MSCF Mathematics Preparatory Course
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Solutions to Homework #2 Exercises

3.3 (a) f is a rational function, and since $x^3 + 2x^2 + 5x + 1 \geq 1$ for all $x \in [0,1]$, we certainly don't have $x^3 + 2x^2 + 5x + 1 = 0$ for such x. Therefore, by Corollary 3.5, f is continuous on $[0,1]$.

(b) Let $g(x) = 3x^2$; let $h(x) = e^x$; let $k(x) = \sqrt{x}$; let $m(x) = \cos x$; and let $p(x) = -11$. Then k is continuous on $[\cos 1, 1]$ by Exercise 3.1, and m is continuous on $[0,1]$ (we may assume this is known by the comments preceding the chapter 3 exercises). By Corollary 3.12, then, $k \circ m = \sqrt{\cos x}$ is continuous on $[0,1]$. Next since h is continuous on $\mathbf{R}$, we have $h \circ (k \circ m) = e^{\sqrt{\cos x}}$ continuous on $[0,1]$, again by Corollary 3.12. Finally, g and p are continuous on $\mathbf{R}$, since they are polynomials. Therefore,

$$f = g \cdot (h \circ (k \circ m)) - p$$

is continuous on $[0,1]$ by Theorem 3.4. As a bonus, let me include the following argument, which shows how we could prove this part without Corollary 3.12. Occasionally you should challenge yourself to prove something with more primitive tools, just to see if it can be done, and also because sometimes the availability of a "powerful" theorem can cause you to lose sight of the foundational underpinnings of that theorem.

Lemma 1: If f is continuous at x_0, then $g(x) = e^{f(x)}$ is continuous at x_0.

Proof: Let $\varepsilon > 0$ be given. Let $y_0 = f(x_0)$. Then since e^y is continous on $\mathbf{R}$, and in particular at y_0, there exists $\eta > 0$ such that $|e^y - e^{y_0}| < \varepsilon$ whenever $|y - y_0| < \eta$. Then since f is continuous at x_0, we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \eta$ whenever $|x - x_0| < \delta$. Choose such an x, then, and let $y = f(x)$. Then $|y - y_0| = |f(x) - f(x_0)| < \eta$, and therefore

$$|e^{f(x)} - e^{f(x_0)}| < \varepsilon,$$

as desired. The proof is complete. We also need

Lemma 2: If $f : [0,1] \to [0,\infty)$ is continuous at x_0, then $g(x) = \sqrt{f(x)}$ is continuous at x_0.

I'll omit this proof, which would be a modification of the argument one would use to prove Exercise 3.1. Now we can use Lemma 2 to establish that $\sqrt{\cos x}$ is continuous on $[0,1]$; then we would have $e^{\sqrt{\cos x}}$ continuous on $[0,1]$ by Lemma 1, and finally we would apply Theorem 3.4 to conclude that f is continuous on $[0,1]$.

(c) By Exercise 3.1, $\sqrt{x}$ is continuous on $(0, 1]$. We are given that $\sin x$ is continuous on $\mathbf{R}$ and therefore on $[1, \infty)$. The function $1/x$ is a rational function and is continuous on $(0, \infty)$ since $x \neq 0$ on this domain. So by Corollary 3.12, $\sin(1/x)$ is continuous on $(0, 1]$. Finally, by Theorem 3.4, $\sqrt{x}\sin(1/x)$ is continuous on $(0, 1]$. Now we need to show that f is continuous at 0, i.e., we need to show that $\lim_{x\to 0^+} f(x) = f(0) = 0$. Let $\varepsilon > 0$ be given, and let $\delta = \varepsilon^2$. Then if $x \in (0, \delta)$, we have

$$|f(x) - 0| = |\sqrt{x}\sin(1/x)| \leq |\sqrt{x}| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon,$$

noting that $|\sin(1/x)| \leq 1$ for all $x > 0$. So f is continuous on $[0, 1]$.

(d) The function $g(x) = \sqrt{x}$ is continuous on $[0, 1]$ by Exercise 3.1. The function $h(x) = 1 - x$ is continuous on $[0, 1]$, since h is a polynomial. So $f(x) = (g \circ h)(x)$ is continuous on $[0, 1]$ by Corollary 3.12. Or we can argue as follows: Let $x_0 \in [0, 1]$. Then $\lim_{x\to x_0} h(x) = \lim_{x\to x_0}(1 - x) = \lim_{x\to x_0} 1 - \lim_{x\to x_0} x = 1 - x_0 = h(x_0)$, using the Algebra of Limits Theorem. So h is continuous at x_0, since $\lim_{x\to x_0} h(x) = h(x_0)$. Furthermore, if $x_0 \in [0, 1]$, then $h(x_0) \in [0, 1]$ as well. Since g is continuous on $[0, 1]$ by Exercise 3.1, g is continuous at $h(x_0)$. By Proposition 3.11, $\lim_{x\to x_0} f(x) = g(\lim_{x\to x_0} h(x)) = g(h(x_0)) = f(x_0)$, so that f is continuous at x_0.

