MSCF Mathematics Preparatory Course
August 2006
Homework #4

Do exercises 7.1(b), 7.1(c), 7.2, 7.5(a), and those below. (A solution to Problem 7.5(b) will be provided in case you are inclined to do that part in your copious spare time.)

Problem A: Let T, K, and σ be positive constants, let r be a nonnegative constant, and let $x > 0$ and $t \in [0, T)$ be given. Define $\tau = T - t$, which is positive. Let

$$\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

be the standard normal density, and let $N(d)$ be the cumulative standard normal distribution

$$N(d) = \int_{-\infty}^{d} \varphi(y)\,dy.$$

Let

$$d_+(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right]$$

and let

$$d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right]$$

Compute

$$c(t, x) = \int_{-\infty}^{\infty} e^{-r\tau}\left(x \exp\left\{\sigma\sqrt{\tau}\,y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+ \varphi(y)\,dy.$$

You should obtain the Black-Scholes formula

$$c(t, x) = xN\big(d_+(\tau, x)\big) - Ke^{-r\tau}N\big(d_-(\tau, x)\big).$$

Remarks: (i) Firstly, it may bother you that we have written $c(t, x)$ when the variable t does not explicitly appear in either of the expressions for $c(t, x)$, above. However, t implicitly appears through the variable τ, and our work in this problem is simplified slightly since we can work with τ instead of $T - t$. (ii) For a European call expiring at time T with strike price K, if the time-t stock price is x, then the Black-Scholes price at time $t \in [0, T)$ is

$$c(t, x) = xN\big(d_+(T - t, x)\big) - Ke^{-r(T-t)}N\big(d_-(T - t, x)\big).$$

Problem B: Show that if A is any $m \times n$ matrix, then

$$(Ax) \cdot y = x \cdot (A^T y)$$

for all $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

Problem C: Let A be an $m \times n$ matrix. Show that if $z \in \mathbf{R}^m$ satisfies $Ay = z$ for some $y \in \mathbf{R}^n$, and $x \in \mathbf{R}^m$ satisfies $A^T x = 0$, then $x \cdot z = 0$.

Problem D: Let $\alpha \in \mathbf{R}$ and $x_0 \in \mathbf{R}^n$ be given. Define

$$H = \{x \in \mathbf{R}^n : x \cdot x_0 = \alpha\}$$

Show that H is a subspace of $\mathbf{R}^n$ if and only if $\alpha = 0$.

Problem E: Let Q be an $n \times n$ orthogonal matrix.

(a) Show that $(Qx) \cdot (Qy) = x \cdot y$ for all $x, y \in \mathbf{R}^n$.

(b) Show that $\|Qx\| = \|x\|$ for all $x \in \mathbf{R}^n$.

(c) Suppose λ is a real eigenvalue of Q. Show that either $\lambda = 1$ or $\lambda = -1$.