MSCF Mathematics Preparatory Course
August 2006
Solutions to Homework #1 Exercises

1.2

(a) Suppose $a_1, a_2 \in A$ with $g(f(a_1)) = g(f(a_2))$. Then we must have $f(a_1) = f(a_2)$ since g is one-to-one. But then since f is one-to-one, $a_1 = a_2$. Therefore $g \circ f$ is one-to-one. To show that $g \circ f$ is onto, choose any $c \in C$. Since g is onto C, there exists $b \in B$ for which $g(b) = c$. But then since f is onto B, there exists $a \in A$ for which $f(a) = b$. Therefore, $g(f(a)) = c$, and we have found $a \in A$ such that $(g \circ f)(a) = c$.

(b) Suppose $a_1, a_2 \in A$ are such that $f(a_1) = f(a_2)$. Then we certainly have $g(f(a_1)) = g(f(a_2))$ (otherwise $g \circ f$ is not a function). But since $g \circ f$ is one-to-one, this implies $a_1 = a_2$. Therefore, f is one-to-one. Now we show that g is onto. Choose any $c \in C$. Then there exists $a \in A$ such that $g(f(a)) = c$, since $g \circ f$ is onto. Let $b = f(a)$. Then $g(b) = c$. Hence we have found an element $b \in B$ which maps to c under g. So g is onto C.

(c) Let $f : \{w\} \to \{x, y\}$ be given by $f(w) = x$. Let $g : \{x, y\} \to \{z\}$ be given by $g(x) = z$ and $g(y) = z$ (which, observe, is the only possible function with the given domain and target space). Then f is not onto, since there does not exist any element a of the set $\{w\}$ for which $f(a) = y$. Also, g is not one-to-one, since $g(x) = g(y)$ but $x \neq y$. But $g \circ f : \{w\} \to \{z\}$ is given by $g(f(w)) = z$ and is one-to-one and onto.

1.4 If there exists a one-to-one, onto function $f : A \to B$, then we must have $A \neq \phi$, for there must be at least one element $x \in A$ in order for us to have a function defined on A at all. Now by Definition 1.6, part (ii), since A is finite, there exists $n \in \mathbf{N}$ and a function $g : \{1, \ldots, n\} \to A$ which is one-to-one and onto. Then by Exercise 1.2 (a), the function $f \circ g : \{1, \ldots, n\} \to B$ is one-to-one and onto, and this shows that B is finite.

1.5 By Definition 1.12, A countably infinite implies that there exists a one-to-one, onto function $f : \mathbf{N} \to A$. Then we are given that there exists a one-to-one, onto function $g : A \to B$. So again by Exercise 1.2 (a), the function $g \circ f : \mathbf{N} \to B$ is one-to-one and onto, which shows that B is countably infinite.

1.11 Since A is countably infinite, there exists a one-to-one, onto function $f : \mathbf{N} \to A$. Since B is countably infinite, there exists a one-to-one, onto function $g : \mathbf{N} \to B$. Consider the following "list" of elements of $A \cup B$:

$$f(1), g(1), f(2), g(2), f(3), g(3), \ldots$$

The list certainly includes every element of $A \cup B$ by virtue of f and g being onto. However, the list may involve repetition, since for example if $x \in A$ and $x \in B$, we may have $f(74) = g(1021) = x$. To avoid this, delete any item $g(k)$ on the list if $g(k) = f(i)$ for any $i \in \{1, 2, \ldots, k\}$, and delete any item $f(k)$ on the list if $f(k) = g(i)$ for any $i \in \{1, 2, \ldots, k-1\}$. The function $h : \mathbf{N} \to A \cup B$ induced by the resulting list is then one-to-one and onto, so that $A \cup B$ is countably infinite.

1.14 First suppose $n = 2$. Then

$$\mathbf{Q}^2 = \{(q_1, q_2) : q_1, q_2 \in \mathbf{Q}\}.$$

Since $\mathbf{Q}$ is countably infinite, we can write $\mathbf{Q} = \{q_1, q_2, q_3, q_4, \ldots\}$. (This is true due to the existence of a one-to-one, onto function $f : \mathbf{N} \to \mathbf{Q}$. By setting $q_i = f(i)$ for each i, we generate a "list" of elements of $\mathbf{Q}$.) Now construct a table of ordered pairs like so:

$$\begin{array}{ccccc}
(q_1, q_1) & (q_1, q_2) & (q_1, q_3) & (q_1, q_4) & \cdots \\
(q_2, q_1) & (q_2, q_2) & (q_2, q_3) & (q_2, q_4) & \cdots \\
(q_3, q_1) & (q_3, q_2) & (q_3, q_3) & (q_3, q_4) & \cdots \\
(q_4, q_1) & (q_4, q_2) & (q_4, q_3) & (q_4, q_4) & \cdots \\
(q_5, q_1) & (q_5, q_2) & (q_5, q_3) & (q_5, q_4) & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

