Regarding Exercise 1.14

Some of you did not understand what is meant by the set $\mathbf{Q}^n$. The problem says

$$\mathbf{Q}^n = \{(q_1, q_2, \ldots, q_n) : q_i \in \mathbf{Q} \ \text{ for each } \ 1 = 1, 2, \ldots, n\}$$

The equals sign makes this look like an equation, but it's more properly considered a definition. And since it's a definition of a *set*, we might even consider it an explanation of what the set is. So the "equation" is like an explanatory sentence. When you see a set "explained" in notation like this, you should read the "sentence" in your head as though it is written in words. And remember that the colon separates the *generic form* of members of the set from one or more conditions which must apply.

So in this case, the "sentence" might read

> $\mathbf{Q}^n$ is the set of all objects of the form $(q_1, q_2, \ldots, q_n)$, where q_i belongs to $\mathbf{Q}$ for $i = 1$, and for $i = 2$, and for $i = 3$, and so on up to $i = n$.

Now what is the set $\mathbf{Q}$? It is the set of all rational numbers. So another way to decode the "sentence" $(*)$ is to say:

> $\mathbf{Q}^n$ is the set of all objects of the form $(q_1, q_2, \ldots, q_n)$, where q_i is a rational number for $i = 1$, and for $i = 2$, and for $i = 3$, and so on up to $i = n$.

Here's another translation of $(*)$:

> $\mathbf{Q}^n$ is the set of all objects of the form $(q_1, q_2, \ldots, q_n)$, where q_1 is a rational number, and q_2 is a rational number, and so is q_3, and so on up to q_n.

Now notice something: Each of these explanations assume that the value of n is fixed. And so you must understand that the first sentence of the problem statement tells us this. It says, "Let $n \in \mathbf{N}$." What this means is, "Assume that some positive integer has been given to you. Call it n, and consider its value fixed."

So for instance, if the positive integer 21 were handed to us, then we'd be dealing with the set $\mathbf{Q}^{21}$, which consists of all objects of the type $(q_1, \ldots, q_{21})$; this means that *every* member of $\mathbf{Q}^{21}$ is essentially a (terminating) "list" of 21 rational numbers (with repetition allowed). Not only is every member of $\mathbf{Q}^{21}$ such a "list", but the set $\mathbf{Q}^{21}$ consists of *all possible* such "lists". (Each is what we'd call a 21-*tuple*, and in general, given n, we call such a "list" an n-*tuple*.)

Suppose n happens to be 3. Then $\mathbf{Q}^n$ would be $\mathbf{Q}^3$, which is the set of all possible triplets of rational numbers. Here are some examples of members of $\mathbf{Q}^3$:

$$\left(-\frac{1}{80}, 52, -\frac{7}{6}\right), \quad \left(-\frac{2}{283}, 0, 0\right), \quad (-1, -1, -1), \quad \left(2, 3, \frac{1}{2}\right), \quad \left(3, 2, \frac{1}{2}\right)$$

Now you must envision every possible way of filling three slots with rational numbers, repetition allowed. That's how you would "generate" the set $\mathbf{Q}^3$.

To prove **1.14**, you need an argument which applies to every possible value of n. If $n = 1$, then we have $\mathbf{Q}^1$, which would just be $\mathbf{Q}$, already known to be countable. So you may as well assume $n \geq 2$, but you need a single argument that holds true whatever value of n might be chosen. This is somewhat like a magician saying, "Pick a card, any card." The magician needs to be able to execute the trick *no matter what card might be picked.* It's no good to have a trick that can only be carried out if the volunteer picks a club, or an ace. The trick has to work every time. Similarly, you need a proof that would cover any specific case, such as $n = 171$, or $n = 90,455$, or whatever.

Finally, the set $\mathbf{Q}^n$ is not the union of sets $\mathbf{Q}^1$, $\mathbf{Q}^2$, $\mathbf{Q}^3$, and so on up to $\mathbf{Q}^k$ for whatever value of k you might specify. The reason this is nonsense is that every member of $\mathbf{Q}^{21}$ is a "list" of 21 rational numbers. But now, for example, let's look at the set $\mathbf{Q}^3$. Every member of $\mathbf{Q}^3$ is a *triplet* of rational numbers. A triplet is a completely different object than a list-of-21. The members of $\mathbf{Q}^3$ and of $\mathbf{Q}^{21}$ are *of entirely different type.* So there is no way that $\mathbf{Q}^{21}$ can be the union of $\mathbf{Q}^3$ and *any* other set. You would have to have sets whose members are all 21-tuples in order to form the union of such sets and obtain $\mathbf{Q}^{21}$.

For the same reason, we cannot say (for example) that $\mathbf{Q}^2$ is a subset of $\mathbf{Q}^3$, which is in turn a subset of $\mathbf{Q}^4$, etc. No. A set A is a subset of B if every member of A is also in B. It is not true that every member of $\mathbf{Q}^2$ is also a member of $\mathbf{Q}^3$. In fact, *no* element of $\mathbf{Q}^2$ belongs to $\mathbf{Q}^3$, because every member of $\mathbf{Q}^2$ is a *pair* of rational numbers, and the set $\mathbf{Q}^3$ does not contain any such object.

Neither can we say that $\mathbf{Q}^n$ is the union of n "copies" of the set $\mathbf{Q}$. Refer to Exercise 1.9, where *union* is defined, and note that it says $A \cup B$ consists of everything that is either in the set A or is in the set B (including anything that happens to belong to *both* sets). So to claim that $\mathbf{Q}^2 = \mathbf{Q} \cup \mathbf{Q}$ (for example) would be to say that $\mathbf{Q}^2$ consists of everything that is either in $\mathbf{Q}$ or is in $\mathbf{Q}$. And that is a rather silly statement, isn't it? And it just reduces to saying that $\mathbf{Q}^2$ is made up solely of everything that belongs to $\mathbf{Q}$. And that is not true.