Nonlocal Front Propagation Problems in Bounded Domains with Neumann-type Boundary Conditions and Applications

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Key-words: Front propagation,nonlocal reaction-diffusion equations, asymptotic behavior, geometrical approach, level-set approach, Neumann boundary condition, angle boundary condition, viscosity solutions.

Abstract

This paper is concerned with the asymptotic behavior as $\varepsilon \to 0$ of the solutions of nonlocal reaction-diffusion equations of the form $u_t - \Delta u + \varepsilon^{-2} f(u, \varepsilon \int_0 u) = 0$ in $O \times (0,T)$ associated with nonlinear oblique derivative boundary conditions. We show that such behavior is described in terms of an interface evolving with normal velocity depending not only on its curvature but also on the measure of the set it encloses. To this purpose we introduce a weak notion of motion of hypersurfaces with nonlocal normal velocities depending on the volume they enclose, which extends the geometric definition of generalized motion of hypersurfaces in bounded domains introduced by G. Barles and the first author in [BDL] to solve a similar problem with local normal velocities depending on the normal direction and the curvature of the front. We also establish comparison and existence theorems of viscosity solutions to initial-boundary value problems for some singular degenerate nonlocal parabolic pde's with nonlinear Neumann-type boundary conditions

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Introduction

In this paper we study the limiting behaviour, as $\varepsilon \to 0$, of the solution of the following equation

$$u_{\varepsilon,t} - \Delta u_\varepsilon + b(x) \cdot Du_\varepsilon + \varepsilon^{-2} f(u_\varepsilon, \varepsilon \int_O u_\varepsilon) = 0 \quad \text{in } O \times (0,T), \tag{1}$$

where O is a smooth bounded domain of $I\!R^N$, $T > 0$, $u^\varepsilon : \overline{O} \times [0,T] \to I\!R$ is the solution, the nonlinearity $f(u,v)$ is smooth and $f_0(u) := f(u,0)$ is the derivative of a double-well potential W. A typical example is $f_0(u) := 2u(u^2-1)$ and $f(u,v) = f_0(u) + vh(u)$. The equation (1) is obtained as a limit as a system of reaction-diffusion equations often referred to as a Belousov-Zhabotinskii model (see, e.g. [CHL]). We consider (1) together with a nonlinear boundary condition of the form

$$G(x,t,Du) = 0 \quad \text{on } \partial O \times (0,T), \tag{2}$$

where $G : \partial O \times (0,T) \times I\!R^N \to I\!R$ is a continuous function satisfying : for any $T > 0$, there exists a constant $\nu(T) > 0$ such that, for all $\lambda > 0$, $x \in \partial O$, $t \in [0,T]$, $p \in I\!R^N$, one has

$$G(x,t,p+\lambda n(x)) - G(x,t,p) \geq \nu(T)\lambda \ .$$

Typical examples of such boundary conditions, besides the homogeneous Neumann boundary condition, are the oblique derivative boundary condition

$$\frac{\partial u_\varepsilon}{\partial \gamma} = 0 \quad \text{on } \partial O \times (0,T) \ , \tag{3}$$

where $\gamma : \partial O \times (0,T) \to I\!R^N$ is a Lipschitz continuous vector field such that $\gamma(x,t) \cdot n(x) > 0$ on $\partial O \times (0,T)$, $n(x)$ being the unit exterior normal vector to ∂O at x and the capillarity type boundary condition

$$\frac{\partial u}{\partial n} = \theta(x,t)|Du| \quad \text{on } \partial O \times (0,T) \ , \tag{4}$$

where $\theta : \partial O \times (0,T) \to I\!R$ is, say, a locally Lipschitz continuous function such that $|\theta(x,t)| < 1$ on $\partial O \times (0,T)$. Finally we impose an initial data

$$u_\varepsilon(x,0) = g(x) \quad \text{on } \overline{O} \times \{0\} \ , \tag{5}$$

where $g \in C(\overline{O})$.

When $f(u,v)$ is independent of v, i.e, $f(u,v) = f_0(u)$ the equation (1) reduces to the Allen-Cahn equation [AC] modelling the motion of the sharp interface -the antiphase boundary- between regions of different phases of a material. The main feature of the solution of the Allen-Cahn equation is that the zero level set of the solution approximates (as $\varepsilon \to 0$) the motion by mean curvature. Formal derivation of this connection was carried out by Allen-Cahn [AC], Keller, Rubinstein, Stenberg [KRS], Fife [F] and many others.

A first rigorous, but partial, proof of this result was proposed by Chen [C] in the case when the motion by mean curvature is classical i.e. when the fronts are smooth hypersurfaces evolving smoothly. This means in fact a small time result since it is well-known that, for motion by mean curvature, singularities develop in finite time.

In order to rigorously prove and even formulate the result for all time, a suitable notion of generalized motion by mean curvature is needed in order to define it past the development of singularities. This question was solved in a rather general way by the "level-set approach" (see Osher and Sethian [OS] Evans and Spruck [ES] and Chen, Giga and Goto [CGG].) Then a different but related approach using the properties of the (signed) distance to the front was introduced (see Soner [S] and Barles, Soner and Souganidis [BSS].) For a general review of these theories, their relationship as well as other related facts we refer to Souganidis [Sou1, Sou2].

In order to treat the case of more complicated reaction-diffusion equations, Barles and Souganidis introduced in [BS] a more geometrical approach. Recently Barles and the first author in [BDL] apply this approach to the the asymptotics of reaction-diffusion equations in bounded domains and with Neumann-type boundary condition, thus extending the result obtained by Katsoulakis, Kossioris, and Reitich [KKR] for convex domains and with homogeneous Neumann boundary condition.

