MSCF Mathematics Preparatory Course
August 2006
Solutions to Homework #3 Exercises

4.6 (a) Let $r \in \mathbf{R}$ be given. Since $\lim_{x\to-\infty} f(x) = -\infty$, there exists $z_1 \in \mathbf{R}$ such that $f(x) < r-1$ for all $x < z_1$. Then since $\lim_{x\to\infty} f(x) = \infty$, there exists $z_2 \in \mathbf{R}$ such that $f(x) > r+1$ for all $x > z_2$. Let $z_1' = z_1 - 1$ and $z_2' = z_2 + 1$. Then $f(z_1') < r - 1$ and $f(z_2') > r+1$, and therefore $f(z_1') < r < f(z_2')$, and since f is continuous on $\mathbf{R}$, in particular f is continuous on $[z_1', z_2']$, so we can apply the Intermediate Value Theorem (3.20) on this interval and conclude that there exists $x_0 \in (z_1', z_2')$ for which $f(x_0) = r$. (Note that this argument establishes that the range of f is actually all of $\mathbf{R}$.)

(b) A polynomial p of odd degree satisfies either

$$\lim_{x\to-\infty} p(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} p(x) = \infty$$

or

$$\lim_{x\to-\infty} p(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} p(x) = -\infty$$

In the first case, as all polynomials are continuous on $\mathbf{R}$, we can apply (a) to p, with $r = 0$, and get the result. In the second case, an argument similar to that in (a) shows that the range of p is $\mathbf{R}$, so that in particular, p "hits" the value 0.

4.9 (a) Let $g(x) = \sin x$, and let $h(x) = x$. Pick any $x_0 \in \mathbf{R} - \{0\}$, and note that g and h are differentiable at x_0. Since $h(x_0) = x_0 \neq 0$, f is differentiable at x_0 by Theorem 4.5(iii). To show the differentiability of f at 0, we consider

$$\frac{f(x) - f(0)}{x - 0} = \frac{\frac{\sin x}{x} - 1}{x} = \frac{\sin x - x}{x^2}.$$

Letting $k(x) = \sin x - x$ and $m(x) = x^2$, we see that $\lim_{x\to0} k(x) = \lim_{x\to0} m(x) = 0$, and k and m are both defined and differentiable on any deleted neighborhood of 0. We have $k'(x) = \cos x - 1$, and $m'(x) = 2x$. Now $\lim_{x\to0} k'(x) = \lim_{x\to0} m'(x) = 0$, and k' and m' are both defined and differentiable in any deleted neighborhood of 0; in fact, $k''(x) = -\sin x$, and $m''(x) \equiv 2$. Also note $\lim_{x\to0} \dfrac{k''(x)}{m''(x)} = 0$. Applying l'Hôpital's Rule (Theorem 4.14) to k' and m', then, we have $\lim_{x\to0} \dfrac{k'(x)}{m'(x)} = 0$. But now having established the existence of this last limit, we apply the theorem again to k and m to find $\lim_{x\to 0} \frac{k(x)}{m(x)} = 0$. We conclude that f is differentiable at 0, and that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

So we have found that f is differentiable on $\mathbf{R}$, and for $x \neq 0$, we would have, by the Quotient Rule, $f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{h(x)^2} = \frac{x\cos x - \sin x}{x^2}$. So the formula for f' is

$$f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

(b) Let $x_0 \in \mathbf{R} - \{0\}$, and note that $g(x) = x$, $h(x) = \cos x$, $k(x) = \sin x$, and $m(x) = x^2$ are all continuous at x_0, and $m(x_0) \neq 0$. So f' is continuous at x_0 by the Algebra of Continuous Functions (3.4). But to show the continuity of f' at 0, we need $\lim_{x\to 0} f'(x) = f'(0) = 0$. By the Algebra of Limits, we have that $\lim_{x\to 0}(x\cos x - \sin x) = 0\cdot 1 - 0 = 0$, and also $\lim_{x\to 0} x^2 = 0$. Applying the Sum Rule and the Product Rule, we have $(gh - k)'(x) = -x\sin x + \cos x - \cos x = -x\sin x$. Then $m'(x) = 2x$, so $\lim_{x\to 0} \frac{(gh-k)'(x)}{m'(x)} = \lim_{x\to 0} \frac{-x\sin x}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0$. Applying l'Hôpital's Rule to $gh - k$ and m, we find

$$\lim_{x\to 0} \frac{(gh-k)(x)}{m(x)} = \lim_{x\to 0} f'(x) = 0,$$

as desired. So f' is continuous at 0 as well, and therefore f is continuous on all of $\mathbf{R}$.

