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Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions

Dejan Slepčev

Department of Mathematics, University of Texas at Austin, Austin, TX 78712-1082, USA

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Abstract

The convergence of schemes for propagation of fronts in a bounded domain moving with normal velocities is studied. The velocities considered depend on the principal curvatures, the normal direction, the location, as well as some nonlocal properties of the front. Most of the schemes considered are in essence threshold dynamics type approximation schemes, modified for Neumann boundary conditions and nonlocal terms. The existence and uniqueness of appropriately defined viscosity solutions of the level-set equations describing the nonlocal motions is also shown.

Keywords: Front propagation; Approximation scheme; Threshold dynamics; Nonlocal parabolic equations; Viscosity solutions

1. Introduction

In the present paper we study the motion of fronts by normal velocities in bounded domains. At the boundary of the domain fronts are orthogonal to it. The velocities considered can depend on nonlocal terms, in addition to the curvature, the normal direction and the location of the front. For example, they can depend on the size of the set that the front encloses.

The general form of velocities that we study is:

$$v = v(x, t, n, Dn, \Omega_t),$$

E-mail address: dslepcev@math.utexas.edu (D. Slepčev).

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where n is the unit outward normal vector to the front, Dn is the derivative of (an arbitrary, unit length, smooth extension of) n, and the front at time t is the boundary of the set Ω_t. We also require that v is nonincreasing in Dn and nondecreasing in Ω_t (with respect to set inclusion).

The main focus of the paper is on threshold dynamics type approximations schemes for the given velocities. To that end, we modify and generalize the threshold dynamics schemes studied by Ishii et al. [17]. We refer to that paper for a more thorough introduction on threshold dynamics type schemes.

Motion of fronts (interfaces, hypersurfaces) by a normal velocity depending on the (local) geometry of the fronts has been extensively studied in the past decade. One of the main mathematical difficulties is that fronts can develop singularities. Fortunately, for a large class of velocities, there is a weak formulation, based on the level set approach, that provides definition of fronts past the singularities. The level set approach was introduced by Osher and Sethian [19] for numerical computations. Viscosity solutions of equations of the level-set approach were used by Evans and Spruck [11] (for motion by mean curvature), and Chen et al. [5] (for a large class of motions) to rigorously define generalized (weak) front propagation.

Fronts propagating in generalized sense have been shown to govern asymptotic behavior of solutions to a number of reaction-diffusion equations. The first result in this direction was obtained for the Allen–Cahn equation by Evans et al. (see [10]), and a large number of results followed; see for example [21] and [3] (where a more geometric, but equivalent definition of generalized front propagation was given) and the references therein for more details.

In a different development, Chen et al. [6] have studied the limiting behaviour of an Allen–Cahn equation with a nonlocal term. The equation was obtained as a limit of a system of reaction-diffusion equations often referred to as the Belousov–Zhabotinskii model. It was shown that the interfaces (as long as they are smooth hypersurfaces) that appear in the solutions to the Allen–Cahn equation move (in the limit) by a normal velocity that in addition the curvature of the front, depends on the size on the set bounded by the front ('interior of the front'). Recently, Kim showed that the result of [6] holds even if the fronts develops singularities. She used viscosity solutions (whose definition was extended to handle equations with nonlocal terms) to define generalized propagation of fronts by velocities depending nonlocally on the front.

In this paper we continue studying front propagation with nonlocal velocities. We use slightly different level-set equations than ones in [18]. The ones studied here are truly geometric, meaning that every level set moves with the given normal velocity. The appropriate definition of viscosity solutions is given and existence and uniqueness of viscosity solutions are proven in the second section. The definition of front propagation with nonlocal terms given in Section 3 is equivalent to the one given in [18]. This can be shown using the arguments given in [3] and [18] to prove the appropriate analogue of Theorem 2.2 in [18].

Beginning with Section 4 we turn our attention to approximation schemes. In [17], Ishii et al., studied threshold dynamics type approximation schemes for front propagation of fronts in $\mathbb{R}^N$. They adapted threshold dynamics models used in cellular automata modeling growth processes and excitable media introduced by Gravner and

Griffeath [14] and generalized and extended the scheme introduced by Bence et al. [4] for motion by mean curvature.

In the present paper we employ the same strategy as Ishii, Pires and Souganidis, to prove the convergence of the schemes for both local and nonlocal motions in bounded domain U. Most of the schemes we study are threshold dynamics type, although the convergence of a particular (nonthreshold) adaptation of the scheme introduced by Bence et al., in [4], is also proven.

The strategy is the following: at any time fronts moving within the set $\bar{U}$ are given as boundaries of subsets of $\bar{U}$. The algorithms considered are given as mappings $M_h : \mathscr{B} \to \mathscr{B}$ where $\mathscr{B}$ is the set of measurable subsets of $\bar{U}$. For a front given as the boundary of $A \subset \mathbb{R}^N$ the approximate position after time h is the boundary of $M_h A$. To each of these schemes for front propagation we associate an approximation scheme, S_h, for the level set equation of the given motion as follows: for φ a bounded function on $\bar{U}$ we define

$$S_h\varphi(x) := \sup\{\lambda \in \mathbb{R}\colon x \in M_h\{y \in \bar{U}\colon \varphi(y) \geqslant \lambda\}\}.$$

Using general results on convergence of monotone approximation schemes for viscosity solutions of parabolic equations we establish the convergence of the scheme S_h which implies convergence of scheme M_h.

The general criteria for convergence of monotone approximation schemes for nonlocal parabolic equations are given in Section 4. The basic idea for the proof comes from [2], although the presence of nonlocal terms forces us to introduce auxiliary schemes that are used in formulating the conditions and in the proofs.

In Section 5 we extend the threshold schemes of [17] to handle the boundary conditions, and also to handle a slightly larger class of motions. The convergence results are analogous to those of [17]. We also prove the convergence of an adaptation to Neumann boundary conditions of the scheme for mean curvature motion given in [4].

In Section 6 we construct threshold type schemes (in which there is an additional scaling of the test measures depending on the set that is being updated) for nonlocal motions which do not depend on the curvature of the front. Although the velocities considered are monotone in the appropriate arguments, the simplest approximation schemes are not monotone with respect to the set inclusion. Their convergence is proven by constructing two, more complicated, approximation schemes (that bound the approximation scheme of interest from above and below) whose convergence we show directly. At the end we consider schemes for curvature dependent nonlocal motions.

Note: After this paper was completed we have learned of new research by Ishii and Ishii [16] on threshold type approximation schemes for motion by mean curvature on bounded domains with Neumann boundary data. The schemes studied by Ishii and Ishii are similar to some of the schemes we studied in Section 5.2. However, they are not immediately extendable to anisotropic motions.

2. Existence and uniqueness

Let U be a bounded domain in $\mathbb{R}^N$ with C^1 boundary that satisfies the outside sphere condition. Let m be the Lebesgue measure on U. For two measurable subsets A, B of

U we say that $A \sim B$ if $m(A \triangle B) = 0$. Let $\mathscr{B}$ be the set of equivalence classes of measurable subsets of U with respect to relation $\sim$. We consider $\mathscr{B}$ with topology that comes from the metric $d(A,B) = m(A \triangle B)$. Let F be a function

$$F : \bar{U} \times [0,\infty) \times \mathbb{R}^N \setminus \{0\} \times S^N \times \mathscr{B} \to \mathbb{R}$$

that satisfies:

(F1) F is degenerate elliptic:

$$F(x,t,p,X,K) \geqslant F(x,t,p,Y,K) \quad \text{if } X \leqslant Y$$

(F2) F is nonincreasing in its set argument:

$$F(x,t,p,X,K) \geqslant F(x,t,p,X,L) \quad \text{if } K \subseteq L$$

(F3) F is geometric:

$$F(x,t,\lambda p,\lambda X + \mu p \otimes p,K) = \lambda F(x,t,p,X,K)$$

(F4) F is continuous (on its domain).
(F5) $-\infty < F_*(x,t,0,O,K) = F^*(x,t,0,O,L) < \infty$.

Here F^* denotes the upper semicontinuous envelope of F, while F_* is the lower semicontinuous envelope of F.

Let us consider the following equation:

$$u_t(x,t) + F(x,t,Du(x,t),D^2u(x,t),\{y\colon u(y,t) \geqslant u(x,t)\}) = 0 \quad \text{in } U \times (0,T), \tag{1a}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial U \times (0,T), \tag{1b}$$

$$u(x,0) = u_0(x) \quad \text{in } U. \tag{1c}$$

Here ν is the outward unit normal.

Definition 2.1. An upper semicontinuous function $u : \bar{U} \times [0,T) \to \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of Eq. (1) if for all $x \in \bar{U}$, $u(x,0) \leqslant u_0^*(x)$ and for all $(x,t) \in \bar{U} \times (0,T)$ and all functions $\varphi \in C^\infty(\bar{U} \times (0,T))$ such that $u - \varphi$ has maximum at (x,t), if $x \in U$ or $x \in \partial U$ and $\partial\varphi/\partial\nu(x,t) > 0$ then

$$\varphi_t(x,t) + F_*(x,t,D\varphi(x,t),D^2\varphi(x,t),\{y\colon u(y,t) \geqslant u(x,t)\}) \leqslant 0. \tag{2}$$

A lower semicontinuous function $v : \bar{U} \times [0,T) \to \mathbb{R} \cup \{\infty\}$ is a viscosity supersolution of Eq. (1) if for all $x \in \bar{U}$, $v(x,0) \geqslant u_{0_*}(x)$ and for all $(x,t) \in \bar{U} \times (0,T)$ and all functions $\varphi \in C^\infty(\bar{U} \times (0,T))$ such that $v - \varphi$ has minimum at (x,t), if $x \in U$ or $x \in \partial U$ and $\partial\varphi/\partial\nu(x,t) < 0$ then

$$\varphi_t(x,t) + F^*(x,t,D\varphi(x,t),D^2\varphi(x,t),\{y\colon u(y,t) > u(x,t)\}) \geqslant 0. \tag{3}$$

Note the difference in the choice of the 'test sets' in the definitions of a subsolution and a supersolution. That is an essential ingredient in extending the viscosity solutions to nonlocal, geometric parabolic equations. (If the supersolutions were defined with $\geqslant$ instead of $>$ in the definition of the test set, the existence of solutions, among other things, would not hold.) A function $u : \bar{U} \times [0, T) \to \mathbb{R}$ is a viscosity solution of (1) if u^* is a viscosity subsolution and u_* is a viscosity subsolution of (1). Since all the solutions in this paper are in viscosity sense, in the rest of the paper we are not using the term viscosity solution. Instead we just say solution.

Let us list some of the properties of subsolutions and supersolutions.

(P1) If u is a subsolution (resp. supersolution) of (1a) and $\rho : \mathbb{R} \to \mathbb{R}$ nondecreasing then $(\rho \circ u)^*$ is also a subsolution (resp. $(\rho \circ u)_*$ is a supersolution).

(P2) *Stability*: If $\{u_n\}_{n=1,2,\dots}$ is a sequence of subsolutions (resp. supersolutions) of (1a) bounded from above (resp. below) then $u = \limsup^* u_n$ is also a subsolution (resp. $u = \liminf_* u_n$ is a supersolution).

Here $\limsup^*_{n\to\infty} u_n(x,t) := \sup\{\limsup_{n\to\infty} u_n(x_n,t_n) : (x_m,t_m) \to (x,t) \text{ as } m \to \infty\}$. Note that $\limsup^*$ of a sequence of functions is an upper semicontinuous function.

For smooth strictly increasing functions property (P1) can be checked directly, using the definition of a subsolution. For general ρ (P1) then follows by approximating and using property (P2). Note that property (P1) implies that if u is a subsolution than so is $\operatorname{sign}^*(u)$. Let us verify property (P2):

Proof (of P2). Let $(x_0, t_0) \in \bar{U} \times (0, T)$. Let $\varphi \in C^\infty(\bar{U} \times (0, T))$ such that $u - \varphi$ has maximum at (x_0, t_0). We can assume that the maximum is strict and that $u(x_0, t_0) = 0$. Then from the definition of u follows that there is a subsequence of $\{u_n\}_{n=1,2,\dots}$ that we also denote by $\{u_n\}_{n=1,2,\dots}$ such that $u_n - \varphi$ has maximum at (x_n, t_n) and

$$(x_n, t_n) \to (x_0, t_0) \quad \text{and} \quad u_n(x_n, t_n) \to u(x_0, t_0) \quad \text{as } n \to \infty.$$

Let us first consider the case $x_0 \in U$. By starting from a large enough index if necessary we can assume than $x_n \in U$ for all n. So since u_n are subsolutions

$$\varphi_t(x_n, t_n) + F_*(x_n, t_n, D\varphi(x_n, t_n), D^2\varphi(x_n, t_n), \{u_n \geqslant u_n(x_n, t_n)\}) \leqslant 0. \tag{4}$$

To continue the proof we need the following fact: Let f_n be a sequence of measurable functions on U and $f \geqslant \limsup^* f_n$ and a_n a sequence converging to 0. Then

$$m(\{f_n \geqslant a_n\} \setminus \{f \geqslant 0\}) \to 0 \quad \text{as } n \to \infty. \tag{5}$$

To show this first note that

$$\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{f_i \geqslant a_i\} \subseteq \{f \geqslant 0\}.$$

Let $\varepsilon > 0$ then there exists n_0 such that for all $n \geqslant n_0$

$$m\left(\bigcup_{i=n}^{\infty} \{f_i \geqslant a_i\} \setminus \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} \{f_i \geqslant a_i\} \right) < \varepsilon .$$

So for $n \geqslant n_0$

$$m(\{f_n \geqslant a_n\} \setminus \{f \geqslant 0\}) \leqslant m\left(\bigcup_{i=n}^{\infty}\{f_i \geqslant a_i\} \setminus \bigcap_{m=1}^{\infty}\bigcup_{i=m}^{\infty}\{f_i \geqslant a_i\}\right) < \varepsilon\,.$$

Note that (5) would not hold if $\geqslant$ was replaced by $>$.

A consequence of (5), applied to $f_n := u_n$ and $a_n := u_n(x_n, t_n)$ is that the sets $\{u_n \geqslant a_n\} \cup \{u \geqslant 0\}$ converge to $\{u \geqslant 0\}$ as n goes to infinity. From inequalities (4) we obtain

$$\begin{aligned}0 &\geqslant \limsup_{n\to\infty}(\varphi_t(x_n,t_n) + F_*(x_n,t_n,D\varphi,D^2\varphi,\{u_n \geqslant a_n\}))\\ &\geqslant \limsup_{n\to\infty}(\varphi_t(x_n,t_n) + F_*(x_n,t_n,D\varphi,D^2\varphi,\{u_n \geqslant a_n\} \cup \{u \geqslant 0\}))\\ &\geqslant \varphi_t(x_0,t_0) + F_*(x_0,t_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0),\{u \geqslant 0\}).\end{aligned}$$

If $x_0 \in \partial U$ we need to consider only the functions φ for which $\partial\varphi/\partial v(x_0,t_0) > 0$. But then $\partial\varphi/\partial v > 0$ in some neighborhood of (x_0,t_0) which makes the rest of the proof same as above. □

The following lemma is an analogue (for nonlocal equations) of Proposition 2.2 in [1]. It gives a criterion for a function to be a subsolution. The requirements are weaker than the ones in the definition, which make the lemma useful when verifying that a certain function is a subsolution. Its proof is a straightforward adaptation of the proof given in [1].