All the results mentioned above concern with the asymptotics of "local" reaction diffusion equation. The first asymptotic result for "nonlocal" equations of the form (1) was provided by Chen, Hilhorst, Logak [CHL]. In [CHL] the authors proved that the limiting behaviour of the solution u^ε of

$$\begin{cases} u_{\varepsilon,t} - \Delta u_\varepsilon + \varepsilon^{-2} f(u_\varepsilon, \varepsilon \int_O u_\varepsilon) = 0 & \text{in } O \times (0,T), \\ \dfrac{\partial u}{\partial n}(x,t) = 0 & \text{on } \partial O \times (0,T), \\ u_\varepsilon(x,0) = g(x) & \text{on } \overline{O}, \end{cases} \tag{6}$$

is governed by the motion of an interface $(\Gamma_t)_t$ with the following normal velocity V_ν

$$V_\nu = -\mathrm{tr} D\nu + c_0(\lambda(\Omega_t^+) - \lambda(\Omega_t^-)) \tag{7}$$

where ν and $D\nu$ denote the exterior normal vector to Γ_t and its derivative , Ω_t^+ is the region enclosed by Γ_t, $\Omega_t^- = \overline{O} \setminus (\Gamma_t \cup \Omega_t^+)$, $\lambda(\Omega_t^\pm)$ is the Lebesgue measure of $\Omega_t^\pm$ and

c_0 is a certain constant depending only on the velocity of the traveling wave solution associated with f. Yet their asymptotic result is proved under the assumptions that $(\Gamma_t)_t$ is a family of smooth hypersurfaces evolving smoothly according to the law (7) and never touches the boundary ∂O, therefore their convergence result holds as long as a smooth evolution of the motion exists.

The aim of this paper is to rigorously establish *for all time* the connection between the equations (1) set in bounded domains with nonlinear Neumann boundary conditions and the motion of fronts with normal velocity (7) and satisfying a contact angle boundary condition on the boundary. In order to do that the first step is to find a way to interpret the evolution with "nonlocal" normal velocities past the development of the singularities, moreover the weak definition has to be sufficiently flexible in order to be able to justify the appearance of an interface in the asymptotic analysis of the reaction-diffusion equations.

To this purpose we consider the motion of hypersurfaces Γ_t with general normal velocity

$$V_\nu = \tilde{v}(x, t, \nu, D\nu, \Omega_t) \tag{8}$$

under the assumption that $\tilde{v}$ is decreasing with respect to $D\nu$ and increasing with respect to Ω_t (which is the region enclosed by Γ_t) in order to guarantee that the resulting evolutions satisfy the geometric maximum-type principle (the so called avoidance-inclusion property). A typical example is the evolution law (7) with $c_0 \geq 0$. When $\tilde{v}$ is not monotone with respect to either arguments such property mail fail (see e.g. the counter-example in [Ca]). In [Ca] Cardaliaguet proposed a weak definition of motion of compact hypersurfaces in $\mathbb{R}^N$ by nonlocal velocities of the form (8) including also the case when $\tilde{v}$ is not increasing with respect to Ω_t. In this last case he showed the existence of approximate solutions to the problem which converge to the generalized solution of the problem only under suitable regularity assumptions on $\tilde{v}$.

In this paper we follow an approach which is very close to the one developed for local motions by Barles and Souganidis in [BS] and later modified by Barles and the first author [BDL] to treat problems with Neumann-type boundary conditions. We recall that both approaches in [BS] and [BDL] consist in considering the evolution of open sets instead of hypersurfaces (which is quite natural from the point of view of the applications) and rely on the "monotonicity property" of the front propagations which, roughly speaking, can be expressed in the following way : if $(\Omega^1_s)_{s\in(a,b)}$, $(\Omega^2_s)_{s\in(a,b)}$ are two families of open subsets evolving with the same normal velocity then, if $\Omega^1_t \subset \Omega^2_t$ for some $t \in (a, b)$, one has

$$\Omega^1_s \subset \Omega^2_s \quad \text{for any } s \in [t, b).$$

The key points used in [BS] and in [BDL] are that (i) it is enough to test against families of smooth open subsets evolving smoothly, (ii) this has to be done only on small time interval and (iii) one can use families whose normal velocities are smaller or bigger than the considered normal velocity.

At this level of generality these basic ideas apply more or less readily in our framework.

Indeed we extend the notion of viscosity solution to initial-boundary value problems for nonlocal parabolic pde's with nonlinear Neumann boundary conditions (already introduced by one of the authors in [Sl] in the case of homogeneous Neumann boundary condition). Then we provide a general comparison result between semicontinuous viscosity sub- and supersolutions to Neumann-type problems for a large class of nonlocal degenerate parabolic (possibly singular) pde's which includes the case of nonlocal "geometric" equations such as for example

$$\begin{cases} u_t - \mathrm{tr}[(I - \hat{Du} \otimes \hat{Du})D^2u] - c_0|Du|\mu(x,t,u) = 0 & \text{in } O \times (0,T), \\ G(x,t,Du) = 0 & \text{in } \partial O \times (0,T), \end{cases} \tag{9}$$

where $\mu(x,t,u) := \lambda(\Omega^+_{t,x}(u)) - \lambda((\Omega^+_{t,x}(u))^c)$, $\Omega^+_{t,x}(u) = \{u(\cdot,t) > u(x,t)\}$, for all $p \neq 0$ $\hat{p} = |p|^{-1}p$ and $p \otimes p$ denotes the symmetric matrix defined by $(p \otimes p)_{ij} = p_i p_j$, for all $1 \leq i,j \leq N$. As a consequence of the comparison result we get by means of the Perron Method (which can be applied with minor changes to the case of nonlocal equations) the existence of a unique continuous viscosity solution. Moreover under the additional assumption that G is homogeneous of degree 1 in p, the level-set approach can be extended to geometric pde's (9), namely we can define the front Γ_t moving by (8) as the zero level set of a the unique solution u of (9) (with the convention that the set Ω_t enclosed by Γ_t is given by $\{u(\cdot,t) > 0\}$).