(e) Consider the function $g(x) = \frac{1-x^2}{1-x}$. Since g is a rational function, g is continuous on $\mathbf{R} - \{1\}$. Let $h : [0, 1] \to \mathbf{R}$ be given by $h(x) = \tan x$. Since $h(x) = \frac{\sin x}{\cos x}$, and since $\sin x$ and $\cos x$ are given to be continuous on $\mathbf{R}$, h is continuous at any point at which $\cos x \neq 0$; since $\cos x > 0$ for all $x \in [0, 1]$, h is continuous on $[0, 1]$. Now let $x_0 \in [0, 1] - \{\frac{\pi}{4}\}$. Since $h(x_0) \neq 1$, we have that g is continuous at $h(x_0)$. Therefore by Corollary 3.12, f is continuous at x_0. We have established above that h is continuous on $[0, 1]$; in particular, h is continuous at $\frac{\pi}{4}$. Therefore, $\lim_{x\to\pi/4} \tan x = \tan(\pi/4) = 1$. Then

$$\lim_{x\to\pi/4} f(x) = \lim_{x\to\pi/4} \frac{1 - \tan^2 x}{1 - \tan x}$$

$$= \lim_{x\to\pi/4} \frac{(1 - \tan x)(1 + \tan x)}{1 - \tan x} = \lim_{x\to\pi/4}(1 + \tan x)$$

$$= 1 + \lim_{x\to\pi/4} \tan x = 1 + 1 = 2 = f(\pi/4).$$

This shows that f is continuous at $\pi/4$. Hence f is continuous on $[0, 1]$.

3.5 A polynomial of degree 3 on $\mathbf{R}^4$ satisfying the given conditions takes the form

$$P(x) = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_2x_3x_4 + x_1x_3x_4$$
$$+x_1x_2x_4 + x_1x_2x_3 + c_{1100}x_1x_2 + c_{1010}x_1x_3$$
$$+c_{1001}x_1x_4 + c_{0110}x_2x_3 + c_{0101}x_2x_4 + c_{0011}x_3x_4$$
$$+c_{1000}x_1 + c_{0100}x_2 + c_{0010}x_3 + c_{0001}x_4 + c_{0000}.$$

3.8 (a) We take $x_0 = 0$ in this example, and let $f, g : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} x & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

Then first note that $g \circ f : \mathbf{R} \to \mathbf{R}$ is just the zero function, i.e., $g \circ f \equiv 0$. So we certainly have $\lim_{x\to 0}(g \circ f)(x) = 0$. However, $\lim_{x\to 0} f(x) = 0$, so we have $L = 0$ here. But $g(L) = g(0) = 1$, which is not the value of $\lim_{x\to 0}(g \circ f)(x)$.

(b) Take $x_0 = 0$ again, and g as above, but now let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Then by Exercise 2.2, $\lim_{x\to 0} f(x)$ exists and equals 0. So we have $L = 0$. Then g indeed has a limit at L, viz., $\lim_{x\to 0} g(x) = 0$. But $\lim_{x\to 0}(g \circ f)(x)$ does not exist: Note that the formula for $g \circ f$ is

$$(g \circ f)(x) = \begin{cases} 0 & \text{if} \quad \sin\left(\frac{1}{x}\right) \neq 0 \\ 1 & \text{if} \quad \sin\left(\frac{1}{x}\right) = 0 \ \text{ or } \ x = 0 \end{cases}$$

Now no matter how small a deleted neighborhood of $x_0 = 0$ we consider, we have infinitely many values of x for which $f(x) = 0$, so that we then have $g(f(x)) = g(0) = 1$. But we also have infinitely many values of x at which $f(x) \neq 0$,and at all these values of x, we have $g(f(x)) = 0$. So 1 and 0 are both "candidates" for the limit, and therefore $\lim_{x\to 0}(g \circ f)(x)$ does not exist.

