our last case for Part II is as follows.

Winners	Critical
12	12
123	12
124	12
125	12
2345	2345
1345	1345
1245	12
1235	12
1234	12

An example is $[6; 3, 3, 1, 1, 1]$, and we have

$$\beta = \left(\frac{4}{11}, \frac{4}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right).$$

Part III: Now suppose there are exactly two winning 2-player coalitions. Then they must be $\{P_1, P_2\}$ and $\{P_1, P_3\}$. It follows that the coalitions $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_2, P_5\}$, $\{P_1, P_3, P_4\}$, and $\{P_1, P_3, P_5\}$ automatically win. So we can have no fewer than five 3-player winning coalitions. On the other hand, we can have no more than eight, since the coalitions $\{P_3, P_4, P_5\}$ and $\{P_2, P_4, P_5\}$ must lose.

Case 26: With eight 3-player winning coalitions, we have the following.

Winners	Critical
12	12
13	13
123	1
124	12
125	12
134	13
135	13
145	145
234	234
235	235
2345	23
1345	1
1245	1
1235	none
1234	none

An example is $[5; 3, 2, 2, 1, 1]$, and we have

$$\beta = \left(\frac{5}{13}, \frac{3}{13}, \frac{3}{13}, \frac{1}{13}, \frac{1}{13}\right).$$

Case 27: We may have seven 3-player winning coalitions by striking either $\{P_2, P_3, P_5\}$ or $\{P_1, P_4, P_5\}$ from the list in Case 26. In the former event, we have the following.

Winners	Critical
12	12
13	13
123	1
124	12
125	12
134	13
135	13
145	145
234	234
2345	234
1345	1
1245	1
1235	1
1234	none

An example is $[8; 5, 3, 3, 2, 1]$, and we have

$$\beta = \left(\frac{11}{25}, \frac{1}{5}, \frac{1}{5}, \frac{3}{25}, \frac{1}{25} \right).$$

Case 28: In case $\{P_1, P_4, P_5\}$ is struck from the list in Case 26, we obtain the following.

Winners	Critical
12	12
13	13
123	1
124	12
125	12
134	13
135	13
234	234
235	235
2345	23
1345	13
1245	12
1235	none
1234	none

An example is $[7; 4, 3, 3, 1, 1]$, and we have

$$\beta = \left(\frac{9}{25}, \frac{7}{25}, \frac{7}{25}, \frac{1}{25}, \frac{1}{25} \right).$$

Case 29: We may have six 3-player winning coalitions in one of two ways; one way is to strike $\{P_2, P_3, P_5\}$ from the list in Case 28. Then we have the following.

Winners	Critical
12	12
13	13
123	1
124	12
125	12
134	13
135	13
234	234
2345	234
1345	13
1245	12
1235	1
1234	none

An example is $[10; 6, 4, 4, 2, 1]$, and we have

$$\beta = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}, 0 \right).$$

Case 30: Another way we may we have six 3-player winning coalitions is by striking $\{P_2, P_3, P_4\}$ from the list in Case 27. Then we have the following.

Winners	Critical
12	12
13	13
123	1
124	12
125	12
134	13
135	13
145	145
2345	2345
1345	1
1245	1
1235	1
1234	1

An example is $[6;4,2,2,1,1]$, and we have

$$\beta = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right).$$

Case 31: If we have five 3-player winning coalitions, then these are exactly those we identified at the beginning of Part II.

Winners	Critical
12	12
13	13
123	1
124	12
125	12
134	13
135	13
2345	2345
1345	13
1245	12
1235	1
1234	1

An example is $[8;5,3,3,1,1]$, and we have

$$\beta = \left(\frac{11}{23}, \frac{5}{23}, \frac{5}{23}, \frac{1}{23}, \frac{1}{23}\right).$$

Part IV: If we have exactly three 2-player winning coalitions, then $\{P_1, P_2\}$ and $\{P_1, P_3\}$ must win, and the third may be either $\{P_2, P_3\}$ or $\{P_1, P_4\}$.

Case 32: Suppose the 2-player winners are $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_1, P_4\}$, and their complements are the only losing 3-player coalitions. Then we have the following.

Winners	Critical
12	12
13	13
14	14
123	1
124	1
125	12
134	1
135	13
145	14
234	234
2345	234
1345	1
1245	1
1235	1
1234	none

An example is $[6; 4, 2, 2, 2, 1]$, and we have

$$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0\right).$$

Case 33: Suppose instead that $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_2, P_3\}$ win. Then their complements are guaranteed to lose. But observe also that the seven remaining 3-player coalitions are guaranteed to win, since they all contain two of the players P_1, P_2, P_3. So in this case we have the following.

Winners	Critical
12	12
13	13
23	23
123	none
124	12
125	12
134	13
135	13
234	23
235	23
2345	23
1345	13
1245	12
1235	none
1234	none

An example is $[6; 3, 3, 3, 1, 1]$, and we have

$$\beta = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right).$$

Case 34: The only case remaining under Part IV is the case in which $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_1, P_4\}$ are the winning 2-player coalitions, their complements lose, and the coalition $\{P_2, P_3, P_4\}$ also loses. Then we have the following.

Winners	Critical
12	12
13	13
14	14
123	1
124	1
125	12
134	1
135	13
145	145
2345	2345
1345	1
1245	1
1235	1
1234	1

An example is $[7; 5, 2, 2, 2, 1]$, and we have

$$\beta = \left(\frac{13}{23}, \frac{3}{23}, \frac{3}{23}, \frac{3}{23}, \frac{1}{23}\right).$$

Part V: Finally, suppose there are four 2-player winning coalitions. This is the maximum number possible with 5 players and, as we mentioned at the beginning of the proof, this can only occur if the winners are $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_4\}$, and $\{P_1, P_5\}$. Now only one case remains.

Case 35: The winning coalitions are now completely determined, since the complements of $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_1, P_4\}$, and $\{P_1, P_5\}$ lose, while the six

remaining 3-player coalitions are guaranteed to win. We come to the case which gives P_1 maximum Banzhaf power, corresponding to the scenario in the proof of Theorem 3.

Winners	Critical
12	12
13	13
14	14
123	1
124	1
125	12
134	1
135	13
145	145
2345	2345
1345	1
1245	1
1235	1
1234	1

An example is $[4; 3, 1, 1, 1, 1]$, and we have

$$\beta = \left(\frac{7}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}\right).$$

The proof is complete.

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