In analogy with what was done in [BS] and [BDL] the geometric approach we want to apply to our problem is based on the idea that given the geometric definition it suffices to justify the assumptions when everything is smooth. Even though we can still consider small-time smooth approximations of the fronts in our case, local-in-space approximations by smooth functions as in [BS] and [BDL] does not apply directly to problems like (1). Indeed the main difficulty in our analysis lies in finding a proper approximation of the nonlocal term in the equation, both in the definition of the generalized fronts and in the asymptotic analysis.

In this paper we focus on nonlocal velocities depending on the Lebesque measure of the region Ω_t (or on any finite measure which is absolutely continuous with respect to the Lebesgue one) and in this particular case we can find appropriate approximations of the volume of the set Ω_t (see Theorem 2.2). This fact motivates our definition, namely we say, roughly speaking, that a family of open sets $(\Omega_t)_t$ is a generalized solution for motions with the nonlocal velocity V_ν if and only if it is a generalized solution, according to the geometric definition introduced in [BDL], for motion with local velocities obtained by replacing the volume of the set Ω_t with smooth functions of time which are less (resp. bigger) than the volume of Ω_t.

We also show that, as in [BS] and [BDL], our approach is essentially equivalent to the level-set approach (see Theorem 2.3).

We then extend the abstract method introduced in [BS] and [BDL] to justify the appearance of moving interfaces in the asymptotic limits of problems like (1). One of the main assumptions which allows us to apply readily this method and to have a rather simple proof of the asymptotics is that the nonlinearity f is supposed to be nonincreasing with respect to v. Indeed under this condition we can replace the nonlocal term in the reaction-diffusion equations with some suitable smooth functions of the time and to use in this way the asymptotic result obtained in [BDL] in the case of equations with x, t and ε- dependent $f's$. We remark that under this monotonicity assumption we are able to prove a comparison result for viscosity solutions of (1) which automatically yields, by means of the Perron Method, the existence of a continuous solution. Finally we show how it would be possible, by using the same approach, to extend the asymptotics under a suitable relaxation of the monotonicity condition of f with respect to v. In particular we have in mind the case when this condition holds only in the interval between the two stable equilibria of W, and one of the main example is $f(u, v) = 2(u + v)(u^2 - 1)$.

Two main issues that we would like to understand and investigate in the future is the asymptotics when $f_v \geq 0$ (namely f it is still monotone with respect to v but in the opposite way) and to see if our approach can be applied to reaction-diffusion systems like the one studied in [SS] and [HLS].

This paper is organized as follows. In Section 1 we extend the definition of viscosity solutions to nonlocal second order degenerate parabolic pde's introduced by one of the authors in [Sl]. We also prove the comparison between discontinuous viscosity sub- and supersolutions of initial-boundary value problems for a large class of nonlocal degenerate, possibly singular, parabolic pde's (including the "geometric" ones) with nonlinear Neumann-type boundary conditions, thus extending the comparison result obtained in [Sl] in the case of homogeneous Neumann boundary conditions. In Section 2 we introduce the new definition for motions with nonlocal velocities and angle boundary condition and show its connection with the level set approach. Section 3 is devoted to the application of the new definition to the study of the asymptotics of reaction-diffusion equations: we first present a general abstract method and then we apply it to the model case of the nonlocal Allen-Cahn Equation (1) with a nonlinear Neumann boundary condition.

Acknowledgement : The authors would like to thank Prof. P. Souganidis for proposing the problem and the first author wishes to thank also Prof. G. Barles for the helpful discussions.

1 Viscosity solutions for nonlocal equations, existence and uniqueness

Let $O \subset \mathbb{R}^N$ be a bounded open set with C^1 boundary and $\mathcal{B}$ the set of all measurable subsets of $\overline{O}$. The topology on $\mathcal{B}$ is generated by the distance $d(A,B) = \lambda(A \triangle B)$, where λ is the Lebesgue measure.

Let F be a real-valued, locally bounded function on $\overline{O} \times [0,\infty) \times \mathbb{R}^N \times \mathcal{S}(N) \times \mathcal{B}$, which is continuous in $\overline{O} \times [0,\infty) \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}(N) \times \mathcal{B}$, where $\mathcal{S}(N)$ is the set of real symmetric $N \times N$ matrices and let G be a real-valued, continuous function on $\partial O \times (0,\infty) \times \mathbb{R}^N$. We consider nonlocal degenerate (and possibly singular) parabolic equations, with nonlinear Neumann-type boundary conditions, of the following form:

$$\begin{cases} (i) & u_t + F(x,t,Du,D^2u,\{y \in \overline{O}: \ u(y,t) \geq u(x,t)\}) = 0 & \text{in } O \times (0,T), \\ (ii) & G(x,t,Du) = 0 & \text{in } \partial O \times (0,T), \\ (iii) & u(x,0) = u_0(x) & \text{in } \overline{O}, \end{cases} \tag{10}$$