Lemma 2.2. *An upper semicontinuous function $u : \bar{U} \times [0,T) \to \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of Eq.* (1) *if for all $x \in \bar{U}$, $u(x,0) \leqslant u_0^*(x)$ and for all $(x,t) \in \bar{U} \times (0,T)$ and all functions $\varphi \in C^\infty(\bar{U} \times (0,T))$ such that $u - \varphi$ has maximum at (x,t), if $x \in U$ and $D\varphi(x,t) \neq 0$ or $D^2\varphi(x,t) = 0$, or if $x \in \partial U$ and $\partial\varphi/\partial v(x,t) > 0$ the following inequality holds*:

$$\varphi_t(x,t) + F_*(x,t,D\varphi(x,t),D^2\varphi(x,t),\{y\colon u(y,t) \geqslant u(x,t)\}) \leqslant 0. \tag{6}$$

An analogous statement is true for supersolutions.

To show the comparison we also need to assume:

(F6) There exist positive constant c such that for all $X, Y \in S^N$ and nonnegative numbers v, μ, ξ for which the inequality

$$\langle Xl, l\rangle + \langle -Ym, m\rangle \leqslant v|l-m|^2 + \mu(|l|^2 + |m|^2) + \xi|l-m|(|l| + |m|)$$

holds for every $l, m \in \mathbb{R}^N \setminus \{0\}$, the inequality

$$F(x,t,p,X,K) - F(x,t,q,Y,K) \geqslant -c(v|\hat{p}-\hat{q}|^2 + \mu + \xi|\hat{p}-\hat{q}| + |p-q|)$$

holds for all $p, q \in \mathbb{R}^N \setminus \{0\}$ and all $(x,t,K) \in \bar{U} \times (0,T) \times \mathscr{B}$. Here $\hat{p} = p/|p|$.

(F7) For given constant $C > 0$, there exists a function $\omega : [0,\infty] \to [0,\infty]$ such that $\omega(0+) = 0$ and, if $|p| \leqslant \alpha|x-y|$ and $\|X\| \leqslant C\alpha$, then, for all $K \in \mathscr{B}$

$$F(x,t,p,X,K) - F(y,t,p,X,K) \leqslant \omega(\alpha|x-y|^2 + |x-y|).$$

Remark. We need the conditions (F6) and (F7) instead of the usual condition ('F8') in order to handle the special test function used in proving comparison for equations with Neumann boundary conditions.

('F8') There exists a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ and if

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$F(y,t,\alpha(x-y),Y,K) - F(x,t,\alpha(x-y),X,K) \leqslant \omega(\alpha|x-y|^2 + |x-y|)$$

for all $x, y \in \bar{U}$, $t \in (0,T)$, $X, Y \in S^N$ and $K \in \mathscr{B}$.

Note that (F7) implies ('F8'). Although the conditions (F6) and (F7) are more restrictive than ('F8') many equations satisfy them; for example equations in which

$$F(p,X,K) = -\text{trace}(A(\hat{p})X) + B(\hat{p})m(K),$$

where $\hat{p} = p/|p|$, $A(\cdot)$ are symmetric matrix, $A(\cdot) \geqslant 0$ and $\sqrt{A(\cdot)}$ is a Lipschitz continuous function on S^{N-1} and $B(\cdot) \leqslant 0$.

Now we are ready to show the comparison for Eq. (1).

Theorem 2.3. *Let F be a function satisfying* (F1)–(F7). *Let u be a subsolution and v a supersolution of* (1a) *and* (1b). *If $u(x,0) \leqslant v(x,0)$ in $\bar{U}$ then $u(x,t) \leqslant v(x,t)$ on $\bar{U} \times [0,T)$.*

Proof. To prove the comparison for equations with Neumann boundary conditions we follow the approach taken by Giga and Sato in [13]. To handle the Neumann boundary conditions they proved a special version of Crandall–Ishii lemma and designed a special test function. Here we do not go through all details of their argument, as they can be found in [14]; rather we use a simplified problem to point out what needs to be added because of the presence of nonlocal terms.

For that reason let us assume that $u(x,t) \leqslant v(x,t)$ for $(x,t) \in \partial U \times [0,T)$. To show comparison we can now use the usual test function $|x-y|^4$.

Let us assume that the comparison does not hold. Then there exist time $\bar{t} < T$ and $\bar{x} \in U$ such that $u(\bar{x},\bar{t}) > v(\bar{x},\bar{t})$. Let τ be a time such that $\bar{t} < \tau < T$. Then for $\gamma > 0$ small enough the function $u(x,t) - v(x,t) - \gamma/(\tau - t)$ has a positive maximum on $\bar{U} \times [0,\tau)$, which by assumptions we made has to be in $U \times (0,\tau)$. Let

$$w_\varepsilon(x,y,t) = u(x,t) - v(y,t) - \frac{|x-y|^4}{4\varepsilon} - \frac{\gamma}{\tau - t}.$$

Let $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ be a maximum of w_ε. It follows (see [7]) that $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ converge (along a sequence of ε's converging to 0) to (x_0, x_0, t_0) where (x_0, t_0) is a maximum of $u(x,t) - v(x,t) - \gamma/(\tau - t)$ and furthermore $|x_\varepsilon - y_\varepsilon|^4/\varepsilon$ approaches 0 as ε goes to 0.

Let us consider first the case that there exists ε such that $x_\varepsilon = y_\varepsilon$. Then $x_\varepsilon = x_0$ and $t = t_0$. By the parabolic Crandall–Ishii lemma [CIL, Lemma 8.3] there are constants

a, b such that

$$(a, 0, O) \in \bar{\mathscr{P}}^{2,+}u(x_0, t_0) \quad \text{and} \quad (b, 0, O) \in \bar{\mathscr{P}}^{2,-}v(x_0, t_0)$$

and $a - b = (\gamma/(\tau - t_0)^2) > 0$. Since u is a subsolution and v a supersolution,

$$a + F_*(t_0, x_0, 0, O, \{u \geqslant u(x_0, t_0)\}) \leqslant 0 \quad \text{and}$$

$$b + F^*(t_0, x_0, 0, O, \{v > v(x_0, t_0)\}) \geqslant 0.$$

Assumption (F5) now implies that $a - b \leqslant 0$ which is in contradiction with $a - b > 0$.

Now let us consider the general case that is that $x_\varepsilon \neq y_\varepsilon$ for all $\varepsilon > 0$. Let $\alpha = |x_\varepsilon - y_\varepsilon|^2/\varepsilon$. From Crandall–Ishii lemma follows that there exist numbers a, b, symmetric matrices X, Y such that

$$-30\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 30\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

$$(a, \alpha(x_\varepsilon - y_\varepsilon), X) \in \bar{\mathscr{P}}^{2,+}u(x_\varepsilon, t_\varepsilon)$$

$$(b, \alpha(x_\varepsilon - y_\varepsilon), Y) \in \bar{\mathscr{P}}^{2,-}v(y_\varepsilon, t_\varepsilon)$$

and

$$a - b = \frac{\gamma}{(\tau - t_\varepsilon)^2}.$$

Hence

$$a + F(x_\varepsilon, t_\varepsilon, \alpha(x_\varepsilon - y_\varepsilon), X, \{u \geqslant u(x_\varepsilon, t_\varepsilon)\}) \leqslant 0$$

$$b + F(y_\varepsilon, t_\varepsilon, \alpha(x_\varepsilon - y_\varepsilon), Y, \{v > v(y_\varepsilon, t_\varepsilon)\}) \geqslant 0.$$

From $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ being a maximum of w_ε now follows that for all $x \in U$

$$u(x, t_\varepsilon) - v(x, t_\varepsilon) \leqslant u(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, t_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^4}{4\varepsilon}$$

$$u(x, t_\varepsilon) - u(x_\varepsilon, t_\varepsilon) < v(x, t_\varepsilon) - v(y_\varepsilon, t_\varepsilon)$$

$$\{u \geqslant u(x_\varepsilon, t_\varepsilon)\} \subseteq \{v > v(y_\varepsilon, t_\varepsilon)\}.$$

Property (F2) now leads to

$$a - b \leqslant F(y_\varepsilon, t_\varepsilon, \alpha(x_\varepsilon - y_\varepsilon), Y, \{u \geqslant u(x_\varepsilon, t_\varepsilon)\})$$
$$\qquad - F(x_\varepsilon, t_\varepsilon, \alpha(x_\varepsilon - y_\varepsilon), X, \{u \geqslant u(x_\varepsilon, t_\varepsilon)\})$$

which using property (F7) implies

$$a - b < \omega\left(\frac{|x_\varepsilon - y_\varepsilon|^4}{\varepsilon} + |x_\varepsilon - y_\varepsilon|\right).$$

Since we know that $|x_\varepsilon - y_\varepsilon|^4/\varepsilon$ converges to 0 as ε goes to 0 the last inequality implies that $a - b \leqslant 0$ which leads to contradiction. □

We can now use property (P2) to, via Perron's method (see [8]), establish existence of solutions.

Theorem 2.4. *Let F be a function satisfying* (F1)–(F7) *and u_0 a continuous function on $\bar{U}$. Then there exist a unique continuous solution of* (1).

Existence of continuous solutions is obtained using the Perron's method while the uniqueness follows from the comparison. If the function u_0 is not continuous, Perron's method can still be used to obtain existence, but the solution may not be unique.

3. Generalized front propagation with nonlocal velocities

Let us first recall the definition of the motion of a smooth front by a given normal velocity. Let $\{\Omega_t\}_{t\geqslant 0}$ be a family of open subsets of $\bar{U}$. Let $\Gamma_t := \partial\Omega_t$. Assume that $\{\Gamma_t\}_{t\geqslant 0}$ is a smooth family of $N-1$ dimensional submanifolds of $\bar{U}$ such that $\Gamma_t \perp \partial U$ on $\partial U \cap \Gamma_t$. For a general velocity v, that depends on the properties of the front, $v : \bar{U} \times \mathscr{B} \to \mathbb{R}$ we say that the family $\{(\Omega_t, \Gamma_t)\}_{t\geqslant 0}$ propagates with velocity v if for all smooth curves $\gamma : [a,b] \to U$ such that for all $t \in [a,b]$ $\gamma(t) \in \Gamma_t$ the following holds:

$$\left\langle \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t), n(\gamma(t)) \right\rangle = v(\gamma(t), \Omega_t)$$

for all $t \in [a,b]$. Here n is the outward (with respect to Ω_t) normal vector to Γ_t at the given point.

To define motion for fronts that may not be smooth for all times, we use the level set approach. For it to work it is necessary restrict the class of velocities that are considered. To be able to put appropriate restrictions on the velocities, let us consider the velocities written as $v(x, n(x), Dn(x), \Omega_t)$ where Dn is the gradient of (arbitrary smooth extension of) n. Let

$$F(x,p,X,A) := -|p|\, v\left(x, -\hat{p}, -\frac{1}{|p|}(I - \hat{p}\otimes\hat{p})X(I - \hat{p}\otimes\hat{p}), A\right).$$

Consider the equation

$$\begin{aligned} u_t + F(x, Du, D^2u, \{u(\cdot,t) \geqslant u(x,t)\}) = 0 \qquad & \text{in } U \times (0,T),\\ \frac{\partial u}{\partial \nu} = 0 \qquad & \text{on } \partial U \times (0,T),\\ u(\cdot,0) = g(\cdot) \quad & \text{in } U. \end{aligned} \tag{7}$$

Here g is a continuous function on $\bar{U}$. Suppose that the equation has a smooth solution u. Let

$$\Omega_t := \{x \in \bar{U} : u(x,t) > 0\} \quad \text{and} \quad \Gamma_t := \{x \in \bar{U} : u(x,t) = 0\}. \tag{8}$$

Suppose also that $Du \neq 0$ on Γ_t for all $t \in (0,T)$. It is elementary to check that (Ω_t, Γ_t) propagates with velocity v.

To be able to define the generalized propagation using the level sets of viscosity solutions of the Eq. (7) we need to impose conditions that guarantee existence and uniqueness of such solutions. Hence, let us assume that F satisfies conditions (F1)–(F7).

Definition 3.1. Let Ω_0 be an open subset of $\bar{U}$ and Γ_0 its boundary. Let g be a continuous function on $\bar{U}$ such that $\Omega_0=\{x\in\bar{U}\colon g(x)>0\}$ and $\Gamma_0=\{x\in\bar{U}\colon g(x)=0\}$. Let u be the unique viscosity solution of the level set Eq. (7) and Ω_t, Γ_t given by (8). We then say that the family $\{(\Omega_t,\Gamma_t)\}_{t\in[0,T)}$, is the generalized propagation of (Ω_0,Γ_0) by velocity v.

For this to be a valid definition it must not depend on the choice of the initial data, g. Proof of that can be found, for example, in [21]. Note that all the level sets of the solution u propagate with given velocity.

This definition allows us to study fronts when they are not smooth, develop singularities or change topological type. However, there are some difficulties. In particular although $\Gamma_0=\partial\Omega_0$ it may happen that the set Γ_t is not equal to $\partial\Omega_t$ and that $m(\Gamma_t)>0$. If that happens we say that the front fattens. Fattening is related to nonuniqueness of front propagation. Possibility of front fattening forces us to define the following mappings, which are used later in statements of convergence results.

For given velocity v, and time t let $X_t:\mathscr{F}\to\mathscr{F}$ and $N_t:\mathscr{O}\to\mathscr{O}$ (here $\mathscr{F}$ are the closed and $\mathscr{O}$ the open subsets of $\bar{U}$) be defined in the following way: for a closed set $A\subseteq\bar{U}$ let $\Omega_0=\operatorname{int}(A)$ and $\Gamma_0=A\setminus\Omega_0$ (respectively for an open set B let $\Omega_0=B$ and $\Gamma_0=\partial B$). Let (Ω_t,Γ_t) be the generalized propagation of (Ω_0,Γ_0) at time t. Then $X_t(A):=\Omega_t\cup\Gamma_t$ and $N_t(B):=\Omega_t$.