(c) To show f' is differentiable at 0, we must show that

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0}$$

exists. So we consider

$$\frac{f'(x) - f'(0)}{x - 0} = \frac{\frac{x\cos x - \sin x}{x^2} - 0}{x - 0} = \frac{x\cos x - \sin x}{x^3}.$$

Letting $g(x) = x\cos x - \sin x$ and $h(x) = x^3$, we see that $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$. Consider $g'(x) = -x\sin x$ and $h'(x) = 3x^2$. Note that $\lim_{x\to 0} g'(x) = \lim_{x\to 0} h'(x) = 0$. Now consider $g''(x) = -x\cos x - \sin x$

and $h''(x) = 6x$. Note that $\lim_{x\to 0} g''(x) = \lim_{x\to 0} h''(x) = 0$. Finally, $g'''(x) = x\sin x - 2\cos x$, and $h'''(x) = 6$. By the Algebra of Limits, $\lim_{x\to 0} \frac{g'''(x)}{h'''(x)} = \frac{0 - 2\cdot 1}{6} = -\frac{1}{3}$. Applying l'Hôpital's Rule three times, then, we find that

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = -\frac{1}{3}.$$

Therefore, f' is differentiable at 0, and $f''(0) = -1/3$.

5.1 (a) Let P be any partition of $[a, b]$ for which P does not contain x_1 and x_2 as partition points, and for which x_1 and x_2 are contained in two different subintervals I_k and I_m. Then

$$S_P^-(f) = \sum_{j=1}^{n} (x_j - x_{j-1}) \cdot \left(\inf_{I_j} f\right)$$

$$= \sum_{j\neq k,m} (x_j - x_{j-1}) \cdot \left(\inf_{I_j} f\right) + (x_k - x_{k-1}) \cdot \left(\inf_{I_k} f\right) + (x_m - x_{m-1}) \cdot \left(\inf_{I_m} f\right) \qquad (*)$$

Now consider $S_P^-(g)$ and note that if we replace f with g in $(*)$, then the last two terms vanish because

$$\inf_{I_k} g = \inf_{I_m} g = 0,$$

using the positivity of f. But the first term in $(*)$ is unchanged. So now altering the partition so that the subintervals I_k and I_m are small, we can, for any given $\varepsilon > 0$, find P so that

$$S_P^-(g) + \varepsilon > S_P^-(f).$$

But we always have $S_P^-(g) \leq S_P^-(f)$, and therefore

$$\sup\{S_P^-(g) : P \text{ is a partition of } [a, b]\}$$

$$= \sup\{S_P^-(f) : P \text{ is a partition of } [a, b]\} = \int_a^b f(x)dx$$

But now observing that for any P, $S_P^+(g) = S_P^+(f)$, we also have

$$\inf\{S_P^+(g) : P \text{ is a partition of } [a, b]\}$$

$$= \inf\{S_P^+(f) : P \text{ is a partition of } [a, b]\} = \int_a^b f(x)dx$$

Therefore, g is integrable, and

$$\int_a^b g(x)dx = \int_a^b f(x)dx$$

(b) Let $E = \{x_1, \ldots, x_k\}$ be points of $[a,b]$ and let $g : [a,b] \to \mathbf{R}$ be given by

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in E - [a,b] \\ r_i & \text{if} \quad x = x_i \in E \end{cases}$$

where r_i are any numbers. Then f has been changed at finitely many points to produce g. The idea now is to choose partitions P of $[a,b]$ so that the subintervals containing points of E are "small". Since f is bounded on $[a,b]$, there exist $m, M \in \mathbf{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a,b]$, and if $m \leq r_i \leq M$ for all i, then we have $m \leq f(x) \leq M$ as well. Otherwise, we can at least say

$$\min\{m, r_1, r_2, \ldots, r_k\} \leq g(x) \leq \max\{M, r_1, r_2, \ldots, r_k\},$$

which we write as $m' \leq g(x) \leq M'$. So now observe that for all $x \in [a,b]$, we have

$$|f(x) - g(x)| \leq M' - m',$$

from which we obtain

$$|\sup_I f - \sup_I g| \leq M' - m' \quad \text{and} \quad |\inf_I f - \inf_I g| \leq M' - m'$$

where I is any subinterval of $[a,b]$, or even $[a,b]$ itself. So now given any $\varepsilon > 0$, choose a partition P so that the subintervals containing points of E each have length less than $\dfrac{\varepsilon}{2k(M' - m')}$. Then since each $x_i \in E$ can be contained in at most two of the subintervals (which occurs only if x_i happens to be a common endpoint of two adjacent subintervals), there are at most $2k$ subintervals containing points of E, and their total length is less than $\dfrac{\varepsilon}{M' - m'}$. Now write $P = \{y_0, y_1, \ldots, y_n\}$, and then