Some examples of such operators F are:

$$F(x,t,p,X,K) := -\mathrm{Trace}[(I - \hat{p} \otimes \hat{p})A(x,t,p)X] + p \cdot b(x,t) - c_0|p|(\lambda(K) - \lambda(K^c))$$

with $c_0 > 0$ and

$$F(x,t,p,X,K) := -\mathrm{Trace}[A(x,t,p)X] + H(x,t,p) - |p|\, \beta \left(\int_K \theta(x,t,\hat{p},y)dy \right)$$

where

$$A(x,t,p) = \sigma(x,t,p)\sigma^t(x,t,p)$$

with $\sigma: \overline{\Omega} \times [0,T] \times \mathbb{R}^N \to \mathcal{M}_{N,k}$ (the space of $N \times k$ matrices) is a bounded function, possibly discontinuous at $p = 0$ and locally Lipschitz on $\overline{\Omega} \times [0,T] \times \mathbb{R}^N \setminus \{0\}$ with

$$|D_x\sigma(x,t,p)| \leq C \quad \text{and} \quad |D_p\sigma(x,t,p)| \leq \frac{C}{|p|}$$

for every t, almost every $x \in \Omega$ and $p \in \mathbb{R}^N \setminus \{0\}$. Function b is assumed to be Lipschitz continuous in x and continuous in t, while H is assumed to be locally Lipschitz continuous on $\overline{O} \times [0,T] \times \mathbb{R}^N$, and such that for all t, almost all $x \in \overline{O}$ and $p \in \mathbb{R}^N$

$$|D_xH(x,t,p)| \leq C(1 + |p|) \quad \text{and} \quad |D_pH(x,t,p)| \leq C.$$

Function $\theta \in C(\overline{O} \times [0,T] \times S^{N-1} \times \overline{O}, [0,\infty))$ is assumed to be Lipschitz continuous in x and $\hat{p}$ variable, while β is nondecreasing and Lipschitz continuous.

If these operators are to be geometric (that is to satisfy condition **(A4)** that follows) then we need also to require that A is homogeneous of degree 0 in p and H is homogeneous of degree 1 in p (and also to add $(I - \hat{p} \otimes \hat{p})$ under the trace in the second example).

In (ii) we have typically in mind the following two boundary conditions:

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \partial O \times (0,\infty) \, , \tag{11}$$

where $\gamma : \partial O \times [0,\infty) \to I\!R^N$ is a Lipschitz continuous vector field such that $\gamma(x,t) \cdot n(x) > 0$ on $\partial O \times [0,\infty)$, $n(x)$ being the unit exterior normal to ∂O at x, and the capillarity type boundary condition

$$\frac{\partial u}{\partial n} = \theta(x,t)|Du| \quad \text{on } \partial O \times (0,\infty) \, , \tag{12}$$

where $\theta : \partial O \times [0,\infty) \to I\!R$ is, say, a locally Lipschitz continuous function such that $|\theta(x,t)| < 1$ on $\partial O \times [0,\infty)$.

To define, and prove existence and uniqueness of, viscosity solutions of (10), we follow [Sl] where nonlocal equations with homogeneous Neumann boundary conditions were studied. Using results on (local) parabolic equations with nonlinear Neumann-type boundary conditions by Ishii and Sato [IS] and Barles [B], we extend the results of [Sl] to equations of the form (10).

We now list the requirements on F and G. The assumptions introduced because of the presence of the nonlocal term are the monotonicity with respect to set inclusion and the continuity of F (with respect to topology on $\mathcal{B}$). The precise basic assumptions are:

(A1) The function F is locally bounded on $\overline{O} \times [0,\infty) \times I\!R^N \times \mathcal{S}(N) \times \mathcal{B}$, continuous on $\overline{O} \times [0,\infty) \times I\!R^N \setminus \{0\} \times \mathcal{S}(N) \times \mathcal{B}$, and satisfies the degenerate *ellipticity* condition:

$$F(x,r,p,X,K) \leq F(x,r,p,Y,K) \quad \text{whenever } X \geq Y, \tag{13}$$

where "$\geq$" stands for the usual partial ordering of symmetric matrices.

(A2) F is nonincreasing with respect to its set arguments:

$$F(x,r,p,X,K) \leq F(x,r,p,X,L) \quad \text{whenever } L \subseteq K, \tag{14}$$

(A3) For any $T > 0$, the function G is uniformly continuous on $\partial O \times (0,T) \times I\!R^N$ and there exists a constant $\nu(T) > 0$ such that, for all $\lambda > 0$, $x \in \partial O$, $t \in (0,T]$ and $p \in I\!R^N$,

$$G(x,t,p + \lambda n(x)) - G(x,t,p) \geq \nu(T)\lambda \, . \tag{15}$$

The nonlocal equations we consider are in most instances level-set equations of some geometric evolutions. Such equations fall in the class of *geometric* equations, that is the ones that satisfy the following two conditions:

(A4) For any $\lambda > 0$, $\nu \in I\!R$ and $x \in \overline{O}$, $t \in (0, \infty)$, $p \in I\!R^N \setminus \{0\}$, $X \in \mathcal{S}(N)$, $K \in \mathcal{B}$

$$F(x, t, \lambda p, \lambda X + \nu p \otimes p, K) = \lambda F(x, t, p, X, K), \tag{16}$$

where $p \otimes p$ denotes the symmetric matrix defined by $(p \otimes p)_{ij} = p_i p_j$, for all $1 \leq i, j \leq N$.

(A5) For all $\lambda > 0$, $x \in \partial O$, $t \in (0, \infty)$ and $p \in I\!R^N$

$$G(x, t, \lambda p) = \lambda G(x, t, p). \tag{17}$$

To be able to show the uniqueness and existence of solutions we need additional assumptions. Since we are able to extend both the results of [B] and [IS], the conditions required in either of the papers (for local equations) are sufficient. We denote these assumptions by **(A6)** and list them in the Appendix A.