4. Convergence of monotone schemes

We consider approximation schemes to problem (1). To construct them we use an auxiliary scheme

$$S:\mathscr{B}\times\mathbb{R}^+\times\mathbb{R}_0^+\times B(\bar{U})\to B(\bar{U}),$$

where $B(\bar{U})$ is the set of bounded measurable functions on $\bar{U}$. We write $S(A,h,t)u$ for $S(A,h,t,u)$. The actual approximation scheme

$$\hat{S}:\mathbb{R}^+\times\mathbb{R}_0^+\times B(\bar{U})\to B(\bar{U})$$

is then defined by

$$\hat{S}(t,h,u)(x):=\sup\{\lambda\colon S(\{u\geqslant\lambda\},t,h,u)(x)\geqslant\lambda\}$$

for all $x\in\bar{U}$. If only equations with no nonlocal terms are considered then S does not depend on the set argument, and hence S and $\hat{S}$ are essentially the same. For a partition $P:=(0=t_0<t_1<\cdots<t_m=T)$ of $[0,T]$ let us define the approximate solution for problem (1) recursively by

$$u_P(\cdot,t):=\begin{cases} u_0(x) & \text{if } t=0,\\ \hat{S}(t-t_i,t_i)u_h(\cdot,t_i) & \text{if } t\in(t_i,t_{i+1}].\end{cases} \tag{9}$$

The scheme S is assumed to satisfy the following conditions:

(S1) *Monotonicity*: For all $A, B \in \mathscr{B}$, all $h > 0$ and $t \geqslant 0$, and $u, v \in B(\bar{U})$

$$\text{If } u \leqslant v \text{ and } A \subseteq B \quad \text{then } S(A,h,t)u \leqslant S(B,h,t)v.$$

(S2) For any $c \in \mathbb{R}$ and $A \in \mathscr{B}$

$$S(A,h,t)(u+c) = S(A,h,t)u + c \quad \text{and} \quad S(A,h,t)0 = 0.$$

(S3) *Consistency*: For all $A \in \mathscr{B}$, all $t \geqslant 0$, all smooth functions $\varphi \in C^\infty(\bar{U})$ and all points $x \in U$ for which either $D\varphi(x) \neq 0$ or $D\varphi(x) = 0$ and $D^2\varphi(x) = 0$, and all points $x \in \partial U$ for which $\partial\varphi/\partial\nu(x) > 0$

$$\limsup_{h\to 0}{}^{*} \frac{S(A,h,t)\varphi - \varphi}{h}(x) \leqslant -F_*(x,t,D\varphi,D^2\varphi,A)$$

and for all points $x \in U$ for which either $D\varphi(x) \neq 0$ or $D\varphi(x) = 0$ and $D^2\varphi(x) = 0$, and all points $x \in \partial U$ for which $\partial\varphi/\partial\nu(x) < 0$

$$\liminf_{h\to 0}{}_{*} \frac{S(A,h,t)\varphi - \varphi}{h}(x) \geqslant -F^*(x,t,D\varphi,D^2\varphi,A).$$

(S4) *Continuity at $t = 0$*: For every function $g \in C(\bar{U})$ there exists a function $\mu : [0,\infty) \to [0,\infty]$ such that $\mu(0+) = 0$ and for all $t \in [0,T)$ and P

$$\sup_{x\in U} |u_P(x,t) - g(x)| \leqslant \mu(t).$$

From the property (S1) and the definition of $\hat{S}$ follows that for $u, v \in B(\bar{U})$

$$\text{if } u \leqslant v \quad \text{then } \hat{S}(h,t)u \leqslant \hat{S}(h,t)v.$$

Theorem 4.1. *Let F be a function satisfying* (F1)–(F5) *and S an approximation scheme satisfying* (S1)–(S4). *Assume that the comparison holds for Eq.* (1) *and that u_0 is continuous. Then u_P converges uniformly to the unique continuous solution of Eq.* (1) *as $\|P\| \to 0$.*

Here $\|P\|$, the mesh of P, is defined to be $\max_{i=1\dots m}(t_{i+1} - t_i)$.

Note that if F also satisfies (F6) and (F7) then by Theorem 2.3 comparison for Eq. (1) does hold.

Remark. The theorem is also true if condition in (S4) is replaced by the following, weaker condition:

$$\sup_{x\in U} |u_P(x,t) - g(x)| d(x,\partial U) \leqslant \mu(t).$$

The proof is the same as the one below.

Proof. To simplify the notation through the proof we assume that in partition P is regular, that is, that $t_i = ih$. We also write u_h instead of u_P. Let $\bar{u} := \limsup^*_{h\to 0} u_h$ and $\bar{u} := \liminf_{*h\to 0} u_h$. The goal is to show that $\bar{u}$ is a subsolution and $\underline{u}$ is a supersolution of (1). The comparison then implies that $\bar{u} \leqslant \underline{u}$. Since by definition $\underline{u} \leqslant \bar{u}$, we conclude that $\underline{u} = \bar{u}$. Therefore, $u := \bar{u}$ is the unique continuous solution of (1). Since $u = \liminf_{*h\to 0} u_h = \limsup^*_{h\to 0} u_h$ and $\bar{U}$ is compact the convergence is uniform.

Showing that $\underline{u}$ is a supersolution is analogous to showing that $\bar{u}$ is a subsolution, so we only show the latter claim.

Note that $\bar{u}$ is bounded. Let us also show that $\bar{u}(x,0) = u_0(x)$ for all $x \in \bar{U}$. Let x_n be a sequence in $\bar{U}$ converging to x and h_n, t_n sequences converging to 0, such that $\lim_{n\to\infty} u_{h_n}(x_n,t_n) = \bar{u}(x,0)$. Then

$$\begin{aligned}|\overline{u}(x,0) - u_0(x)| &= \lim_{n\to\infty} |u_{h_n}(x_n,t_n) - u_0(x)| \\ &\leqslant \lim_{n\to\infty} |u_{h_n}(x_n,t_n) - u_0(x_n)| + |u_0(x_n) - u_0(x)| \\ &\leqslant \lim_{n\to\infty} \mu(t_n) + |u_0(x_n) - u_0(x)| = 0.\end{aligned}$$

Let $v, w \in B(\bar{U})$. Then for any $x \in \bar{U}$, any $h > 0$, $t \geqslant 0$

$$v(x) - \|w - v\| \leqslant w(x) \leqslant v(x) + \|w - v\|$$

by (S1), (S2)

$$S(h,t)v(x) - \|w - v\| \leqslant S(h,t)w \leqslant S(h,t)v(x) + \|w - v\|$$

$$\|S(h,t)w - S(h,t)v\| \leqslant \|w - v\|. \tag{10}$$

Let $x_0 \in \bar{U}$ and $t_0 \in (0,T)$ and $\varphi \in C^\infty(\bar{U} \times (0,T))$ such that $\bar{u} - \varphi$ has maximum at (x_0,t_0) and that if $x_0 \in \partial U$ then $\partial\varphi/\partial\nu(x_0) > 0$. If $D\varphi(x_0,t_0) = 0$, by Lemma 2.2, we can assume that $D^2\varphi(x_0,t_0) = 0$. We can also assume that $\varphi(x_0,t_0) = 0$ and $\bar{u}(x_0,t_0) = 0$. Then, for appropriate smooth functions h, g, constant a and vector p:

$$\varphi(x,t) = \langle p, x - x_0\rangle + a(t - t_0) + \tfrac{1}{2}\langle X(x - x_0), x - x_0\rangle + h(x) + g(x,t).$$

And $h(x) = O(|x - x_0|^3)$ and $g(x,t) = o(|t - t_0| + |x - x_0|^3)$. Let $\varepsilon > 0$. Then there exists $r \in (0,1)$ such that $\varepsilon(|t - t_0| + |x - x_0|^3) \geqslant g(x,t)$ in $B_r(x_0,t_0)$. Let $M = \|\varphi\|_{C(\bar{U}\times[0,T))} + \|h/|x - x_0|^3\|_{C(\bar{U})} + \|g\|_{C(\bar{U}\times[0,T))} + \|u_0\|$ and $c = \frac{6}{r^3}M$. Let

$$\rho(x) = \langle p, x - x_0\rangle + \tfrac{1}{2}\langle X(x - x_0), x - x_0\rangle + c|x - x_0|^3,$$

$$\tau(t) = a(t - t_0) + \varepsilon|t - t_0| + c|t - t_0|^3,$$

$$\psi(x,t) = \rho(x) + \tau(t).$$

Note that $\psi(x_0,t_0) = \varphi(x_0,t_0) = 0$, $D\psi(x_0,t_0) = D\varphi(x_0,t_0) = p$, $D^2\psi(x_0,t_0) = D\varphi(x_0,t_0) = X$ and that $\psi(x,t) > \varphi(x,t)$ on $\bar{U} \times [0,T) \setminus \{(x_0,t_0)\}$. Therefore $\bar{u} - \psi$ has strict maximum at (x_0,t_0). Also M was chosen so that for any $t \in [0,T)$ and $x \in \bar{U} \setminus B_r(x_0)$ $\psi(x,t) > 2\|u_0\| = 2\sup_{h>0,t\in[0,T)} \|u_h(\cdot,t)\|$. Then, by compactness, there exists a sequence

$$(x_n,t_n,h_n) \to (x_0,t_0,0) \quad \text{as } n \to \infty.$$

Such that $u_{h_n} - \psi$ has maximum at (x_n,t_n) and $u_{h_n}(x_n,t_n) \to \bar{u}(x_0,t_0)$ as $n \to \infty$. To simplify notation, let us write in the rest of the proof u_n instead of u_{h_n}. Let i_n be the greatest integer less than t_n/h_n and $\Delta t_n := t_n - i_n h_n$. Note that $\Delta t_n > 0$.

For all $z \in \bar{U}$

$$u_n(z, i_n h_n) - u_n(x_n, t_n) \leqslant \psi(z, i_n h_n) - \psi(x_n, t_n).$$

By the definition of u_n:

$$0 = \hat{S}(\Delta t_n, i_n h_n)[u_n(\cdot, i_n h_n)](x_n) - u_n(x_n, t_n).$$

Let $a_n := u(x_n, t_n) - (\Delta t_n)^2$. By the definition of S and monotonicity:

$$0 \leqslant S(\{u(\cdot, i_n h_n) \geqslant a_n\}, \Delta t_n, i_n h_n)[u_n(\cdot, i_n h_n)](x_n) - u_n(x_n, t_n) + (\Delta t_n)^2$$
$$\leqslant S\left(\bigcup_{m \geqslant n}\{u(\cdot, i_m h_m) \geqslant a_m\}, \Delta t_n, i_n h_n\right)[\psi(\cdot, i_n h_n)](x_n) - \psi(x_n, t_n) + (\Delta t_n)^2.$$

Let $A := \bigcap_{n \geqslant 1} \bigcup_{m \geqslant n}\{u(\cdot, i_m h_m) \geqslant a_m\}$. Note that $A \subseteq \{\bar{u}(\cdot, t_0) \geqslant 0\}$. Using properties (S2) and (S3) one gets:

$$\begin{aligned}
0 &\leqslant \limsup_{n\to\infty} \frac{1}{\Delta t_n}\left(S\left(\bigcup_{m \geqslant n}\{u(\cdot, i_m h_m) \geqslant a_m\}, \Delta t_n, i_n h_n\right)[\psi(\cdot, i_n h_n)](x_n)\right.\\
&\quad \left. - \psi(x_n, i_n h_n) + \psi(x_n, i_n h_n) - \psi(x_n, t_n) + \Delta t_n\right)\\
&\leqslant \limsup_{n\to\infty} \frac{1}{\Delta t_n} S\left(\bigcup_{m \geqslant n}\{u(\cdot, i_m h_m) \geqslant a_m\}, \Delta t_n, i_n h_n\right)[\rho(\cdot) - \rho(x_n)](x_n)\\
&\quad + \limsup_{n\to\infty} \frac{\tau(i_n h_n) - \tau(t_n)}{i_n h_n - t_n}\\
&\leqslant \limsup_{h\to 0}{}^{*} \frac{1}{h}(S(A, h, t_0)[\rho(\cdot)](x_0) - \rho(x_0))\\
&\quad + \limsup_{n\to\infty} \frac{a(i_n h_n - t_n) + \varepsilon(|i_n h_n - t_0| - |t_n - t_0|) + c(|i_n h_n - t_0|^4 - |t_n - t_0|^4)}{i_n h_n - t_n}\\
&\leqslant -F_*(x_0, t_0, p, X + 2\varepsilon I, A) - a\\
&\quad + \limsup_{n\to\infty} \frac{\varepsilon(|i_n h_n - t_n|) + c|i_n h_n - t_n|(|i_n h_n - t_0| + |t_n - t_0|)^3}{|i_n h_n - t_n|}\\
&\leqslant -F_*(x_0, t_0, p, X + 2\varepsilon I, A) - a + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields

$$0 \leqslant -F_*(x_0, t_0, p, X, \{\bar{u}(\cdot, t_0) \geqslant 0\}) - a. \qquad \square$$

In some applications that follow we define the approximate solution of be piece-wise constant using an operator

$$Q_t^h := \hat{S}((j-1)h, h) \circ \cdots \hat{S}(h, h) \circ \hat{S}(0, h).$$

Where $t \geqslant 0$, $h > 0$ and $j = \lfloor t/h \rfloor$. Using the property (S4) the following corollary is easily obtained.

Corollary 4.2. *If the assumptions of the Theorem* 4.1 *are satisfied then* $Q_t^h u_0$ *converges uniformly to the unique continuous solution of Eq.* (1).

The following lemma is used in proof of convergence of some schemes for nonlocal front propagation.

Lemma 4.3. *Let* F *be a function satisfying* (F1)–(F5) *and assume that comparison holds for the Eq.* (1). *Let* $S_1, S_2, S : \mathbb{R}^+ \times \mathbb{R}_0^+ \times B(\bar{U}) \to B(\bar{U})$ *be approximation schemes for the Eq.* (1). *Assume that for all* $A \in \mathscr{B}$, *all* $h > 0$, *all* $t \geqslant 0$ *and all* $u \in B(\bar{U})$

$$S_1(A,h,t)u \leqslant S(A,h,t)u \leqslant S_2(A,h,t)u$$

and that S_1 *and* S_2 *are convergent. Then* S *is convergent too.*

5. Schemes for front propagation with Neumann boundary conditions

Let U be a closed domain in $\mathbb{R}^N$ with C^1 boundary that satisfies the outside sphere condition. To construct threshold type numerical schemes for motions with Neumann boundary condition we need map τ that "mirrors" regions next to ∂U inside and outside of U. Since ∂U is a C^1 manifold in $\mathbb{R}^n$ it has a partial tubular neighborhood (see [15]) and therefore there exists $\varepsilon > 0$ such that on neighborhood $W := \{x\colon\ \operatorname{dist}(x, \partial U) < \varepsilon\}$ of ∂U on there exists a C^1 function $v : W \to \partial U$ defined as follows: $v(x) = y$ if y is the closest point to x on ∂U. Let $\tau(x) := 2v(x) - x$. Note that $\tau : W \to W$. Also note that τ has the following properties: $\tau \circ \tau = id$, $\tau|_{\partial U} = id$, and $\tau(U \cap W) \bigcap U = \emptyset$. For a measurable set $A \subset \mathbb{R}^N$ (having subsets of $\bar{U}$ in mind) let $T(A) := (A \cap \bar{U}) \cup \tau(A \cap W \cap U)$.