$$|S_P^+(f) - S_P^+(g)| = \left| \sum_{j=1}^{n} (y_j - y_{j-1}) \cdot \left(\sup_{I_j} f \right) - \sum_{j=1}^{n} (y_j - y_{j-1}) \cdot \left(\sup_{I_j} g \right) \right|$$

$$= \left| \sum_{j=1}^{n} (y_j - y_{j-1}) \cdot \left(\sup_{I_j} f - \sup_{I_j} g \right) \right| \qquad (*)$$

Now

$$\sup_{I_j} f - \sup_{I_j} g = 0 \text{ if } I_j \cap E = \phi.$$

Otherwise,

$$\left| \sup_{I_j} f - \sup_{I_j} g \right| \leq M' - m',$$

so $(*)$ is less than or equal to $M' - m'$ times the total of the lengths of the subintervals containing points of E, and thus

$$|S_P^+(f) - S_P^+(g)| < \varepsilon.$$

Similarly, for any given $\varepsilon > 0$, we can find a partition P for which

$$|S_P^-(f) - S_P^-(g)| < \varepsilon.$$

Hence

$$\inf\{S_P^+(g) : P \text{ is a partition of } [a,b]\}$$

$$= \inf\{S_P^+(f) : P \text{ is a partition of } [a,b]\} = \int_a^b f(x)dx$$

and

$$\sup\{S_P^-(g) : P \text{ is a partition of } [a,b]\}$$

$$= \sup\{S_P^-(f) : P \text{ is a partition of } [a,b]\} = \int_a^b f(x)dx$$

Therefore, g is integrable, and

$$\int_a^b g(x)dx = \int_a^b f(x)dx$$

5.2 (a) Let $\varepsilon > 0$ be given.

$$\text{Choose } x_1 \in (0,1) \text{ such that } x_1 < \frac{\varepsilon}{2} \quad (*)$$

Let $E' = [x_1, 1] \cap E$, and observe that this set is equal to the set

$$\left\{\frac{1}{N}, \frac{1}{N-1}, \frac{1}{N-2}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right\}$$

for some $N \in \mathbf{N}$. Now choose a partition P which includes the point x_1, so that $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ and then note that the subinterval I_1 contains infinitely many points of E, while the other subintervals contain the N points left over (i.e., the points of E'). Choose P in such a way that any subinterval containing a point of E' has length less than $\frac{\varepsilon}{4N}$. There could be a total of $2N$ such subintervals, so their total length is less than $\varepsilon/2$. Also, the length of I_1 is less than $\varepsilon/2$, by $(*)$. So the total length of all subintervals containing points of E is less than ε. On each such subinterval, $\inf f = 0$ and $\sup f = 1$, but on all other subintervals of P, $\inf f = \sup f = 0$. Hence $S_P^-(f) = 0$, while $0 < S_P^+(f) < \varepsilon$. But now since ε is arbitrary,

$$\inf\{S_P^+(f) : P \text{ is a partition of } [0,1]\} = 0,$$

and certainly

$$\sup\{S_P^-(f) : P \text{ is a partition of } [0,1]\} = 0$$

since all lower sums are zero. So by Definition 5.2, f is integrable on $[0,1]$, and $\int_0^1 f(x)dx = 0$.

(b) Though f turns out to be integrable with A taken to be the set E in part (a), in general it is not true that f is integrable for any choice of countably infinite subset A of $[0,1]$, for if we take $A = [0,1] \cap \mathbf{Q}$, then we have $f : [0,1] \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } \ x \text{ is rational} \\ 0 & \text{if } \ x \text{ is irrational} \end{cases}$$

So now no matter what partition P we choose, every subinterval contains both rational and irrational numbers, and therefore, $\inf f = 0$ and $\sup f = 1$ on every subinterval. Therefore the lower sum equals 0 and the upper sum equals 1. So the supremum of all possible lower sums is 0, and the infimum of all possible upper sums is 1, and since these two numbers do not agree, f is not integrable on $[0,1]$.