We will show that assumptions **(A1)-(A6)** imply the existence and uniqueness of solutions of (10) with continuous initial data. In the remainder of the paper we will refer to any set of assumptions that includes **(A4)** and **(A5)** and implies existence and uniqueness of (10) for all $u_0 \in C(\overline{O})$ as "the assumptions of the level-set approach".

We recall that if $f : A \to I\!R$, where A is a subset of some $I\!R^k$, the upper- and lower-semicontinuous envelopes f^* and f_* of f are given by

$$f^*(y) = \limsup_{z \to y} f(z) \quad \text{and} \quad f_*(y) = \liminf_{z \to y} f(z).$$

We define the viscosity solutions to the problem (10), analogously to [Sl].

Definition 1.1 *An upper-semicontinuous function $u : \overline{O} \times [0, T) \longrightarrow I\!R \cup \{-\infty\}$ is a viscosity subsolution of (10) if for all $x \in \overline{O}$, $u(x, 0) \leq u_0^*(x)$ and for all $(x, t) \in \overline{O} \times (0, T)$ and all functions $\varphi \in C^\infty(\overline{O} \times (0, T))$ such that $u - \varphi$ has maximum at (x, t), if $x \in O$ or $x \in \partial O$ and $G(x, t, D\varphi(x, t)) > 0$ then*

$$\varphi_t(x, t) + F_*(x, t, D\varphi(x, t), D^2\varphi(x, t), \{y : u(y, t) \geq u(x, t)\}) \leq 0. \tag{18}$$

A lower-semicontinuous function $v : \overline{O} \times [0, T) \longrightarrow I\!R \cup \{\infty\}$ is a viscosity supersolution of (10) if for all $x \in \overline{O}$, $v(x, 0) \geq u_{0*}(x)$ and for all $(x, t) \in \overline{O} \times (0, T)$ and all functions $\varphi \in C^\infty(\overline{O} \times (0, T))$ such that $v - \varphi$ has minimum at (x, t), if $x \in O$ or $x \in \partial O$ and $G(x, t, D\varphi(x, t)) < 0$ then

$$\varphi_t(x, t) + F^*(x, t, D\varphi(x, t), D^2\varphi(x, t), \{y : u(y, t) > u(x, t)\}) \geq 0. \tag{19}$$

A function $u : \overline{O} \times [0,T) \longrightarrow I\!R$ is a viscosity solution of (10) if u^* is a subsolution and u_* is a supersolution.

We remark that using different test sets in the definition of sub and supersolutions is necessary for having the desired properties of viscosity solutions (in particular stability and existence).

Viscosity sub and supersolutions defined as above, have the following properties:

(P1) *Stability*: If $\{u_n\}_{n=1,2,..}$ is a sequence of subsolutions (resp. supersolutions) of (10 *i,ii*) bounded from above (resp. below) then $u = \limsup^* u_n$ is also a subsolution (resp. $u = \liminf_* u_n$ is a supersolution).

We recall the half-relaxed limits of a sequence of functions $u_n \colon \overline{O} \times [0,T] \to I\!R$ are defined by

$$\limsup{}^* \; u_n(x,t) := \limsup_{\substack{(y,s)\to(x,t)\\ n\to\infty}} u_n(y,s) \text{ and } \liminf{}_* \; u_n(x,t) := \liminf_{\substack{(y,s)\to(x,t)\\ n\to\infty}} u_n(y,s).$$

(P2) If u is a subsolution (resp. supersolution) of (10 *i,ii*) and $\rho : I\!R \longrightarrow I\!R$ nondecreasing then $(\rho \circ u)^*$ is also a subsolution (resp. $(\rho \circ u)_*$ is a supersolution).

Proofs of these properties can be found in [Sl]. Here we just remark that the second property is a consequence of the geometric nature of the equations (assumptions **(A4)** and **(A5)**), and just means that the equation is invariant under relabelings of level sets that preserve inclusion. The proofs of comparison and existence also make use of the following fact about stability of level sets of semicontinuous functions.

Lemma 1.1 *Let $f : \overline{O} \times (a,b) \longrightarrow I\!R^N$ be a lower (resp. upper) semicontinuous function. For every $x \in \overline{O}$, $t \in (a,b)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\lambda(\{f(\,\cdot\,,t) > f(x,t))\} \;\setminus\; \{f(\,\cdot\,,s) > f(x,t)+\delta\}) < \varepsilon$$
$$(\textit{resp.} \quad \lambda(\{f(\,\cdot\,,s) \geq f(x,t)-\delta\} \;\setminus\; \{f(\,\cdot\,,t) \geq f(x,t)\}) < \varepsilon\ )$$

for $s \in (t-\delta, t+\delta)$.

Proof. As the two claims are proven analogously, let us show only the first one. We can assume that $f(x,t) = 0$. Assume that for some $\varepsilon > 0$ and $t \in (a,b)$ no appropriate δ can be found. Then there exist sequences δ_n converging to 0 and s_n converging to t as n goes to infinity such that for all $n \in I\!N$

$$\lambda(\{f(\,\cdot\,,t) > 0)\} \setminus \{f(\,\cdot\,,s_n) > \delta_n\}) \geq \varepsilon$$

Since f is lower semicontinuous,

$$\{f(\,\cdot\,,t)>0\}\subseteq \liminf_{n\to\infty}\{f(\,\cdot\,,s_n)>\delta_n\}=\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}\{f(\,\cdot\,,s_i)>\delta_i\}.$$

Note that there exists n_0 such that for all $n>n_0$

$$\lambda\left(\bigcup_{k=1}^{\infty}\bigcap_{i=k}^{\infty}\{f(\,\cdot\,,s_i)>\delta_i\}\setminus\bigcap_{i=n}^{\infty}\{f(\,\cdot\,,s_i)>\delta_i\}\right)<\varepsilon.$$

So for $n>n_0$

$$\lambda\left(\{f(\,\cdot\,,t)>0\}\setminus\{f(\,\cdot\,,s_n)>\delta_n\}\right)<\varepsilon.$$

Contradiction. $\square$

The main result of this Section is the following Theorem in which we prove a comparison result between viscosity sub- and supersolutions of (10). Although the equations we have in mind are geometric the comparison holds for more general equations, and we indicate that below.