5.1. Motions not depending on curvature

Let μ be a measure on $\mathbb{R}^N$ such that $\mu(\mathbb{R}^N) = 1$. Although not necessary, this assumption simplifies the presentation, and does not restrict the class of motions that can be obtained. Let $\theta \in (0,1)$ be the threshold.

For $p \in S^{N-1}$ let us define the outward normal velocity

$$v(p) := \sup\{v\colon\ \mu(\{x\colon\ \langle -p, x \rangle \geqslant v\}) \geqslant \theta\}. \tag{11}$$

We also need the following assumption on μ:

(μ1) Measure μ is either compactly supported or for every $p \in S^{N-1}, v(p)$ is the unique number for which $\mu(\{x \mid \langle -p, x \rangle \geqslant v\}) = \theta$.

Example. Let $N = 2$, and μ determined by $\mu(\{(-1,0)\}) = \mu(\{(1,0)\}) = \frac{1}{2}$. Then $v(p) = |\langle p, (1,0) \rangle|$. We remark that this motion cannot be obtained if only measures that are absolutely continuous with respect to Lebesgue measure are considered (see [20]).

It is not hard to prove now that v is a continuous function of p. We also define function $F:\mathbb{R}^N \to \mathbb{R}$ as follows:

$$F(p) := \begin{cases} -|p|\,v(-\hat{p}) & \text{if } p \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Where $\hat{p} = p/|p|$. Note that F is continuous and satisfies the conditions (F1)–(F7).

For $h > 0$ let $\mu_h(\cdot) := \mu(h^{-1}\cdot)$. Define an update scheme $M_h : \mathscr{B} \to \mathscr{B}$ by

$$M_h A := \{x \in \bar{U}\colon\ \mu_h(T(A) - x) \geqslant \theta\}. \tag{13}$$

$\partial M_h A$ is the approximate position of the front started from ∂A moving with normal velocity v after time h. Note that M_h is monotone in the sense that if $A \subseteq B \subseteq \bar{U}$ then $M_h(A) \subseteq M_h(B)$. Let M_h^k be the kth iterate of M_h. To make the scheme defined for all times let us define

$$C_t^h := M_h^j \quad \text{where } j = \lfloor t/h \rfloor.$$

Following the procedure from [17] (and the references within) we construct an approximation scheme for the level set Eq. (7) using the (set valued) updates of the threshold scheme for approximating propagating fronts. We define mapping $S_h : B(\bar{U}) \to B(\bar{U})$ by

$$S_h\varphi(x) := \sup\{\lambda \in \mathbb{R}\colon\ x \in M_h\{\varphi \geqslant \lambda\}\}.$$

Note that the S used here is same as the S in Section 4, only simpler, and since it has no set valued parameter is also equal to $\hat{S}$. We use Q_t^h to define approximate solutions. Recall that

$$Q_t^h = S_h^j \quad \text{where } j = \left\lfloor \frac{t}{h} \right\rfloor.$$

Let us prove that for all $x \in \bar{U}$

$$x \in M_h\{\varphi \geqslant S_h\varphi(x)\}. \tag{14}$$

By definition of S_h and monotonicity of M_h yields that for any $\varepsilon > 0$

$$x \in M_h\{\varphi \geqslant S_h\varphi(x) - \varepsilon\}.$$

and definition of M_h implies

$$\lim_{\varepsilon \to 0} \mu_h(T(\{\varphi \geqslant S_h\varphi(x) - \varepsilon\}) - x) \geqslant \theta.$$

Therefore

$$\mu_h(T(\{\varphi \geqslant S_h\varphi(x)\}) - x) = \mu_h\left(\bigcap_{\varepsilon > 0} T(\{\varphi \geqslant S_h\varphi(x) - \varepsilon\}) - x\right) \geqslant \theta$$

which implies the claim.

The definition of S_h and (14) now imply that:

$$S_h\varphi(x) \geqslant \lambda \quad \text{if and only if } x \in M_h(\{\varphi \geqslant \lambda\}). \tag{15}$$

Using the results of Section 4 we first establish convergence of approximate solutions of the level set equation. That convergence is then used to establish convergence of approximate fronts towards the front moving by given normal velocity. The convergence results that we prove are analogous to ones proven in [17], only that here we are proving them for problem satisfying Neumann boundary conditions on U, while in [17] results were proven for problem on $\mathbb{R}^N$. If there is no fattening that convergence is in Hausdorff metric, but possibility of fronts fattening forces us to formulate the result in more complicated manner.

Theorem 5.1. *Let g be a continuous function on $\bar{U}$ and let u be the unique continuous solution of* (7) *with F given by* (12). *Then as $h \to 0$, for all $T \in [0,\infty)$,*

$$Q_t^h g(x) \to u(x,t) \quad \textit{uniformly on } \bar{U} \times [0,T].$$

For a set $A \in \mathscr{B}$ and $\varepsilon > 0$ let

$$A_\varepsilon := \{x \in \bar{U}:\ \operatorname{dist}(x, \bar{U} \setminus A) > \varepsilon\} \quad \text{and} \quad A^\varepsilon := \{x \in \bar{U}:\ \operatorname{dist}(x, A) < \varepsilon\}.$$

Theorem 5.2. *For all $\varepsilon > 0$ and $T > 0$ and $A \in \mathscr{B}$ there exists a $\delta > 0$ such that, for all $h \in (0,\delta)$ and $t \in [0,T]$:*

$$N_t A_\varepsilon \subset C_t^h A \subset N_t A^\varepsilon \quad \textit{and} \quad X_t \overline{A_\varepsilon} \subset C_t^h A \subset X_t \overline{A^\varepsilon}.$$

Theorem 5.3. *For every $\varepsilon > 0$ and every closed set $A \subseteq \bar{U}$ and open set $A' \subset U$ there exists $\delta > 0$ such that for all $h \in (0,\delta)$*

$$\bigcup_{t \geqslant 0} C_t^h(A) \times \{t\} \subset \bigcup_{t \geqslant 0} X_t(A) \times \{t\} + B(0,\varepsilon),$$

$$\bigcup_{t \geqslant 0} N_t(A') \times \{t\} + B(0,\varepsilon) \subset \bigcup_{t \geqslant 0} C_t^h(A') \times \{t\}.$$

The last two theorems follow in a general way from the convergence obtained in Theorem 5.1. The proofs can be found in [17].

Proof of Theorem 5.1. We only need to check if conditions of Theorem 4.1 are satisfied. It is easy to see that F satisfies the conditions given, so let us check if S_h does too.

Since M_h is monotone and S_h is defined “level set by level set” for any nondecreasing function $\rho \in C(\mathbb{R})$ the following holds for all $\varphi, \psi \in B(\bar{U})$

$$S_h(\rho \circ \varphi) = \rho \circ (S_h \varphi)$$

and

$$\text{if } \varphi \leqslant \psi \quad \text{then } S_h \varphi \leqslant S_h \psi.$$

The latter of the properties and the definition of S_h now yield that for any $\varphi \in B(\bar{U})$ and $c \in \mathbb{R}$

$$S_h(\varphi + c) = S_h\varphi + c \quad \text{and} \quad S_h 0 = 0.$$

What is left to be proven are properties (S3) and (S4). We do that in the following two lemmas. Property (S3) is immediate consequence of Lemma 5.4. A proof that Lemma 5.5 implies (S4) is given for an analogous claim in the section on motions depending on curvature after Lemma 5.8, and can also be found (for a similar claim) in [17]. □

Lemma 5.4. *Let $\varphi \in C^1(\mathbb{R}^N)$. For all $z \in U$ and all $z \in \partial U$ for which $\partial\varphi/\partial\nu(z) > 0$, and all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x \in B(z,\delta) \cap \bar{U}$ and $h \in (0,\delta)$,*

$$S_h\varphi(x) \leqslant \varphi(x) + (-F(D\varphi(z)) + \varepsilon)h$$

and analogously for all $z \in U$ and all $z \in \partial U$ for which $\partial\varphi/\partial\nu(z) < 0$, and all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x \in B(z,\delta) \cap \bar{U}$ and $h \in (0,\delta)$,

$$S_h\varphi(x) \geqslant \varphi(x) + (-F(D\varphi(z)) - \varepsilon)h.$$

Proof. Since the proof of the latter statement is analogous to the proof of the former so we only prove the former one.

Let φ, z satisfy the assumptions of the lemma. Let $p := D\varphi(z)$ and $E := \{y\colon \langle p, y\rangle \geqslant -F(p) + \varepsilon/2\}$. Then by definition of F and v, $\mu(E) < \theta$. There exists $R > 0$ such that

$$\mu(\mathbb{R}^N \setminus B(0,R)) < \theta - \mu(E). \tag{16}$$

Since φ is continuously differentiable there exists $\delta > 0$ such that the following hold: For all $h \in (0,\delta)$, all $x \in B(z,\delta)$, and $y \in B(0,R)$

$$\varphi(x + hy) \leqslant \varphi(x) + \langle p, y\rangle h + \frac{\varepsilon h}{2} \tag{17}$$

and if $z \in U$ then $B(z, R\delta + \delta) \subset U$ and if $z \in \partial U$ then $B(z, R\delta + \delta) \subset T(\bar{U})$ and for all $y \in B(z, R\delta + \delta)$, $\langle D\varphi(y), n(v(y))\rangle > 0$. Note that in the case that $z \in \partial U$ this implies that, for $y \in B(z, R\delta + \delta) \setminus U$, $\varphi(y) \geqslant \varphi(v(y)) = \varphi(v(\tau(y))) \geqslant \varphi(\tau(y))$.

Let $h \in (0,\delta)$ and $x \in B(z,\delta)$. Let

$$L := \{\varphi \geqslant \varphi(x) + (-F(p) + \varepsilon)h\} \quad \text{and} \quad A := L \cap \bar{U}.$$

In the case that $z \in \partial U$ the inequality $\varphi \geqslant \varphi \circ \tau$ on $B(x, R\delta) \setminus U$ implies

$$T(A) \cap B(x, R\delta) \subset L \cap B(x, R\delta).$$

This is the key fact used in dealing with the boundary condition (see inequalities in (18)), that makes considering the case $z \in \partial U$ analogous to $z \in U$. And furthermore, since showing this inclusion depended only on properties of test function φ and properties of T, the same argument can be used in a variety of schemes. In particular we refer to it when proving convergence of schemes in the following sections.

See also Fig. 1.

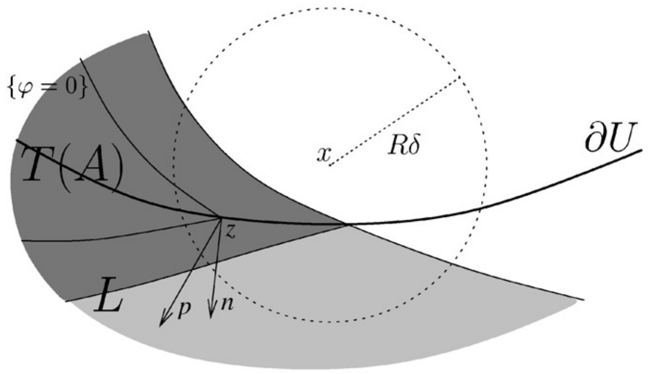

Fig. 1. Generic configuration in $z \in \partial U$ case.

By (15) it is enough to show that

$$x \notin M_h A$$

which is by definition of M_h equivalent to

$$\mu_h(T(A) - x) < \theta.$$

We claim that $L \cap B(x, Rh) \subset x + hE$. Let $x + hy$ be an arbitrary element of $L \cap B(x, Rh)$. Using the definition of L and (17) we get:

$$\varphi(x + hy) \geqslant \varphi(x) + (-F(p) + \varepsilon)h,$$

$$h\langle p, y\rangle + \frac{\varepsilon h}{2} \geqslant -F(p)h + \varepsilon h,$$

$$\langle p, y\rangle \geqslant -F(p) + \frac{\varepsilon}{2}.$$

So $x + hy \in x + hE$. Therefore,

$$\mu_h(L \cap B(x, Rh) - x) \leqslant \mu_h(hE) = \mu(E).$$

But if $z \in U$ then $T(A) \cap B(x, Rh) = L \cap B(x, Rh)$ and if $z \in \partial U$ then $T(A) \cap B(x, Rh) \subset L \cap B(x, Rh)$. So in both cases, by using (16), we obtain

$$\begin{aligned} \mu_h(T(A) - x) &\leqslant \mu_h(L \cap B(x, Rh) - x) + \mu_h(\mathbb{R}^N \setminus B(0, Rh)) \\ &\leqslant \mu(E) + \mu(\mathbb{R}^N \setminus B(0, R)) < \theta. \qquad \Box \end{aligned} \tag{18}$$

Lemma 5.5. *There exist constants $\delta > 0$, $R > 0$ such that for all $y \in \bar{U}$ for the function $\varphi(x) := |x - y|$ (respectively $\varphi(x) := -|x - y|$) the following inequalities hold, for all $x \in \bar{U}$ and $h \in (0, \delta)$*

$$S_h\varphi(x) - \varphi(x) \leqslant Rh \quad (\textit{respectively } S_h\varphi(x) - \varphi(x) \geqslant -Rh).$$

Proof. Let R be such that $\mu(B(0,R/3)) > 1-\theta$. Let δ be such that $U + B(0,3R\delta) \subset T(\bar{U})$. We claim that for any $\gamma \in (0,R\delta)$ and all $x \in \bar{U}$

$$B(x,\gamma) \setminus U \subset T(B(x,3\gamma) \cap \bar{U}).$$

Let z be an arbitrary element of $B(x,\gamma) \setminus U$. Then $\mathrm{dist}(x,z) \leqslant \gamma$ and hence $\mathrm{dist}(z,v(z)) \leqslant \gamma$. Hence, $\mathrm{dist}(z,\tau(z)) \leqslant 2\gamma$, which implies that $\mathrm{dist}(x,\tau(z)) \leqslant 3\gamma$. Therefore $\tau(z) \in B(x,3\gamma) \cap \bar{U}$, which yields that $z \in T(B(x,3\gamma) \cap \bar{U})$.

Let $y \in \bar{U}$. By translating the domain we can assume that $y = 0$. Let $h \in (0,\delta)$. Let us show that

$$S_h\varphi(x) - \varphi(x) \leqslant Rh.$$

We only need to show that $x \notin M_h\{\varphi \geqslant |x| + Rh\}$ which is equivalent to

$$\mu_h(T(\{\varphi \geqslant |x| + Rh\}) - x) < \theta.$$

But using the above we get

$$\begin{aligned} \mu_h(T(\{\varphi \geqslant |x| + Rh\}) - x) &\leqslant \mu_h(\mathbb{R}^N \setminus T(B(x,Rh)) - x) \\ &\leqslant \mu_h(\mathbb{R}^N \setminus B(0,Rh/3)) = 1 - \mu(B(0,R/3)) \\ &< \theta. \end{aligned}$$

The proof that for $\varphi(x) = -|x-y|$, $S_h\varphi(x) - \varphi(x) \leqslant Rh$ is analogous. □

5.2. Motions depending on curvature

Let μ be a measure on $\mathbb{R}^N$ that has the form $\mathrm{d}\mu = f\,\mathrm{d}m$ where f is a function and m is the Lebesgue measure. As before, we require that $\mu(\mathbb{R}^N) = 1$. We also need the following assumptions on μ:

(μ2) $f(x) = f(-x)$.
(μ3) $0 < \int_{p^\perp} f(x)\,\mathrm{d}\mathscr{H}^{N-1}$.
(μ4) The functions $p \mapsto \int_{p^\perp} f(x)\,\mathrm{d}\mathscr{H}^{N-1}$ and $p \mapsto \int_{p^\perp} x_i x_j f(x)\,\mathrm{d}\mathscr{H}^{N-1}$ are continuous for all $i,j \in \{1,2,\dots,N\}$.
(μ5) $\int_{\mathbb{R}^N} |x|^2\,\mathrm{d}\mu < \infty$.