5.7 First note that since f is continuous on $[a,b]$, we know $\lim_{x \to x_0} f(x) = f(x_0)$. So the number $f(x_0)$, which is positive, plays the role of L in Exercise 2.5(a). By that result, there is a positive number α and a deleted neighborhood of x_0 on which $f(x) > \alpha$. In fact, if $L > 0$ in Exercise 2.5(a), one can give an argument in which α is taken to be $L/2$, and show that the existence of the limit at x_0 implies that $f(x) > L/2$ holds on some deleted neighborhood of x_0. It follows that f remains positive on this deleted neighborhood. Now in our case, $L = f(x_0)$, and so we have $f(x) > \frac{1}{2}f(x_0)$ on some deleted neighborhood of x_0; but then if x is taken to be x_0, it is certainly true that $f(x_0) > \frac{1}{2}f(x_0)$, and so we in fact have some interval on which $f(x) > \frac{1}{2}f(x_0) > 0$. If x_0 is equal to either a or b, then the interval is of the form $[x_0, x_0 + \delta)$ or $(x_0 - \delta, x_0]$, but otherwise, the interval can be chosen to be of the form $(x_0 - \delta, x_0 + \delta)$. We shall assume this is the case; the arguments for the other special cases are similar. Then observe that we also have $f(x) > \frac{1}{2}f(x_0) > 0$ true on the closed interval $I = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$. Therefore, $\inf_I \geq \frac{1}{2}f(x_0)$. So we choose P to be the following partition of $[a,b]$:

$$P = \{a, x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}, b\}$$

Then by the nonnegativity of f on $[a,b]$, the lower and upper sums associated with this P are nonnegative. The lower sum, $S_P^-(f)$, involves three subintervals, so there are three terms in the sum, each of which is nonnegative. The middle term involves I, which has length δ, and we have

$$S_P^-(f) \geq \delta \cdot \left(\inf_I f\right) \geq \frac{\delta f(x_0)}{2} \qquad (*)$$

But now if P is any partition of $[a,b]$, and we consider the corresponding lower sum, we can throw out all terms which involve subintervals that do not intersect I, and we find that the inequality $(*)$ still holds. Therefore the supremum of all possible upper sums is greater than or equal to $\frac{\delta f(x_0)}{2}$. Since f is continuous on $[a,b]$, we know f is integrable on $[a,b]$ by Proposition 5.4. Therefore the integral is defined, and we also know from Defintion 5.2 that

$$\int_a^b f(x)dx = \sup\{S_P^-(f) : P \text{ is a partition of } [a,b]\},$$

which is greater than or equal to $\frac{\delta f(x_0)}{2}$, a positive number. So the value of the integral is positive.

6.3 By Proposition 6.3, there exists $M \in \mathbf{R}$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} \leq M \qquad (*)$$

for all $x \in [a, x_0)$. Let

$$m = \sup_{x \in [a,x_0)} \frac{f(x) - f(x_0)}{x - x_0},$$

i.e., m is the least value of M for which $(*)$ holds. Now define $g : [a,b] \to \mathbf{R}$ by $g(x) = m(x - x_0) + f(x_0)$. (Note that g is of the form $cx + d$ with $c = m$ and $d = f(x_0) - mx_0$.) Then we have $g(x_0) = f(x_0)$, and we must show that $g(x) \leq f(x)$ for all $x \in [a,b] - \{x_0\}$. Suppose $x \in [a, x_0)$. Then by $(*)$ and the definition of m,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq m \qquad (**)$$

Since $x - x_0 < 0$, multiplying $(**)$ by $x - x_0$ yields

$$f(x) - f(x_0) \geq m(x - x_0),$$

so that $f(x) \geq m(x - x_0) + f(x_0) = g(x)$, as desired. Now suppose $x \in (x_0, b]$. Referring again to the definition of m, note that if we choose any $\varepsilon > 0$, we can find $z \in [a, x_0)$ for which

$$\frac{f(z) - f(x_0)}{z - x_0} > m - \varepsilon.$$

Then since $z < x_0 < x$, by Proposition 6.2 we have

$$m - \varepsilon < \frac{f(z) - f(x_0)}{z - x_0} < \frac{f(x) - f(x_0)}{x - x_0}$$

Multiplying by $x - x_0$ (which is now positive), we deduce

$$f(x) - f(x_0) > (m - \varepsilon)(x - x_0),$$

which yields

$$f(x) > g(x) - \varepsilon(x - x_0) \qquad (***)$$

So we find that $(***)$ holds for *any* given $\varepsilon > 0$. We claim that this implies $f(x) \geq g(x)$, for suppose this were not true, and $f(x) < g(x)$. Then there would exist $\varepsilon' > 0$ so that $f(x) < g(x) - \varepsilon'$ is also true. But now let $\varepsilon = \varepsilon'/(x - x_0)$. Then we would have $f(x) < g(x) - \varepsilon(x - x_0)$, contradicting $(***)$. Hence $f(x) \geq g(x)$.