Theorem 1.1 (Comparison). *Assume* **(A1)**-**(A3)**, **(A5)** *and* **(A6)**. *Let u,v be respectively a viscosity sub- and supersolution of (10). If $u(x,0)\le v(x,0)$ for all $x\in\overline{O}$, then $u(x,t)\le v(x,t)$ for all $x\in\overline{O}$ and all $t\in[0,T)$.*

Proof. Under the set of assumptions that includes **(A6a)** the proof of comparison given in [IS] extends, following [Sl], to nonlocal equations without any difficulties.

However, there are some technical difficulties if **(A6b)** is satisfied but nevertheless enough control on the nonlocal term can be obtained so that the test function of [B], as well as techniques of [B], can be used. For simplicity we present the proof assuming that the boundary condition G is independent of time. Handling of the nonlocal term is still the same if G depends on time.

Let u be a subsolution and v a supersolution of (10). Assume that the comparison principle does not hold, that is, that for some $(x',t')\in\overline{O}\times(0,T)$, $u(x',t')>v(x',t')$. Let τ be a number in (t',T). Choose $\gamma>0$ so that $u(x',t')-v(x',t')-\dfrac{\gamma}{\tau-t'}>0$. Let (x_0,t_0) be a point where $u(x,t)-v(x,t)-\dfrac{\gamma}{\tau-t}$ reaches its maximum on $\overline{O}\times[0,\tau)$.

For $\varepsilon>0$, $\eta>0$, and $\alpha>0$, consider the function

$$w_{\eta,\varepsilon,\alpha}(x,y,t):=u(x,t)-v(y,t)-\zeta_{\eta,\varepsilon,\alpha}(x,y)-\frac{\gamma}{\tau-t}$$

where $\zeta_{\varepsilon,\eta,\alpha}$ is the test function constructed by Barles in [B]. The test function is rather complicated, so we only present its form, and list its properties:

$$\zeta_{\eta,\varepsilon,\alpha}(x,y):=[(\psi_{\eta,\varepsilon}(x,y))^+]^6-\alpha(\tilde{d}(x)+\tilde{d}(y))$$

where $\tilde{d} \in W^{3,\infty}(\overline{O}, [0,1])$ coincides with the distance to ∂O in a neighborhood of ∂O, and $\psi_{\eta,\varepsilon}$ is a C^2 function defined in the Appendix of [B]. It satisfies the following inequalities for ε and η small enough:

$$-C\eta\varepsilon + \frac{|x-y|^2}{C\varepsilon^2} \leq \psi_{\eta,\varepsilon}(x,y) \leq C\eta\varepsilon + C\frac{|x-y|^2}{\varepsilon^2} \tag{20}$$

for some constant C independent of η and ε. Moreover, for any $x, y \in \overline{O}$ such that $|x-y| < \eta\varepsilon$,

$$\begin{array}{rcl} -C\eta\varepsilon + \dfrac{1}{C}\dfrac{|x-y|}{\varepsilon^2} & \leq & |D_x\psi_{\eta,\varepsilon}(x,y)|, \\ |D_y\psi_{\eta,\varepsilon}(x,y)| & \leq & C\dfrac{|x-y|}{\varepsilon^2} + C\eta\varepsilon, \\ |D_x\psi_{\eta,\varepsilon}(x,y) + D_y\psi_{\eta,\varepsilon}(x,y)| & \leq & C\dfrac{|x-y|^2}{\varepsilon^2} + C\eta\varepsilon, \\ -\dfrac{C}{\varepsilon^2}I \;\leq\; D^2\psi_{\eta,\varepsilon}(x,y) & \leq & \dfrac{C}{\varepsilon^2}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + C\eta I, \\ G(x, D_x\psi_{\eta,\varepsilon}(x,y)) & > & 0 \quad \text{if } x \in \partial O, \\ G(y, D_y\psi_{\eta,\varepsilon}(x,y)) & < & 0 \quad \text{if } y \in \partial O. \end{array} \tag{21}$$

Let $(\bar{x}_{\eta,\varepsilon,\alpha}, \bar{y}_{\eta,\varepsilon,\alpha}, \bar{t}_{\eta,\varepsilon,\alpha})$ be the point where $w_{\eta,\varepsilon,\alpha}$ reaches its maximum on $\overline{O} \times \overline{O} \times [0,\tau)$. For the sake of simplicity of notations, from now on we drop the indexes were possible. Standard argument shows that $|\bar{x}-\bar{y}|^2/\varepsilon^2 \to 0$ as $\varepsilon \to 0$ and that for $\varepsilon, \eta, \alpha$ small enough $\bar{t}_{\eta,\varepsilon,\alpha} > 0$. By considering ε small enough (depending on η) we can assume that $|\bar{x}-\bar{y}| \leq \eta\varepsilon$ so that the properties (21) hold for $(x,y) = (\bar{x},\bar{y})$. By parabolic Crandall-Ishii Lemma [CIL, Lemma 8.3] it follows that for any $\delta > 0$ there exist numbers $\bar{a}, \bar{b}$ and symmetric matrices $\bar{X}, \bar{Y}$ such that