Now convert the table into a single list, selecting items by moving through the table in diagonal fashion as in Figure 2 on page 25. Then we get

$$(q_1, q_1), (q_2, q_1), (q_1, q_2), (q_3, q_1), (q_2, q_2),$$

$$(q_1, q_3), (q_4, q_1), (q_3, q_2), (q_2, q_3), (q_1, q_4), \ldots$$

and the list induces a one-to-one onto function from $\mathbf{N}$ to $\mathbf{Q}^2$. So $\mathbf{Q}^2$ is countably infinite. Now to show $\mathbf{Q}^3$ is countably infinite, we can essentially use the same strategy by writing $\mathbf{Q} = \{q_1, q_2, q_3, q_4, \ldots\}$ and $\mathbf{Q}^2 = \{p_1, p_2, p_3, p_4, \ldots\}$ (now that we know that $\mathbf{Q}^2$ is countably infinite and can be so represented). Then by regarding a generic element (r, s, t) of $\mathbf{Q}^3$ as (r, u), with $r \in \mathbf{Q}$ and $u = (s, t) \in \mathbf{Q}^2$, one can construct a table similar to the one above and then convert it to a list, thereby inducing a one-to-one, onto function from $\mathbf{N}$ to $\mathbf{Q}^3$. Continuing in this manner, one can ultimately establish the existence of a one-to-one onto function from $\mathbf{N} \to \mathbf{Q}^n$ for any given n.

1.15 Define $f : I \to J$ as follows: Given $x \in I$, write $x = 0.d_1 d_2 d_3 d_4 d_5 d_6 \ldots$, where the d_i are the digits in a decimal expansion of x, chosen if necessary to avoid $d_i = 9$ for all i greater than or equal to some k. Let

$$f(x) = (0.d_1 d_3 d_5 d_7 \ldots, 0.d_2 d_4 d_6 d_8 \ldots).$$

Then $f(x)$ is an element of J since each coordinate is between 0 and 1. To show that f is onto J, suppose we choose any $(s,t) \in J$. Then since s and t each belong to $(0,1)$, we can consider their decimal expansions and write

$$(s,t) = (0.s_1s_2s_3s_4\ldots, 0.t_1t_2t_3t_4\ldots). \qquad (*)$$

Then $x = 0.s_1t_1s_2t_2s_3t_3\ldots$ satisfies $f(x) = (s,t)$. So we can always find some element of I which maps to a given element of J; therefore f is onto.

To show that f is one-to-one, suppose $x, y \in I$ with $x \neq y$, and we will show that $f(x) \neq f(y)$. Write $x = 0.d_1d_2d_3d_4\ldots$ and $y = 0.c_1c_2c_3c_4\ldots$. Then there exists some i for which $d_i \neq c_i$. Let $(s,t) = f(x)$ and consider (s,t) as in $(*)$. Let $(u,v) = f(y)$ and write

$$(u,v) = (0.u_1u_2u_3u_4\ldots, 0.v_1v_2v_3v_4\ldots).$$

If i is odd, then we have

$$s_{\frac{i+1}{2}} \neq u_{\frac{i+1}{2}},$$

so that $s \neq u$ and hence $f(x) \neq f(y)$. But if i is even, then we have

$$t_{\frac{i}{2}} \neq v_{\frac{i}{2}},$$

so that $t \neq v$ and hence $f(x) \neq f(y)$. Therefore, f is one-to-one.

2.1 Let $\varepsilon > 0$ be given. Then since $\lim\limits_{x\to 0} f(x) = 0$, there exists $\delta > 0$ so that if $x \in C$ and $0 < |x| < \delta$,

$$|f(x)| = |f(x) - 0| < \varepsilon/B.$$

But then for such x, we have

$$|f(x)g(x) - 0| = |f(x)g(x)| = |f(x)||g(x)| < (\varepsilon/B)\cdot B = \varepsilon.$$

Therefore, $\lim\limits_{x\to 0} f(x)g(x) = 0$, as desired.

2.2 We note that for any $x \neq 0$, $|sin(\frac{1}{x})| \leq 1$. Therefore $g : \mathbf{R} - \{0\} \to \mathbf{R}$, defined by $g(x) = \sin(\frac{1}{x})$, is bounded on any deleted neighborhood of 0. Since $\lim\limits_{x\to 0} x = 0$, we have
$ds \lim_{x\to 0} f(x) = 0$ by Exercise 2.1.