3.10 First note that for any $x, x_0 \in \mathbf{R}^n$, we have, as a consequence of the triangle inequality, Proposition 2.13 (iii),

$$\|x_0\| = \|(x_0 - x) + x\| \leq \|x_0 - x\| + \|x\|,$$

so that

$$\|x_0\| - \|x\| \le \|x_0 - x\| = \|x - x_0\| \quad (*)$$

But if we reverse the roles of x and x_0, and repeat the argument, we get

$$\|x\| = \|(x - x_0) + x_0\| \le \|x - x_0\| + \|x_0\|,$$

and then

$$\|x\| - \|x_0\| \le \|x - x_0\| \qquad (**)$$

Since both inequalities $(*)$ and $(**)$ hold, we in fact have

$$|\ \|x\| - \|x_0\|\ | \le \|x - x_0\|$$

So now to show that f is continuous on $\mathbf{R}^n$, we need to show that for every $x_0 \in \mathbf{R}^n$,

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \|x\| = f(x_0) = \|x_0\|$$

holds. So let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then if $x \in B(x_0, \delta) - \{x_0\}$, we have

$$|f(x) - f(x_0)| = |\ \|x\| - \|x_0\|\ | \le \|x - x_0\| < \delta = \varepsilon,$$

as desired. So the result is proved.

4.1 Let $x_0 \in (0, \infty)$ be given. Consider for a moment the functions $g, h : (0, \infty) \to \mathbf{R}$ given by $g(x) = -1$ for all $x > 0$, and $h(x) = x_0 x$. Then g, h are polynomials (restricted to $(0, \infty)$), and by Corollary 3.5, g and h are continuous at x_0; moreover, the rational function defined by g/h is continuous at x_0. Therefore,

$$\lim_{x \to x_0} \frac{g(x)}{h(x)} = \lim_{x \to x_0} \frac{-1}{x_0 x} = \frac{g(x_0)}{h(x_0)} = -1/x_0^2$$

So now we consider

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\frac{x_0 - x}{x_0 x}}{x - x_0} = \lim_{x \to x_0} \frac{-1}{x_0 x} = \frac{-1}{x_0^2},$$

and so we have the existence of the limit, and therefore f is differentiable at x_0, with $f'(x_0) = -1/x_0^2$.

4.2 (a) We investigate

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

by Exercise 2.2. So this shows that $g'(0)$ exists and equals 0.

(b) To show that f is not differentiable at 0, we investigate

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x\sin(\frac{1}{x})}{x} = \lim_{x\to 0} \sin\left(\frac{1}{x}\right),$$

which does not exist by Example 2.4. So $f'(0)$ does not exist. However, f is continuous at 0, as

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0 = f(0),$$

again using Exercise 2.2.

(c) Basically what happens here is that the graph of f is "sandwiched" between the lines $y = x$ and $y = -x$, so that, intuitively speaking, the graph of f is allowed to "bounce" too much near $x_0 = 0$ to allow for a "tangent line" at the point $(0, 0)$. But the graph of g is "sandwiched" between the curves $y = x^2$ and $y = -x^2$, and this forces a "horizontal tangent line" at $(0, 0)$. (An informal explanation, as was requested.)

(d) We have already computed $g'(0)$ and found it to be 0. So now, if $x_0 \neq 0$, we can compute $g'(x_0)$ using the Algebra of Derivatives and the Chain Rule (Theorem 4.7), since the underlying limit involves deleted neighborhoods of x_0 small enough to avoid 0. So we find, for $x_0 \neq 0$,

$$\begin{aligned} g'(x_0) &= x_0^2\left(\cos\left(\frac{1}{x_0}\right)\right)\left(\frac{-1}{x_0^2}\right) \\ &= -\cos\left(\frac{1}{x_0}\right) + 2x_0\sin\left(\frac{1}{x_0}\right), \end{aligned}$$

so that the formula for $g' : \mathbf{R} \to \mathbf{R}$ is

$$g'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

But this function is not continuous at 0, because it is not true that $\lim_{x\to 0} g'(x) = g'(0)$. In fact, $\lim_{x\to 0} g'(x)$ does not exist. Note that, by a similar argument as in Exercise 2.2,

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$$

does not exist. However,

$$\lim_{x\to 0} \left(2x\sin\left(\frac{1}{x}\right)\right)$$

does exist and equals 0. But since the limit of the other term does not exist, $\lim_{x\to 0} g'(x)$ does not exist.

4.4 Choose any $x_1, x_2 \in (a, b)$. Without loss of generality, suppose that $x_1 < x_2$. Then $[x_1, x_2] \subset (a, b)$, and the differentiability of f on (a, b) implies the differentiability of f on $[x_1, x_2]$, and in view of Lemma 4.6, we also have that f is continuous on $[x_1, x_2]$. Applying the Mean Value Theorem (4.10) to f on $[x_1, x_2]$, then, there exists $z \in (x_1, x_2)$ such that

$$f'(z) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But $f' \equiv 0$ on (a, b) means that $f'(z) = 0$. Therefore, $f(x_2) - f(x_1) = 0$, so that $f(x_1) = f(x_2)$. So arbitrarily choosing any two points $x_1, x_2 \in (a, b)$, we find that f must take the same value at those two points. Therefore, f is constant on (a, b). (*Note that the proof above is direct; if you found a proof by contradiction, look over your argument to see if it can modified to give a direct proof.*)