$$\begin{array}{rcl} (\bar{a}, \bar{p}, \bar{X}) & \in & \overline{P}^{2,+}u(\bar{x},\bar{t}) \\ (\bar{b}, \bar{q}, \bar{Y}) & \in & \overline{P}^{2,-}v(\bar{y},\bar{t}) \\ \bar{a} - \bar{b} & = & \dfrac{\gamma}{(\tau-\bar{t})^2} \\ -\left(\dfrac{1}{\delta} + \|D^2\zeta_{\eta,\varepsilon,\alpha}(\bar{x},\bar{y})\|\right) I & \leq & \begin{pmatrix} \bar{X} & 0 \\ 0 & -\bar{Y} \end{pmatrix} \leq \left(I + \delta D^2\zeta_{\eta,\varepsilon,\alpha}(\bar{x},\bar{y})\right) D^2\zeta_{\eta,\varepsilon,\alpha}(\bar{x},\bar{y}) \end{array} \tag{22}$$

where $\bar{p} = D_x\zeta_{\eta,\varepsilon,\alpha}(\bar{x},\bar{y})$ and $\bar{q} = -D_y\zeta_{\eta,\varepsilon,\alpha}(\bar{x},\bar{y})$. The properties of $\zeta_{\eta,\varepsilon,\alpha}$ insure that

$$G(\bar{x},\bar{p}) > 0 \quad \text{if } \bar{x} \in \overline{O} \quad \text{and} \quad G(\bar{y},q) < 0 \quad \text{if } \bar{y} \in \overline{O}$$

Therefore

$$\begin{array}{rrcl} & \bar{a} + F_*(\bar{x},\bar{t},\bar{p},\bar{X},\{u(\,\cdot\,,\bar{t}) \geq u(\bar{x},\bar{t})\}) & \leq & 0 \\ \text{and} & \bar{b} + F^*(\bar{y},\bar{t},\bar{q},\bar{Y},\{v(\,\cdot\,,\bar{t}) > v(\bar{y},\bar{t})\}) & \geq & 0. \end{array} \tag{23}$$

Since norms of $\bar{X}_\alpha$, $\bar{Y}_\alpha$, $\bar{p}_\alpha$ and $\bar{q}_\alpha$ have bounds independent of α, they, as well as $\bar{x}_\alpha$, $\bar{y}_\alpha$, and $\bar{t}_\alpha$ converge along a subsequence as $\alpha \to 0$. Let us denote again the limiting quantities by $\bar{X}, \bar{Y}, \bar{p}, \bar{q}, \bar{x}, \bar{y}$, and $\bar{t}$. Note that $u(\bar{x}_\alpha, \bar{y}_\alpha) \to u(\bar{x}, \bar{t})$ and $v(\bar{y}_\alpha, \bar{t}_\alpha) \to v(\bar{y}, \bar{t})$ as $\alpha \to 0$. along the same sequence. Otherwise $w_\alpha(\bar{x}, \bar{y}, \bar{t}) > w_\alpha(\bar{x}_\alpha, \bar{y}_\alpha, \bar{t}_\alpha)$ for α small enough, which would contradict the fact that w_α reaches its maximum at $(\bar{x}_\alpha, \bar{y}_\alpha, \bar{t}_\alpha)$.

Lemma 1.1 now implies that

$$\lim_{\alpha \to 0} \lambda(\{u(\,\cdot\,, \bar{t}_\alpha) \geq u(\bar{x}_\alpha, \bar{t}_\alpha)\} \backslash \{u(\,\cdot\,, \bar{t}) \geq u(\bar{x}, \bar{t})\}) = 0,$$

$$\lim_{\alpha \to 0} \lambda(\{v(\,\cdot\,, \bar{t}) > v(\bar{y}, \bar{t})\} \backslash \{v(\,\cdot\,, \bar{t}_\alpha) > v(\bar{y}_\alpha, \bar{t}_\alpha)\}) = 0$$

where the limit is taken along the sequence. Therefore since F_* is lower and F^* is upper semicontinuous, and both are monotone in the set valued variable, (22) and (23) hold for new quantities with $\alpha = 0$ and $w_{\eta,\varepsilon,0}$ has a maximum at $(\bar{x}, \bar{y}, \bar{t})$.

Therefore for all $x \in \overline{O}$

$$u(x, \bar{t}) - v(x, \bar{t}) \leq u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}) - [(\psi_{\eta,\varepsilon}(\bar{x}, \bar{y}))^+]^6 + [(\psi_{\eta,\varepsilon}(x, x))^+]^6 \,.$$

Let us first consider the case that for all η and ε small enough

$$[(\psi_{\eta,\varepsilon}(\bar{x}, \bar{y}))^+]^6 - [(\psi_{\eta,\varepsilon}(x, x))^+]^6 > 0 \quad \text{for all } x \in \overline{O}.$$

Then $\{u(\,\cdot\,, \bar{t}) \geq u(\bar{x}, \bar{t})\} \subseteq \{v(\,\cdot\,, \bar{t}) > v(\bar{y}, \bar{t})\}$ which gives us the desired control on the nonlocal term. Obtaining a contradiction from (23) and (H5-2) follows a classical argument, and we refer the reader to [B] for details. Now consider the case that there is a sequence of η and ε converging to zero, such that there always exists $x \in \overline{O}$ so that $[(\psi_{\eta,\varepsilon}(\bar{x}, \bar{y}))^+]^6 \leq [(\psi_{\eta,\varepsilon}(x, x))^+]^6$. Then from (20) it follows that

$$\frac{|\bar{x} - \bar{y}|^2}{C\varepsilon^2} - C\eta\varepsilon \leq \psi_{\eta,\varepsilon}(\bar{x}, \bar{y}) \leq \psi_{\eta,\varepsilon}(x, x) \leq C\eta\varepsilon.$$

Therefore $|\bar{x} - \bar{y}| < 2C\varepsilon\sqrt{\eta\varepsilon}$. The estimates on derivatives of $\psi_{\eta,\varepsilon}$ that we listed, then yield that $|D_x \psi_{\eta,\varepsilon}(\bar{x}, \bar{y})|$ and $|D_y \psi_{\eta,\varepsilon}(\bar{x}, \bar{y})|$ are $O(\varepsilon^{-1/2})$ and $\|D^2 \psi_{\eta,\varepsilon}\|$ is $O(\varepsilon^{-2})$.