Here $p^\perp$ is the orthogonal complement of vector p in $\mathbb{R}^N$. Let $R:(0,1) \to (0,\infty)$ such that

$$R(\rho) \to \infty \quad \text{and} \quad \sqrt{\rho}R(\rho) \to 0 \quad \text{as } \rho \to 0.$$

For $U \in O(N)$ ($O(N)$ is the group of $N \times N$ orthogonal matrices) let us define $f_U : \mathbb{R}^N \to \mathbb{R}$ by $f_U(x) := f(U^*x)$. Here U^* is the transpose of U.

($\mu 6$) For all functions $g:\mathbb{R}^{N-1}\to\mathbb{R}$ of the form $g(\cdot)=a+\langle A\cdot,\cdot\rangle$ where $a\in\mathbb{R}$ and $A\in S^{N-1}$

$$\sup_{U\in O(N)}\sup_{0<r<\rho}\left|\int_{B(0,R(\rho))} g(\xi)f_U(\xi,rg(\xi))\,\mathrm{d}\xi-\int_{\mathbb{R}^{N-1}} g(\xi)f_U(\xi,0)\,\mathrm{d}\xi\right|\to 0$$

as $\rho\to 0$.

Note that if f is continuous with compact support then conditions ($\mu 2$)–($\mu 5$) are satisfied.

For $c\in\mathbb{R}$ we construct scheme for fronts propagating with velocity $v:S^{N-1}\times\mathscr{S}^N\to\mathbb{R}$

$$v(p,X)=\left(\int_{p^\perp} f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)\right)^{-1}\left(-\tfrac{1}{2}\int_{p^\perp}\langle Xx,x\rangle f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)+c\right).$$

Let us now define $F:\mathbb{R}^N\setminus\{0\}\times\mathscr{S}^N\to\mathbb{R}$ by

$$F(p,X):=-|p|v\left(\hat{p},\frac{X}{|p|}\right).$$

An update of the set valued scheme $M_h:\mathscr{B}\to\mathscr{B}$ for $h>0$ is defined by

$$M_hA:=\{x\in\bar{U}:\ \mu_{\sqrt{h}}(T(A)-x)\geqslant\tfrac{1}{2}-c\sqrt{h}\}.$$

C_t^h, S_h and Q_t^h are defined as before. The analogues of the Theorems 5.1, 5.2 and 5.3 hold. To show that we just need to verify conditions (S3) and (S4), that is the consistency and the continuity at $t=0$. They follow from the following three lemmas:

Lemma 5.6. *Let $\varphi\in C^\infty(\mathbb{R}^N)$. For all $z\in U$ for which $D\varphi(z)\neq 0$ and all $z\in\partial U$ for which $\partial\varphi/\partial\nu(z)>0$, and for all $\varepsilon>0$, there exists $\delta>0$, such that for all $x\in B(z,\delta)\cap\bar{U}$ and $h\in(0,\delta)$,*

$$S_h\varphi(x)\leqslant\varphi(x)+(-F(D^2\varphi(z),D\varphi(z))+\varepsilon)h$$

and analogously for all $z\in U$ for which $D\varphi(z)\neq 0$ and all $z\in\partial U$ for which $\partial\varphi/\partial\nu(z)<0$, and for all $\varepsilon>0$, there exists $\delta>0$, such that for all $x\in B(z,\delta)\cap\bar{U}$ and $h\in(0,\delta)$,

$$S_h\varphi(x)\geqslant\varphi(x)+(-F(D^2\varphi(z),D\varphi(z))-\varepsilon)h.$$

This lemma establishes the consistency of the scheme (condition (S3)) when $D\varphi(z)\neq 0$. Its proof is analogous to the proof of Lemma 3.1 in [17], while the boundary of U is handled in the same way as in Lemma 4.4. The consistency of the scheme for $D\varphi(z)=0$ is a consequence of the following lemma.

Lemma 5.7. *Let $z\in U$ and $\varphi\in C^\infty(\mathbb{R}^N)$ such that $D\varphi(z)=0$ and $D^2\varphi(z)=0$. Then for any $\varepsilon>0$ there exists $\delta>0$ such that for all $h\in(0,\delta)$ and all $x\in B(z,\sqrt{h})$*

$$-2\varepsilon\leqslant\frac{S_h\varphi(x)-\varphi(x)}{h}\leqslant 2\varepsilon.$$

Proof. Let z, φ and ε satisfy the above conditions. We can assume that $z=0$ and that $\varphi(0)=0$. Let $C>1$ be such that $\mu(B(0,C))>\frac{3}{4}$. Choose $\delta>0$ so that it satisfies the following conditions: $B(z,2C\sqrt{\delta})\subset U$, $4c\sqrt{\delta}<1$ (where c was given in the definition of the scheme), and if $|y|<2C\sqrt{\delta}$ then $|\varphi(y)|<(\varepsilon/C^2)|y|^2$. This can be done since $\varphi(y)=o(|y|^2)$. Let $h\in(0,\delta)$ and $x\in B(0,\sqrt{h})$. Note that $|\varphi(x)|<\varepsilon h$.

Therefore, to prove the lemma it suffices to show that $-\varepsilon h\leqslant S_h\varphi(x)\leqslant\varepsilon h$. As the proofs of the two inequalities are analogous let us just show that $S_h\varphi(x)\leqslant\varepsilon h$. Using the definition of S_h and monotonicity of M_h, it is enough to show that $x\notin M_h\{\varphi\geqslant\varepsilon h\}$. Since $c\sqrt{h}<\frac{1}{4}$, by definition of M_h it suffices to show that $\mu_{\sqrt{h}}(T(\{\varphi\geqslant\varepsilon h\})-x)\leqslant\frac{1}{4}$.

Note that if $|y|<2C\sqrt{h}$ then $|\varphi(y)|<(\varepsilon/C^2)C^2h=\varepsilon h$. Therefore $\{\varphi\geqslant\varepsilon h\}\subseteq\mathbb{R}^n\setminus B(x,C\sqrt{h})$. Since $B(x,C\sqrt{h})\subset U$ this implies $T(\{\varphi\geqslant\varepsilon h\})\subseteq\mathbb{R}^n\setminus B(x,C\sqrt{h})$. Consequently

$$\mu\left(\frac{T(\{\varphi\geqslant\varepsilon h\})-x}{\sqrt{h}}\right)\leqslant\mu\left(\frac{\mathbb{R}^n\setminus B(0,C\sqrt{h})}{\sqrt{h}}\right)=\mu(\mathbb{R}^n\setminus B(0,C))<\frac{1}{4}.\qquad\square$$

Lemma 5.8. *There exist constants $\delta>0$, $R>0$ such that for all $M>0$ and $y\in U$ for any $C>2M/d(y,\partial U)$ for the function $\varphi(x):=\min\{C\sqrt{|x-y|^2+1},M\}$ (respectively $\varphi(x):=\max\{-C\sqrt{|x-y|^2+1},-M\}$) the following inequality*:

$$S_h\varphi(x)-\varphi(x)\leqslant RCh\quad(\textit{respectively } S_h\varphi(x)-\varphi(x)\geqslant-RCh)$$

holds for all $x\in\bar{U}$ and all $h\in(0,\delta)$.

This lemma can be proven by making minor modifications of the proof of the Lemma 3.2 in [17].

Let us show now that Lemma 5.8 implies the weak form of the continuity at $t=0$ (condition S4) of the scheme S_h. Let $g\in C(\bar{U})$. Let $M:=2\|g\|_C+\operatorname{diam}(U)+1$. For simplicity let us assume that partition $P=(0,h,2h,\dots,T)$. For every $\varepsilon>0$ there exists $C_\varepsilon>1$ such that for all $x,y\in\bar{U}$

$$-\varepsilon-C_\varepsilon(\sqrt{|x-y|^2+1}-1)\leqslant g(x)-g(y)\leqslant\varepsilon+C_\varepsilon(\sqrt{|x-y|^2+1}-1).$$

For fixed $y\in U$ let $\hat{C}_\varepsilon:=\max\{M/d(y,\partial U),C_\varepsilon\}$. Then

$$g(x)-g(y)\leqslant\varepsilon+\min\{\hat{C}_\varepsilon(\sqrt{|x-y|^2+1}-1),M\}.$$

Applying a step of the scheme S_h and using the Lemma 5.5 we obtain:

$$S_hg(x)-g(y)\leqslant\varepsilon+\min\{\hat{C}_\varepsilon(\sqrt{|x-y|^2+1}-1),M\}+R\hat{C}_\varepsilon h.$$

By induction

$$u_P(x,t)-g(y)\leqslant\varepsilon+\min\{\hat{C}_\varepsilon(\sqrt{|x-y|^2+1}-1),M\}+R\hat{C}_\varepsilon t.$$

The last inequality evaluated for $x=y$ yields:

$$u_P(y,t)-g(y)\leqslant\varepsilon+R\hat{C}_\varepsilon t\leqslant\varepsilon+RC_\varepsilon\frac{M}{d(y,\partial U)}t.$$

Let $\mu(t) := \inf_{\varepsilon>0} \operatorname{diam}(U)\varepsilon + RC_\varepsilon t$. Note that $\lim_{t\to 0+}\mu(t) = 0$. The last inequality now reads:

$$u_P(y,t) - g(y) \leqslant \frac{\mu(t)}{d(y,\partial U)}.$$

In the same fashion estimate from below is obtained and so for all $y \in U$

$$|u_P(y,t) - g(y)| d(y,\partial U) \leqslant \mu(t).$$

5.2.1. An adaptation of BMO scheme for mean curvature motion

The scheme for mean curvature motions of fronts in $\mathbb{R}^N$ developed by Bence, Merriman and Osher, BMO in [4] has been extensively studied. Different proofs of its convergence were given in [9], [1] and [17]. One way to adapt the BMO scheme to motion of subsets of U has already been presented in this section. It is sufficient to take f to be the heat kernel and apply the threshold scheme given above.

In this section we will study another, nonthreshold, adaptation of the BMO scheme for mean curvature motion. This scheme can be modified further (with present techniques) to study other motions, but, for simplicity, we will only present the mean curvature case.

So, for motion by mean curvature $F : \mathbb{R}^N \setminus \{0\} \times \mathcal{S}^N \to \mathbb{R}$ is defined by

$$F(p,X) := -\operatorname{trace}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right).$$

To simplify the presentation let us assume that ∂U is a C^∞ manifold in $\mathbb{R}^N$. Let $\mathscr{C}$ be the set of closed subsets of $\bar{U}$. For $A \in \mathscr{C}$ let us consider the function $u_A : \bar{U} \times \mathbb{R} \to \mathbb{R}$ the solution of the following initial-boundary value problem:

$$\frac{\partial u_A}{\partial t} - \Delta u_A = 0 \quad \text{in } \bar{U} \times (0,T],$$

$$u_A(x,0) = \mathbb{1}_A \quad \text{in } \bar{U},$$

$$\frac{\partial u_A}{\partial \nu}(x,t) = 0 \quad \text{on } \partial U \times (0,T].$$

The approximation scheme $M_h : \mathscr{C} \to \mathscr{C}$ is now defined by

$$M_h A := \{x \in \bar{U}:\ u_A(x,h) \geqslant \tfrac{1}{2}\}. \tag{19}$$

C_t^h, S_h and Q_t^h are defined as before. We claim that the statements in the Theorems 5.1, 5.2 and 5.3 hold for this scheme. The main step is to prove the convergence of the scheme. To prove it we compare the updates of given scheme (carried out on subsets of $\bar{U}$ with the updates of the original BMO. scheme that are carried out on appropriate subsets of $\mathbb{R}^N$. An update of the BMO scheme is constructed in the following way: For given closed set A let $v_A(x,t)$ be the solution of the following Cauchy problem

$$\frac{\partial v_A}{\partial t} - \Delta v_A = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

$$v_A(x,0) = \mathbb{1}_A \quad \text{in } \mathbb{R}^N.$$

The update of the scheme itself is defined by

$$\tilde{M}_h A := \{x \in \mathbb{R}^N\colon v_A(x,h) \geqslant \tfrac{1}{2}\}.$$

$\tilde{S}_h$ is defined in the usual way.

Note that BMO scheme is nothing else than threshold dynamics type scheme on $\mathbb{R}^N$ with $f(x)$ being,

$$f(x) = \frac{1}{(2\sqrt{\pi})^N}\,\mathrm{e}^{\frac{-|x|^2}{4}}$$

and μ the measure with $\mathrm{d}\mu = f\,\mathrm{d}m$.

Those schemes were studied in [17]. The next lemma, that help us compare the two schemes, follows from the proof of consistency of the scheme (Lemma 3.1) given in [17].

Lemma 5.9. *Let $\varphi \in C^2(\mathbb{R}^N)$. For all $z \in \mathbb{R}^N$ for which $D\varphi(z) \neq 0$ and all $\varepsilon > 0$ there exists $\delta > 0$ and $\varepsilon_1 > 0$ such that for all $x \in B(z,\delta)$ and all $h \in (0,\delta)$*

$$\mu_{\sqrt{h}}(A_1 - x) \leqslant \tfrac{1}{2} - \varepsilon_1\sqrt{h},$$

and

$$\mu_{\sqrt{h}}(A_2 - x) \geqslant \tfrac{1}{2} + \varepsilon_1\sqrt{h},$$

where $A_1 := \{y \in \mathbb{R}^N\colon \varphi(y) \geqslant (-F(D^2\varphi(z), D\varphi(z)) + \varepsilon)h\}$ and $A_2 := \{y \in \mathbb{R}^N\colon \varphi(y) \geqslant (-F(D^2\varphi(z), D\varphi(z)) - \varepsilon)h\}$.

We also need the following two lemmas that establish some properties of solutions to heat equation. The first is used for localizing the arguments and the second is used in comparing the schemes at boundary points of U. Elementary proofs of the lemmas are given in Appendix A.