2.7 Let $D = [-1,1]$, and let $f : D \to \mathbf{R}$ and $g : D \to \mathbf{R}$ be given by $f(x) = 0$ for all $x \in D$, and

$$g(x) = \begin{cases} |x| & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$$

Then we have $f(x) < g(x)$ for all $x \in D$. Let $x_0 = 0$. Then $\lim_{x\to 0} f(x) = 0$, which, for good measure, we will prove: If $\varepsilon > 0$ is given, choose any $\delta \in (0,1)$. Then if $x \in (-\delta, \delta) - \{0\}$, we have

$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon,$$

as desired. We also have $\lim_{x\to 0} g(x) = 0$, since if $\varepsilon > 0$ is given, we can choose $\delta = \varepsilon$ (we may as well assume $\varepsilon < 1$), and if $x \in (-\delta, \delta) - \{0\}$, we have

$$|g(x) - 0| = ||x| - 0| = |x| < \delta = \varepsilon.$$

We could not have $L > M$ under these hypotheses, for then if we let $\varepsilon = \dfrac{L - M}{2}$, we can find $\delta', \delta'' > 0$ such that

$$L - \varepsilon < f(x) < L + \varepsilon \qquad (*)$$

for $x \in (x_0 - \delta', x_0 + \delta') - \{x_0\}$ and

$$M - \varepsilon < g(x) < M + \varepsilon \qquad (**)$$

for $x \in (x_0 - \delta'', x_0 + \delta'') - \{x_0\}$. Now take $\delta = \min\{\delta', \delta''\}$, and for $x \in (x_0 - \delta, x_0 + \delta) - \{x_0\}$, $(*)$ and $(**)$ both hold. Taking the right hand inequality in $(**)$ and multiplying by -1, we obtain

$$-g(x) > -M - \varepsilon,$$

which when added to left hand inequality in $(*)$ yields

$$f(x) - g(x) > L - M - 2\varepsilon = L - M - 2\left(\frac{L - M}{2}\right) = 0.$$

But this implies $f(x) > g(x)$ on $(x_0 - \delta, x_0 + \delta) - \{x_0\}$, which is a contradiction. Hence we must have $L \le M$.

2.8 We claim that if $x_0 \notin \mathbf{Z}$, then f does have a limit at x_0, and that $\lim_{x\to x_0} f(x) = x_0 - [x_0] = f(x_0)$. Let $\varepsilon > 0$ be given, and choose $\delta = \min\{\varepsilon, x_0 - [x_0], 1 + [x_0] - x_0\}$. Then if x satisfies $0 < |x - x_0| < \delta$, then we note that $x \in ([x_0], 1 + [x_0])$ by choice of δ, so that $[x] = [x_0]$. Therefore

$$|f(x) - f(x_0)| = |x - [x] - (x_0 - [x_0])|$$

$$= |x - x_0 + ([x_0] - [x])| = |x - x_0| < \delta \le \varepsilon,$$

so that indeed, $\lim_{x\to x_0} f(x) = f(x_0)$.

2.9 (i) First suppose that $\lim_{x \to x_0} f(x) = L$. Then given $\varepsilon > 0$, we can find $\delta > 0$ so that if

$$0 < \|x - x_0\| < \delta, \qquad (*)$$

then $\|f(x) - L\| < \varepsilon$. But if $(*)$ applies, then

$$|f_j(x) - L_j| = \sqrt{|f_j(x) - L_j|^2}$$

$$\leq \sqrt{\sum_{i=1}^{m} |f_i(x) - L_i|^2} = \|f(x) - L\| < \varepsilon.$$

Hence $\lim_{x \to x_0} f_j(x) = L_j$. (ii) On the other hand, if $\lim_{x \to x_0} f_j(x) = L_j$ is true for each $j \in \{1, \ldots, m\}$, then given $\varepsilon > 0$, we can find $\delta_j > 0$ so that $|f_j(x) - L_j| < \frac{\varepsilon}{\sqrt{m}}$ is true whenever x satisfies

$$0 < \|x - x_0\| < \delta_j. \qquad (**)$$

Now let $\delta = \min\{d_j : j = 1, 2, \ldots, m\}$. Then if $0 < \|x - x_0\| < \delta$, we have $(**)$ satisfied for every j, so that

$$\|f(x) - L\| = \sqrt{\sum_{j=1}^{m} |f_j(x) - L_j|^2}$$

$$< \sqrt{\sum_{j=1}^{m} \frac{\varepsilon^2}{m}} = \sqrt{\frac{m\varepsilon^2}{m}} = \varepsilon,$$

as desired. Therefore, $\lim_{x \to x_0} f(x) = L$.