Lemma 5.10. *Let $z \in \bar{U}$ and $r > 0$. Let g be a smooth function on $\partial U \times [0,T]$ supported outside of $B(z,2r)$ and ψ a bounded measurable function on $\bar{U}$ supported outside of $B(z,2r)$. Let u be the solution of the following initial-boundary value problem*:

$$u_t - \Delta u = 0 \quad \text{in } U \times (0,T],$$

$$u(x,0) = \psi(x) \quad \text{in } \bar{U},$$

$$\frac{\partial u}{\partial \nu}(x,t) = g(x,t) \quad \text{on } \partial U \times (0,T].$$

Then for given $\varepsilon > 0$ there exists $\delta > 0$ depending on ψ only through $\|\psi\|_{L^\infty(\bar{U})}$ such that for all $h \in [0,\delta]$

$$\|u\|_{L^\infty(B(z,r)\times(0,h])} \leqslant \varepsilon h.$$

Lemma 5.11. *Let $z\in\mathbb{R}^N$ and φ and ψ smooth functions on $\mathbb{R}^N$ such that $\varphi(z)=\psi(z)=0$, $D\varphi(z)\neq 0$ and $D\psi(z)\neq 0$ and $\langle D\varphi(z), D\psi(z)\rangle > 0$. Let V be a neighborhood of z and let, for fixed constant ε, $u(x,t)$ be the solution of the Cauchy problem*:

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^N\times(0,T],$$

$$u(x,0) = g(x) \quad \text{in } \mathbb{R}^N,$$

where g is such function that $g|_V = \mathbb{1}_{\{\varphi\geqslant\varepsilon\}}$ and $\|g\|_{L^\infty}=1$. Then there exist $\varepsilon_0>0$, $\delta>0$ and W, an open neighborhood of z, such that for all $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, $x\in W$ for which $\psi(x)=0$ and all $t\in(0,\delta)$

$$\langle Du(x,t), D\psi(x)\rangle > 0. \tag{20}$$

Now we are ready to prove consistency of the scheme, that is the Lemma 4.6 for the scheme defined by (19).

Proof. As the proofs of two claims are analogous, we only prove the first one. Let $\varphi\in C^2(\mathbb{R}^N)$.

First, let us consider the case $z\in U$ and $D\varphi(z)\neq 0$. Let $\varepsilon>0$. By Lemma 5.9 there exists $\delta_1>0$ and $\varepsilon_1>0$ such that $B(z,2\delta_1)\subset U$ and for all $x\in B(z,\delta_1)$ and all $h\in(0,\delta_1)$

$$v_A(x,h) = \mu_{\sqrt{h}}(A-x) \leqslant \tfrac{1}{2} - \varepsilon_1\sqrt{h}.$$

Here $A := \{y\in\mathbb{R}^N\colon \varphi(y)\geqslant\varphi(x) + (-F(D^2\varphi(z), D\varphi(z)) + \varepsilon)h\}$ and $A' := A\cap\bar{U}$. Let $u_{A'}(x,t)$ be defined as before and let $w\colon \bar{U}\times[0,T]\to\mathbb{R}$ be $w := u_{A'} - v_A$. Then w satisfies the heat equation with zero initial data and is less than 2 on ∂U. Let η be a smooth cut-off function equal to zero in $B(z,\delta_1)$ and equal to 2 on ∂U. Let $\bar{w}$ be the solution of the heat equation on $U\times(0,T]$ with η being the initial data and equal to 2 on $\partial U\times(0,T]$. By interior regularity there exists $\delta_2\in(0,1)$ such that for all $h\in[0,\delta_2]$ and all $x\in B(z,\delta_2)$, $|\bar{w}(x,h)|\leqslant\varepsilon_1 h$. Note that, by comparison $w\leqslant\bar{w}$. This implies that for all $h\in[0,\delta_2]$ and all $x\in B(z,\delta_2)$

$$u_{A'}(x,h) \leqslant v_A(x,h) + \varepsilon_1 h.$$

Let $\delta := \min\{\delta_1,\delta_2\}$. Let $x\in B(z,\delta)$. As before (by 15) it is enough to show that $x\notin M_h A'$ for all $h\in[0,\delta)$ which is equivalent to $u_{A'}(x,h) < \frac{1}{2}$. And that, follows from the inequalities above:

$$u_{A'}(x,h) \leqslant v_A(x,h) + \varepsilon_1 h \leqslant \tfrac{1}{2} - \varepsilon_1\sqrt{h} + \varepsilon_1 h < \tfrac{1}{2}.$$

Now let us consider the case $z\in\partial U$. Let φ be a smooth function such that $\partial\varphi/\partial\nu(z) > 0$ and let $\varepsilon>0$. Let $u_{A'}$ and v_A be as above. We again compare them, but to localize argument we first modify v_A. Let ψ be a smooth function defined in a neighborhood of z such that its zero set is the boundary of U (near z) and $D\psi(z)$ is the outside normal vector, ν. Lemma 5.11 then implies that there is a neighborhood of z, which we can assume to be a ball of radius $2a$, such that for all $h\in(0,a)$, $x\in\partial U\cap B(z,2a)$

and $t \in (0,a)$, $\partial v_A/\partial \nu(x,t) > 0$. Let η be a smooth cut-off function supported in $B(z,2a)$ and equal to 1 in $B(z,a)$. Let $\tilde{v}_A$ be the solution of the following Cauchy problem

$$\frac{\partial \tilde{v}_A}{\partial t} - \Delta \tilde{v}_A = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

$$\tilde{v}_A(x,0) = \eta \mathbb{1}_A \quad \text{in } \mathbb{R}^N.$$

It is not hard to show that there exists a negative constant c_2, independent of $h \in (0,a)$ and $x \in B(z,2a)$ (recall that A depends on h and x), such that

$$\frac{\partial \tilde{v}_A}{\partial \nu} > c_2 \quad \text{on } \partial U \times (0,r).$$

Applying the Lemma 5.11 once more, we obtain that there exists $\delta_1 > 0$, smaller that a, such that for all $h \in [0,\delta_1)$, $x \in \partial U \cap B(z,\delta_1)$ and $t \in (0,\delta_1)$, $\partial \tilde{v}_A/\partial \nu(x,t) > 0$. Let ρ be a smooth cut-off function on $\partial U \times [0,T]$ such $\rho = 0$ on $(B(z,\delta_1/2) \cap \partial U) \times [0,T]$ and $\rho = c_2$ on $(\partial U \setminus B(z,\delta_1)) \times [0,T]$. Note that $\rho \leqslant \partial \tilde{v}_A/\partial \nu$ on $\partial U \times (0,T)$.

Let $\tilde{\tilde{v}}_{A'}$ be the solution of

$$\frac{\partial \tilde{\tilde{v}}_{A'}}{\partial t} - \Delta \tilde{\tilde{v}}_{A'} = 0 \quad \text{in } U \times (0,T),$$

$$\tilde{\tilde{v}}_A(x,0) = \eta \mathbb{1}_A \quad \text{in } U,$$

$$\frac{\partial \tilde{\tilde{v}}_{A'}(x,t)}{\partial \nu} = \rho(x,t) \quad \text{on } \partial U \times (0,T).$$

As before we need to prove that for δ small enough, for all $x \in B(z,\delta)$ and all $h \in [0,\delta)$ the inequality $u_{A'}(x,h) < \frac{1}{2}$ holds. Recall that Lemma 5.9 there exists δ, which we can assume to be smaller than δ_1, and $\varepsilon_1 > 0$ such that for all $x \in B(z,\delta)$ and all $h \in [0,\delta)$

$$v_A(x,h) < \tfrac{1}{2} - \varepsilon_1 \sqrt{h}.$$

Let $w = \tilde{\tilde{v}}_{A'} - u_{A'}$. Note that w satisfies the conditions of Lemma 5.10. Therefore, by making δ even smaller (if necessary), we get that for all $h \in [0,\delta)$, $x \in \bar{U} \cap B(z,\delta)$

$$\tilde{\tilde{v}}_{A'}(x,h) - u_{A'}(x,h) > -\varepsilon_1 \sqrt{h}.$$

Combining the two inequalities, along with the fact that, by comparison, $\tilde{\tilde{v}}_{A'} \leqslant \tilde{v}_A \leqslant v_A$, yields

$$u_{A'}(x,h) < \tfrac{1}{2}. \qquad \square$$

To complete the proof of convergence of the scheme we need to verify that the Lemmas 5.7 and 5.8 hold for this scheme as well. Note that in both lemmas the arguments are about points in the interior of U, and the boundary conditions play a minor role. Consequently Lemma 5.8 can again be proven by making only minor modifications to the proof of Lemma 3.2 in [17], while the Lemma 5.7 can be proven along the same lines as before.

6. Schemes for front propagation for velocities having nonlocal terms

Let domain U and mapping T be as in Section 5.

6.1. Motions not depending on curvature

Let measure μ, threshold θ and velocity v be as in Section 5.1. We here construct schemes for the following nonlocal velocity:

$$w(x,p,A) := v(p)b(x,A), \tag{21}$$

where $A \in \mathscr{B}$ and $b : \bar{U} \to \mathbb{R} \times \mathscr{B}$ is continuous. Let us now define $F : \bar{U} \times \mathbb{R}^N \times \mathscr{B} \to \mathbb{R}$ as follows:

$$F(x,p,A) := \begin{cases} -|p|w(x,-\hat{p},A) & \text{if } p \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

For F to be nonincreasing in its set argument we assume the following condition:

(w1) $v(p) \geqslant 0$ for all $p \in S^{N-1}$ and b is nondecreasing in its set argument.

Note that the conditions (F1)–(F7) are now satisfied. Recall that for $h \neq 0$ $\mu_h(\cdot) = \mu(h^{-1}\cdot)$. Let $\mu_0(\cdot)$ be the measure defined by $\mu_0(\{0\}) = 1$, $\mu_0(\mathbb{R}^N \setminus \{0\}) = 0$.

The approximation scheme we consider is defined by

$$\begin{aligned} M_h A := &\{x \in \bar{U}\colon b(x,A) \geqslant 0 \text{ and } \mu_{hb(x,A)}(T(A) - x) \geqslant \theta\} \cup \\ &\{x \in A\colon b(x,A) < 0 \text{ and } \mu_{hb(x,A)}(T(A) - x) > 1 - \theta\}. \end{aligned} \tag{23}$$

Let us briefly explain, what lies behind the apparent different treatment of points where $b > 0$ and points where $b < 0$. If $b < 0$ the front is shrinking so to get points that lie inside the updated front, only the points that are already inside the front need to be considered.

Remark. For nonnegative velocities, that is the velocities for which $b \geqslant 0$, the scheme (as well as many constructions and proofs to follow) become much simpler. Then

$$M_h A := \{x \in \bar{U}\colon \mu_{hb(A,x)}(T(A) - x) \geqslant \theta\}. \tag{24}$$

The approximation scheme $\hat{S}_h$ is defined by

$$\hat{S}_h \varphi(x) = \sup\{\lambda\colon x \in M_h\{\varphi \geqslant \lambda\}\}.$$

It is important to note that this scheme is not monotone in its set argument as can be seen from the following example.

Example. Let $\mu = \pi^{-1}\mathbb{1}_{B(0,1)}$, $\theta = 0.4$, and $b(A) = m(A)$. Although exact computations are easy to obtain we hope that explaining the Fig. 2 will be more insightful. The rectangles $A, B \in \mathscr{B}$ have areas $m(A) = 2$, $m(B) = 1$. The circles represent the supports of $\mu_{hb(A)}$ and $\mu_{hb(B)}$. Note that the proportion of the bigger circle in A is less than 0.4 while the proportion of the smaller circle in B is bigger than 0.4. Therefore $x \notin M_h A$ but $x \in M_h B$.

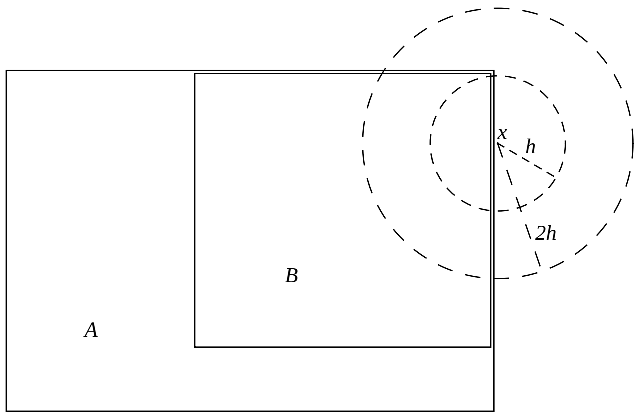

Fig. 2. Example of nonmonotonicity of the scheme M_h.

Our strategy for proving convergence of the scheme $\hat{S}$ is following: We construct two more complicated monotone schemes that bound it from above and below and show their convergence. By Lemma 4.3 that implies the convergence of the scheme $\hat{S}$.

Let

$$\bar{M}_h A := \left\{ x \in \bar{U}\colon b(x,A) \geqslant 0 \text{ and } \sup_{0 \leqslant l \leqslant hb(x,A)} \mu_l(T(A)-x) \geqslant \theta \right\} \cup \left\{ x \in A\colon b(x,A) < 0 \text{ and } \sup_{hb(x,\emptyset) \leqslant l \leqslant hb(x,A)} \mu_l(T(A)-x) > 1-\theta \right\}. \quad (25)$$

For the proof of the convergence we need the auxiliary scheme

$$\bar{M}_h^B A := \left\{ x \in \bar{U}\colon b(x,B) \geqslant 0 \text{ and } \sup_{0 \leqslant l \leqslant hb(x,B)} \mu_l(T(A)-x) \geqslant \theta \right\} \cup \left\{ x \in A\colon b(x,B) < 0 \text{ and } \sup_{hb(x,\emptyset) \leqslant l \leqslant hb(x,B)} \mu_l(T(A)-x) > 1-\theta \right\}. \quad (26)$$

These schemes are monotone in their set arguments, that is: For all $A, B, C \in \mathscr{B}$ and all $h > 0$

$$\text{If } A \subseteq B \quad \text{then } \bar{M}_h A \subseteq \bar{M}_h B,\ \bar{M}_h^C A \subseteq \bar{M}_h^C B \text{ and } \bar{M}_h^A C \subseteq \bar{M}_h^B C. \quad (27)$$

The proofs of monotonicity are straightforward. Let us just illustrate that on one case. Let $A \subset B$, $x \in \bar{M}_h A$ and $b(x,A) < 0 < b(x,B)$. Then $x \in A$ and hence $x \in B$, which implies $\mu_0(B-x) = 1$. Therefore $x \in \bar{M}_h B$.

We now define mapping $\bar{S}_h : \mathscr{B} \times B(\bar{U}) \to B(\bar{U})$ by

$$\bar{S}_h(B)\varphi(x) := \sup\{\lambda \in \mathbb{R}\colon x \in \bar{M}_h^B\{\varphi \geqslant \lambda\}\}.$$

Approximation scheme $\hat{\bar{S}}_h$, defined as in Section 4, is then given by

$$\hat{\bar{S}}_h\varphi(x) = \sup\{\lambda\colon \bar{S}_h(\{\varphi \geqslant \lambda\})\varphi(x) \geqslant \lambda\}.$$

It is straightforward to check that

$$\hat{\bar{S}}_h\varphi(x) = \sup\{\lambda\colon x \in \bar{M}_h\{\varphi \geqslant \lambda\}\}. \tag{28}$$

We also use the scheme $\underline{M}_h$, defined by

$$\underline{M}_h A := \left\{x \in \bar{U}\colon b(x,A) \geqslant 0 \text{ and } \inf_{hb(x,A) \leqslant l \leqslant hb(x,\bar{U})} \mu_l(T(A) - x) \geqslant \theta\right\} \cup \left\{x \in A\colon b(x,A) < 0 \text{ and } \inf_{hb(x,A) \leqslant l \leqslant 0} \mu_l(T(A) - x) > 1 - \theta\right\}. \tag{29}$$

Analogously to the scheme $\bar{M}_h$ we define the auxiliary schemes $\underline{M}_h^B$, $\underline{S}_h$ and the scheme $\hat{\underline{S}}_h$. Note that the analogue of the monotonicity (27) holds for $\underline{M}_h$ and that the analogue of (28) holds for $\hat{\underline{S}}_h$.

To define approximate solutions for $h > 0$ and $t \geqslant 0$ we define Q_t^h, $\bar{Q}_t^h$ and $\underline{Q}_t^h$ by

$$Q_t^h := \hat{S}_h^j, \quad \bar{Q}_t^h := \hat{\bar{S}}_h^j \quad \text{and} \quad \underline{Q}_t^h := \hat{\underline{S}}_h^j \quad \text{where } j = \left\lfloor \frac{t}{h} \right\rfloor.$$

The following properties, that follow immediately from the definitions, are used in the proofs of consistency of the schemes.

$$\text{If } x \notin \bar{M}_h^B\{\varphi \geqslant \lambda\} \quad \text{then } \bar{S}_h(B)\varphi(x) \leqslant \lambda,$$

$$\text{and} \quad \text{if } x \in \underline{M}_h^B\{\varphi \geqslant \lambda\} \quad \text{then } \underline{S}_h(B)\varphi(x) \geqslant \lambda. \tag{30}$$

Also note that from the definitions follows that for all $h > 0$, $A, B \in \mathscr{B}$ and $\varphi \in B(\bar{U})$

$$\underline{M}_h^B A \subseteq M_h^B A \subseteq \bar{M}_h^B A,$$

$$\underline{S}_h(A)\varphi \leqslant S_h(A)\varphi \subseteq \bar{S}_h(A)\varphi. \tag{31}$$

Now we are ready to prove the convergence of the schemes.

Lemma 6.1. *Let g be a continuous function on $\bar{U}$ and let u be the unique continuous solution of* (7) *with F given by* (22). *Then as $h \to 0$, for all $T \in [0, \infty)$,*

$$\bar{Q}_t^h g(x) \to u(x,t) \quad \textit{and} \quad \underline{Q}_t^h g(x) \to u(x,t) \quad \textit{uniformly on } \bar{U} \times [0,T].$$

Proof. We proceed as in the proof of the Theorem 5.1. Monotonicity of $\bar{M}_h^B$ and $\underline{M}_h^B$ and the definition of $\bar{S}_h$ and $\underline{S}_h$ imply that they satisfy properties (S1) and (S2). So we only need to show that properties (S3), (S4) are satisfied. As before we do that in two lemmas. □

Lemma 6.2. *Let $\varphi \in C^1(\mathbb{R}^N)$ and $A \in \mathscr{B}$. For all $z \in U$ and all $z \in \partial U$ for which $\partial\varphi/\partial\nu(z) > 0$, for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x \in B(z,\delta) \cap \bar{U}$ and $h \in (0,\delta)$,*

$$\bar{S}_h(A)\varphi(x) \leqslant \varphi(x) + (-F(z, D\varphi(z), A) + \varepsilon)h, \tag{32}$$

$$\underline{S}_h(A)\varphi(x) \leqslant \varphi(x) + (-F(z, D\varphi(z), A) + \varepsilon)h, \tag{33}$$

and analogously for all $z\in U$ *and all* $z\in\partial U$ *for which* $\partial\varphi/\partial\nu(z)<0$, *there exists* $\delta>0$, *such that for all* $x\in B(z,\delta)\cap\bar{U}$ *and* $h\in(0,\delta)$,

$$\bar{S}_h(A)\varphi(x)\geqslant\varphi(x)+(-F(z,D\varphi(z),A)-\varepsilon)h, \tag{34}$$

$$\underline{S}_h(A)\varphi(x)\geqslant\varphi(x)+(-F(z,D\varphi(z),A)-\varepsilon)h. \tag{35}$$

Proof. From the property (31) follows that it is sufficient to prove (32) and (35). So let us prove (32) first. Let $z\in U$. (The modifications needed in considering the case $z\in\partial U$ are the same as in the proof of Lemma 5.4 and hence we omit that case.) Let $A\in\mathscr{B}$, $\varphi\in C^1(\mathbb{R}^N)$ and $\varepsilon>0$. Let $\bar{b}:=\max\{b(x,\bar{K})\colon x\in\bar{U},K\in\mathscr{B}\}$ and $\bar{v}=\max\{v(p)\colon p\in S^{N-1}\}$. Let $p=D\varphi(z)$.

We split the rest of the proof of (32) into three cases:

Case 1^o: $b(z,A)>0$.

Let $E:=\{y\colon\langle p,y\rangle\geqslant|p|v(-p)+\varepsilon/2\bar{b}\}$. By definition of $v,\mu(E)<\theta$. Note that, since $v\geqslant0$, E is a half-plane not containing the origin, which implies that for $c>1$ $cE\subset E$.

There exists $R>0$ such that

$$\mu(\mathbb{R}^N\setminus B(0,R))<\theta-\mu(E). \tag{36}$$

Since φ is continuously differentiable and $b(\cdot,A)$ is continuous there exists $\delta>0$ such that the following holds: $B(z,R\delta\bar{b}+\delta)\subset U$, $b(z,A)/b(x,A)>1-\min\{\varepsilon/4\bar{b}|p|\bar{v},1\}$ for all $x\in B(z,\delta)$, and for all $h\in(0,\delta)$, all $x\in B(z,\delta)$, and $y\in B(0,R)$

$$\varphi(x+hb(x,A)y)\leqslant\varphi(x)+\langle p,y\rangle hb(x,A)+\frac{\varepsilon h}{4}. \tag{37}$$

Let $h\in(0,\delta)$ and $x\in B(z,\delta)$. Let

$$L:=\{\varphi\geqslant\varphi(x)+(-F(z,p,A)+\varepsilon)h\}\cap\bar{U}.$$

Note that $x\notin L$. By the property (30) it is enough to show that

$$x\notin\bar{M}_h^AL$$

which is, since $b(x,A)>0$, equivalent to showing that

$$\sup_{0\leqslant l\leqslant hb(x,A)}\mu_l(T(L)-x)<\theta.$$

Let us show that $L\cap B(x,Rhb(x,A))\subset x+hb(x,A)E$. Let $x+hb(x,A)y$ be an arbitrary element of $L\cap B(x,Rhb(x,A))$. Using the definition of L and (37) we get:

$$\varphi(x+hb(x,A)y)\geqslant\varphi(x)+(-F(z,p,A)+\varepsilon)h$$

$$hb(x,A)\langle p,y\rangle+\frac{\varepsilon h}{4}\geqslant|p|v(-p)b(z,A)h+\varepsilon h$$

$$\langle p,y\rangle\geqslant|p|v(-p)\frac{b(z,A)}{b(x,A)}+\frac{3\varepsilon}{4\bar{b}}\geqslant|p|v(-p)+\frac{\varepsilon}{2\bar{b}}.$$

So $x + hb(x,A)y \in x + hb(x,A)E$. Therefore for every $l \in (0, hb(x,A)]$

$$\begin{aligned}\mu_l(T(L)-x) &\leqslant \mu_l(L\cap B(x,Rhb(x,A))-x)+\mu_l(\mathbb{R}^N\setminus B(0,Rhb(x,A)))\\ &\leqslant \mu\left(\frac{hb(x,A)}{l}E\right)+\mu\left(\frac{hb(x,A)}{l}(\mathbb{R}^N\setminus B(0,R))\right)\\ &\leqslant \mu(E)+\mu(\mathbb{R}^N\setminus B(0,R))<\theta.\end{aligned}$$

Since $x \notin L$ we also have that $\mu_0(T(L)-x)=0$. Therefore $\sup_{0\leqslant l\leqslant hb(x,A)} \mu_l(T(L)-x)<\theta$, which by definition of $\bar{M}_h^A$ implies that $x \notin \bar{M}_h^A L$.

Case 2^o: $b(z,A)=0$.

Let E, R be as in the first case. Since φ is continuously differentiable there exists $\delta>0$ such that the following holds: $B(z, R\delta\bar{b}+\delta) \subset U$, for all $x\in B(z,\delta)$, $b(x,A)<\min\{\varepsilon/4\bar{b}|p|\bar{v},1\}$ and for all $h\in(0,\delta)$, all $x\in B(z,\delta)$ such that $b(x,A)>0$, and $y\in B(0,R)$

$$\varphi(x+hb(x,A)y)\leqslant\varphi(x)+\langle p,y\rangle hb(x,A)+\frac{\varepsilon h}{4}. \tag{38}$$

Let $h\in(0,\delta)$ and $x\in B(z,\delta)$. Let $L:=\{\varphi\geqslant\varphi(x)+(-F(z,p,A)+\varepsilon)h\}\cap\bar{U}=\{\varphi\geqslant\varphi(x)+\varepsilon h\}\cap\bar{U}$.

As in the first case it is enough to show that $x\notin\bar{M}_h^A L$. If $b(x,A)\leqslant 0$ than this is satisfied since $x\notin L$. So we can assume $b(x,A)>0$. We need to show that

$$\sup_{0\leqslant l\leqslant hb(x,A)}\mu_l(T(L)-x)<\theta.$$

As before, let us show that $L\cap B(x,Rhb(x,A))\subset x+hb(x,A)E$. Let $x+hb(x,A)y$ be an arbitrary element of $L\cap B(x,Rhb(x,A))$. Using the definition of L and (38) we get:

$$\varphi(x+hb(x,A)y)\geqslant\varphi(x)+\varepsilon h,$$

$$hb(x,A)\langle p,y\rangle+\frac{\varepsilon h}{4}\geqslant\varepsilon h,$$

$$\langle p,y\rangle\geqslant\frac{\varepsilon}{4b(x,A)}+\frac{\varepsilon}{2\bar{b}}\geqslant|p|v(-p)+\frac{\varepsilon}{2\bar{b}}.$$

The remainder of the argument is identical to one given in Case 1^o.

Case 3^o: $b(z,A)<0$.

If $p=0$ or $v(-p)=0$ then $x\notin L$ and, as above, that implies the claim. So we can assume that $p\neq0$ and $v(-p)\neq0$. We can now assume that $|p|v(-p)>\varepsilon/\bar{b}$. Let $E:=\{y:\ \langle p,y\rangle\geqslant-|p|v(-p)+\varepsilon/2\bar{b}\}$. By definition of v, $\mu(-E)\leqslant1-\theta$. Note that, E is a half-plain containing the origin, which implies that for $0<c<1$, $-cE\subset-E$.

If μ is not compactly supported, by assumption (μ1) there exists $R>0$ such that

$$\mu(\mathbb{R}^N\setminus B(0,R))<1-\theta-\mu(-E). \tag{39}$$

If μ is compactly supported then let R be the diameter of the support of μ. Since φ is continuously differentiable and $b(\cdot,A)$ is continuous there exists $\delta>0$ such that the following holds: $B(z, R\delta\bar{b} + \delta) \subset U$, $b(z, A)/b(x, A) < 1 + \min\{\varepsilon/4\bar{b}|p|\bar{v}, 1\}$ for all $x \in B(z, \delta)$, and for all $h \in (0, \delta)$, all $x \in B(z, \delta)$, and $y \in B(0, R)$

$$\varphi(x + h|b(x, A)|y) \leqslant \varphi(x) + \langle p, y\rangle h|b(x, A)| + \frac{\varepsilon h}{4}. \tag{40}$$

Let $h \in (0, \delta)$ and $x \in B(z, \delta)$. Let $L := \{\varphi \geqslant \varphi(x) + (-F(z, p, A) + \varepsilon)h\} \cap \bar{U}$. Again, we need to show that $x \notin \bar{M}_h^A L$, which is, since $b(x, A) < 0$, equivalent to showing that

$$\sup_{hb(x,\emptyset) \leqslant l \leqslant hb(x,A)} \mu_l(T(L) - x) \leqslant 1 - \theta.$$

Arguing as in Case 1^o one obtains that $L \cap B(x, Rh|b(x, A)|) \subset x + h|b(x, A)|E$. Therefore for every $l \in [hb(x, \emptyset), hb(x, A)]$

$$\begin{aligned} \mu_l(T(L) - x) &\leqslant \mu_l(L \cap B(x, Rh|b(x, A)|) - x) + \mu_l(\mathbb{R}^N \setminus B(0, Rh|b(x, A)|)) \\ &\leqslant \mu\left(-\frac{hb(x, A)}{l} E\right) + \mu\left(-\frac{hb(x, A)}{l}(\mathbb{R}^N \setminus B(0, R))\right) \\ &\leqslant \mu(-E) + \mu(-\mathbb{R}^N \setminus B(0, R)) \leqslant 1 - \theta. \end{aligned}$$

Therefore $\sup_{hb(x,\emptyset) \leqslant l \leqslant hb(x,A)} \mu_l(T(L) - x) \leqslant 1 - \theta$, which by definition of $\bar{M}_h^A$ implies that $x \notin \bar{M}_h^A L$.

The proof of (35) is dual to the one above and we leave it to the reader. □

Lemma 6.3. *There exist constants $\delta > 0$, $R > 0$ such that for all $y \in \bar{U}$ for the function $\varphi(x) = -|x - y|$ (respectively $\varphi(x) = |x - y|$) the following inequalities hold*

$$\underline{\hat{S}}_h\varphi(x) - \varphi(x) \leqslant Rh \quad (\text{respectively } \underline{\hat{S}}_h\varphi(x) - \varphi(x) \geqslant -Rh), \tag{41}$$

$$\hat{\bar{S}}_h\varphi(x) - \varphi(x) \leqslant Rh \quad (\text{respectively } \hat{\bar{S}}_h\varphi(x) - \varphi(x) \geqslant -Rh). \tag{42}$$

Proof of this lemma is analogous to the proof of Lemma 5.5; only that R in this case is chosen so that $\mu(B(0, R/3\bar{b})) > \max\{\theta, 1 - \theta\}$.

Theorem 6.4. *Let g be a continuous function on $\bar{U}$ and let u be the unique continuous solution of* (7) *with F given by* (22). *Then as $h \to 0$, for all $T \in [0, \infty)$,*

$$Q_t^h g(x) \to u(x, t) \quad \textit{uniformly on } \bar{U} \times [0, T].$$

This theorem follows directly from Lemmas 6.1 and 6.3 and property (31).

Analogues of the Theorems 5.2 and 5.3 follow as before.

6.2. Motions depending on the curvature

In this section we construct an extension of schemes studied in Section 5.2 to some nonlocal motions. So, let μ be a measure such that $\mu(\mathbb{R}^N) = 1$ and that also satisfies conditions $(\mu 2)-(\mu 6)$. Let $b:\mathscr{B}\to\mathbb{R}$ be a nondecreasing (wrt. set inclusion) continuous function. We construct a scheme for front propagation with velocity $v:S^{N-1}\times\mathscr{S}^N\times\mathscr{B}\to\mathbb{R}$

$$v(p,X,A)=\left(\int_{p^\perp} f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)\right)^{-1}$$
$$\left(-\frac{1}{2}\int_{p^\perp}\langle Xx,x\rangle f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)+b(A)\right).$$

We define $F:\mathbb{R}^N\setminus\{0\}\times\mathscr{S}^N\times\mathscr{B}\to\mathbb{R}$ by

$$F(p,X,A):=-|p|v\left(\frac{p}{|p|},\frac{X}{|p|},A\right).$$

An update of the set valued scheme $M_h:\mathscr{B}\to\mathscr{B}$ for $h>0$ is defined by

$$M_hA:=\{x\in\bar{U}:\ \mu_{\sqrt{h}}(T(A)-x)\geqslant\tfrac{1}{2}-b(A)\sqrt{h}\}.$$

The auxiliary scheme is defined by

$$M_h^BA:=\{x\in\bar{U}:\ \mu_{\sqrt{h}}(T(A)-x)\geqslant\tfrac{1}{2}-b(B)\sqrt{h}\}.$$

C_t^h, S_h, $\hat{S}_h$ and Q_t^h are defined as before. We claim that the appropriate analogues of the Theorems 5.1, 5.2 and 5.3 hold. The proof relies on the analogues of the Lemmas 5.6, 5.7 and 5.8. These are proven in the same way as the analogous results for local, curvature dependent motions.

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Appendix A

Proof of Lemma 5.10. Using the normal bundle to ∂U and a cut-off function it is easy to construct a smooth function σ supported outside of $B(z,2r)\times[0,T]$ such that $\partial\sigma/\partial\nu=g$ on $\partial U\times(0,T]$. Let us define functions $f:=-\sigma_t+\Delta\sigma$ and $w=u-\sigma$. The function w is then the solution of the problem:

$$w_t-\Delta w=f\quad\text{in } U\times(0,T],$$

$$w(x,0)=\psi(x)-\sigma(x,0)\quad\text{in } \bar{U},$$

$$\frac{\partial w}{\partial\nu}(x,t)=0\quad\text{on } \partial U\times(0,T].$$

From the theory of linear parabolic partial differential equations (see [12]) we know an explicit formula for the solution of the given initial-boundary value problem:

$$w(x,t) = \int_U N(x,t,\xi,0)(\psi(\xi) - \sigma(\xi))\,d\xi + \int_0^t \int_U N(x,t,\xi,\tau) f(\xi,\tau)\,d\xi\,d\tau. \quad \text{(A.1)}$$

Where N, the Neumann's function for the homogeneous version of the problem above is constructed in the following way: $N(x,t,\xi,\tau) = \Gamma(x,t,\xi,\tau) + V(x,t,\xi,\tau)$ where

$$\Gamma(x,t,\xi,\tau) = \frac{1}{(2\sqrt{\pi})^N}(t-\tau)^{-N/2}\exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right)$$

is the fundamental solution for the heat equation and for $\xi \in U$ and $\tau \in [0,T)$, $V(\cdot,\cdot,\xi,\tau)$ is the solution of the following problem:

$$V_t - \Delta_x V = 0 \quad \text{in } U \times (\tau, T],$$

$$V(\cdot,\tau,\xi,\tau) = 0 \quad \text{in } \bar{U},$$

$$\frac{\partial V}{\partial \nu}(x,t,\xi,\tau) = -\frac{\partial \Gamma}{\partial \nu}(x,t,\xi,\tau) \quad \text{on } \partial U \times (\tau, T].$$

Note that $D\Gamma$ for $x, \xi \in \bar{U}$ satisfies

$$|D\Gamma(x,t,\xi,\tau)| \leqslant \operatorname{diam}(U)(t-\tau)^{-(N+1)/2}\exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right).$$

That, using the estimate given in Lemma 5.3.2 in [12], yields the following estimate for some constant c_2:

$$|V(x,t,\xi,\tau)| \leqslant c_2(t-\tau)^{-(N+1)/2}\exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right).$$

Therefore for $t \in (0,T]$ and $\tau \in [0,t)$ and all $x \neq \xi$ in U

$$|N(x,t,\xi,\tau)| \leqslant c(t-\tau)^{-(N+1)/2}\exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right)$$

for some constant c. The remainder of the proof is straightforward from this estimate and (A.1). □

Proof of Lemma 5.11. Let $c := (\langle D\varphi(z), D\psi(z)\rangle / 2|D\varphi(z)||D\psi(z)|)$. Since φ and ψ are smooth functions there exists $V_1 \subset V$, an open neighborhood of z such that for all $(x,y) \in V_1 \times V_1$

$$\langle D\varphi(x), D\psi(y)\rangle > c|D\varphi(x)|\,|D\psi(y)|. \quad \text{(A.2)}$$

Let $b := 1 + \max_{x\in V} \sum_{1\leqslant i,j\leqslant N} |\partial^2\varphi/\partial x_i \partial x_j|$. Let $a \in (0, 1/b)$ such that $B(z,3a) \subset V_1$. The following estimate is useful later: Let $m \geqslant 0$ be an integer and $\alpha > 1$ then

$$\int_\alpha^\infty w^{2m+1} e^{-w^2}\,dw = \frac{1}{2}\int_{\alpha^2}^\infty s^m e^{-s}\,ds$$

$$= -\frac{1}{2}\left(\sum_{j=0}^m \frac{m!}{j!} s^j\right) e^{-s}\Big|_{\alpha^2}^\infty$$

$$\leqslant (m+1)!\alpha^{2m}\,\mathrm{e}^{-\alpha^2}. \tag{A.3}$$

Note also that if $m < l$ then $\int_\alpha^\infty w^m\,\mathrm{e}^{-w^2}\,\mathrm{d}w < \int_\alpha^\infty w^l\,\mathrm{e}^{-w^2}\,\mathrm{d}w$. The estimate (A.3) implies, after an elementary calculation, that there exists $\delta_1 > 0$ such that for all $t \in (0, \delta_1)$

$$\frac{1}{\sqrt{t}} \int_{\frac{a}{2\sqrt{t}}}^\infty w^{N-1}\,\mathrm{e}^{-w^2}\,\mathrm{d}w < \frac{c}{32N2^N}\exp\left(\frac{-a^2}{8t}\right). \tag{A.4}$$

Let $C := \max_{j=0,1,2} \int_0^\infty w^{N+j}\,\mathrm{e}^{-w^2}\,\mathrm{d}w$. Let

$$\delta := \min\left\{\delta_1, \frac{a^2}{32(\ln 8N - \ln c)}, \left(\frac{c}{256NCb3^N}\right)^2, 1\right\}$$

and

$$c_1 := \min\left\{\frac{a}{8}, \frac{b}{4}, \frac{c}{256NbC3^N}\right\}. \tag{A.5}$$

Let $W := B(z, c_1)$ and let $\varepsilon_0 := \min\{\max_{x\in W\cap\{\psi=0\}} \varphi(x), -\min_{x\in W\cap\{\psi=0\}} \varphi(x)\}$.

Let $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $t \in (0,\delta)$, $x \in W$ such that $\psi(x) = 0$. Let y be a point from the set $\{\varphi = \varepsilon\}$ closest to x. Note that $\mathrm{d}(z, y) < a/4$. Without the loss of generality we can assume that $y = 0$ and that $D\varphi(0)$ is a negative multiple of e_N. Then $x = (0, \ldots, 0, x_N)$. For a vector $q \in \mathbb{R}^N$ let $\check{q} := (q_1, \ldots, q_{N-1})$. To obtain needed estimates we use the following regions of $\mathbb{R}^N$

$$H := \{q\colon q_N < 0\},$$

$$A_1 := \{q\colon q_N < -a\},$$

$$A_2 := \{q\colon q_N \in [-a, a] \text{ and } |\check{q}| > a\},$$

$$A_3 := \{q\colon |\check{q}| \leqslant a \text{ and } |q_N| < b|\check{q}|^2\},$$

$$A_4 := \{q\colon q_N > a\}.$$

For $i = 1,2,3,4$ let $g_i := (g - \mathbb{1}_H)\mathbb{1}_{A_i}$. Let $g_0 := \mathbb{1}_H$. Our assumptions on a and our choice of b yields that $g = g_0 + g_1 + g_2 + g_3 + g_4$. Let $K_N(x,t) := (4\pi t)^{-N/2}\exp(-|x|^2/4t)$ be the heat kernel. For $i = 0,1,2,3,4$ let $u_i(x,t) := \int_{\mathbb{R}^N} g_i(\xi)K_N(x-\xi,t)\,\mathrm{d}\xi$. Note that $u = u_0 + u_1 + u_2 + u_3 + u_4$ and $Du = Du_0 + Du_1 + Du_2 + Du_3 + Du_4$.

Elementary calculation shows that:

$$Du_0(x,t) := \left(0, \ldots, 0, -\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x_N^2}{4t}\right)\right).$$

Elementary calculations and use of (A.5) show that

$$\left|\frac{\partial u_1}{\partial x_N}(x,t)\right| < \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x_N^2}{4t}\right)\exp\left(-\frac{a^2}{8t}\right) < \frac{c}{8N}\,|Du_0(x,t)|.$$

For $j = 1, \ldots, N-1$

$$\left|\frac{\partial u_1}{\partial x_j}(x,t)\right| < \frac{2}{2\sqrt{\pi t}}\exp\left(-\frac{x_N^2}{4t}\right)\exp\left(-\frac{a^2}{8t}\right) < \frac{c}{8N}\,|Du_0(x,t)|.$$

To get the bounds for the gradient of u_2 we use cylindrical coordinates and (A.4)

$$\left|\frac{\partial u_2}{\partial x_N}(x,t)\right| < \int_{-a}^{a}\int_{a}^{\infty}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}(4\pi t)^{-N/2}\frac{|x_N-\xi_N|}{2t}\,r^{N-2}$$

$$\exp\left(-\frac{|r|^2+(x_N-\xi_N)^2}{4t}\right)\mathrm{d}\theta_{n-2}\ldots\mathrm{d}\theta_1\,\mathrm{d}r\,\mathrm{d}\xi_N$$

$$< 2^{N+1}\frac{1}{2\sqrt{\pi t}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} w^{N-2}\,\mathrm{e}^{-w^2}\,\mathrm{d}w < \frac{c}{8N}\,|Du_0(x,t)|.$$

Similarly, for $j = 1,\ldots,N-1$

$$\left|\frac{\partial u_2}{\partial x_j}(x,t)\right| < \int_{-a}^{a}\int_{a}^{\infty}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}(4\pi t)^{-N/2}\frac{1}{2t}\,r^{N-1}$$

$$\exp\left(-\frac{|r|^2+(x_N-\xi_N)^2}{4t}\right)\mathrm{d}\theta_{N-2}\ldots\mathrm{d}\theta_1\,\mathrm{d}r\,\mathrm{d}\xi_N$$

$$< 2^{N+2}\frac{1}{2\sqrt{\pi t}}\int_{\frac{a}{2\sqrt{t}}}^{\infty} w^{N-1}\,\mathrm{e}^{-w^2}\,\mathrm{d}w < \frac{c}{8N}\,|Du_0(x,t)|.$$

Since $x_N < (1/4b)$ then for all $\xi \in A_3$

$$|\xi - x|^2 \geqslant x_N^2 + |\breve{\xi}|^2/2.$$

Using this fact and (A.5) we get

$$\left|\frac{\partial u_3}{\partial x_N}(x,t)\right| < \int_0^{a}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_{-br^2}^{br^2}(4\pi t)^{-N/2}\,\frac{|x_N-\xi_N|}{2t}\,r^{N-2}$$

$$\exp\left(-\frac{x_N^2}{4t}-\frac{r^2}{8t}\right)\mathrm{d}\xi_N\,\mathrm{d}\theta_{N-2}\ldots\mathrm{d}\theta_1\,\mathrm{d}r$$

$$< 4b\exp\left(-\frac{x_N^2}{4t}\right)\int_0^{a} t^{-(N+2)/2}r^N(|x_N|+br^2)\exp\left(-\frac{r^2}{8t}\right)\mathrm{d}r$$

$$< \frac{8b}{\sqrt{\pi t}}\exp\left(-\frac{x_N^2}{4t}\right)\left(x_N 3^{N+1}\int_0^{\infty} w^N\,\mathrm{e}^{-w^2}\,\mathrm{d}w\right.$$

$$\left.+\,3^{N+3}t^{3/2}\int_0^{\infty} w^{N+2}\,\mathrm{e}^{-w^2}\,\mathrm{d}w\right)$$

$$< \frac{c}{8N}\,|Du_0(x,t)|$$

and for $j = 1,\ldots,N-1$

$$\left|\frac{\partial u_3}{\partial x_j}(x,t)\right| < \int_0^{a}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_{-br^2}^{br^2}(4\pi t)^{-N/2}\,\frac{1}{2t}\,r^{N-1}$$

$$\exp\left(-\frac{x_N^2}{4t}-\frac{r^2}{8t}\right)\mathrm{d}\xi_N\,\mathrm{d}\theta_{N-2}\ldots\mathrm{d}\theta_1\,\mathrm{d}r$$

$$< 2b\exp\left(-\frac{x_N^2}{4t}\right)\int_0^a t^{-(N+2)/2}r^{N+1}\exp\left(-\frac{r^2}{8t}\right)\mathrm{d}r$$

$$< 3^{N+2}b\exp\left(-\frac{x_N^2}{4t}\right)\int_0^\infty w^{N+1}\,\mathrm{e}^{-w^2}\,\mathrm{d}w < \frac{c}{8N}\,|Du_0(x,t)|.$$

Estimates for $|Du_4|$ are obtained in the same way as the ones for $|Du_1|$.

It follows that for $j = 1,2,3,4$

$$|Du_j(x,t)| < \frac{c}{4}\,|Du_0|.$$

Using these inequalities, the fact that $Du_0(x,t)$ is a positive multiple of $D\varphi(0)$ and (A.2) we get what was claimed:

$$\langle Du(x,t), D\psi(x)\rangle = \langle Du_0(x,t), D\psi(x)\rangle + \left\langle \sum_{j=1}^{4} Du_j(x,t), D\psi(x)\right\rangle$$

$$\geqslant c|Du_0(x,t)||D\psi(x)| - 4\,\frac{c}{8}\,|Du_0(x,t)||D\psi(x)| > 0. \qquad \square$$

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