Using that $\zeta_{\eta,\varepsilon,0}(x, y) = [(\psi_{\eta,\varepsilon}(x, y))^+]^6$ and that $\psi_{\eta,\varepsilon}(\bar{x}, \bar{y}) = O(\varepsilon)$, along with (22) (with $\delta = \|D^2 \zeta_{\eta,\varepsilon,\alpha}(\bar{x}, \bar{y})\|^{-1}$ for example) implies that $\bar{X}$, $\bar{Y}$, $\bar{p}$, and $\bar{q}$ converge to zero as ε and η converge to zero along the aforementioned sequence. Passing to a subsequence, we can assume that $\bar{x}$, $\bar{y}$, and $\bar{t}$ also converge as ε and η go to 0. We use the same notation for the limits. From (23) then follows, by the semicontinuity of F_* and F^* and their monotonicity in the set-valued argument, that

$$\bar{a} + F_*(\bar{x}, \bar{t}, 0, 0, \overline{O}) \leq 0 \;\text{ and }\; \bar{b} + F^*(\bar{y}, \bar{t}, 0, 0, \emptyset) \geq 0.$$

The fact that $\overline{a} > \overline{b}$ combined with (65) leads to contradiction. $\square$

The existence of a viscosity solution to (10) is obtained via the Perron's method. Although the application of Perron's Method is rather standard we provide a proof for completeness and to outline the difficulties posed by the presence of nonlocal term.

Theorem 1.2 (Existence). *Let F and G be functions satisfying the conditions* **(A1)-(A3)**, **(A5)** *and* **(A6)** *and let u_0 be a continuous function. Then the problem (10) has a unique continuous viscosity solution.*

Proof. Let

$$u(x,t) := \sup\{w(x,t) \ : \ w \text{ a subsolution of } (10)\} \tag{24}$$

We claim that u is the solution to problem (10). Showing that $u^*(x,0) = u_0(x) = u_*(x,0)$ on $\overline{O}$ requires construction of appropriate sub- and supersolutions (barriers), which because of nonlinear boundary conditions poses some difficulties. However the existing constructions given by Ishii and Sato [IS] and mentioned by Barles [B], extend with minimal modifications to equations with nonlocal terms, so we omit them.

Note that, u^* is then, by stability, a viscosity subsolution of (10). The definition of u then implies that $u = u^*$. Let us show that u_* is a supersolution. Assume it is not. Then there exists a smooth function φ on $\overline{O} \times (0,T)$ such that $u_* - \varphi$ has a minimum at (x_0, t_0) and

$$\varphi_t(x_0,t_0) + F^*(x_0,t_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0),\{y \ : \ u_*(y,t_0) > u_*(x_0,t_0)\}) < -2\varepsilon$$

for some $\varepsilon > 0$ (and $G(x_0,t_0,D\varphi(x_0,t_0) < -2\varepsilon$ if $x_0 \in \partial O$). We can assume that $\varphi(x_0,t_0) = u_*(x_0,t_0) = 0$. Consider the function

$$\tilde{u}(x,t) = \max\{u(x,t), \varphi^\delta(x,t)\}$$

where $\varphi^\delta(x,t) := \varphi(x,t) + \delta - |x-x_0|^4 - |t-t_0|^4$ and $\delta > 0$ is to be determined. Since u_* and F_* are lower semicontinuous, using Lemma 1.1, there exists a positive $r < 1$ such that if $(x,t) \in B((x_0,t_0),r)$ then $t_0/2 < t < T - t_0/2$, $|\varphi_t^\delta(x,t) - \varphi_t(x_0,t_0)| < \varepsilon$ and

$$F_*(x,t,D\varphi^\delta,D^2\varphi^\delta,\{u_*(\,\cdot\,,t) > 2Mr\}) < F_*(x_0,t_0,D\varphi,D^2\varphi,\{u_*(\,\cdot\,,t_0) > 0\}) + \varepsilon$$

where $M := \|\varphi\|_{C^1(U)} + 1$ with $U := \overline{O} \times [t_0/2, (T+t_0)/2]$. By making r smaller if necessary we also require that if $x_0 \in O$ then $r <$dist$(0, \partial O)$ and if $x_0 \in \partial O$ then $G(x,t,D\varphi^\delta) < G(x_0,t_0,D\varphi) + \varepsilon$ on $B((x_0,t_0),r)$. Now take any positive $\delta < r^4/2$. Let us show now that $\tilde{u}$ is a subsolution of (10). Let ψ be a smooth function such that $\tilde{u}-\psi$ has a maximum at $(\tilde{x},\tilde{t})$. If $\tilde{u}(\tilde{x},\tilde{t})=u(\tilde{x},\tilde{t})$ then $u-\psi$ has a maximum at $(\tilde{x},\tilde{t})$ and since u is a subsolution

$$\begin{aligned}&\psi_t+F_*(\tilde{x},\tilde{t},D\psi,D^2\psi,\{\tilde{u}(\,\cdot\,,\tilde{t})\geq\tilde{u}(\tilde{x},\tilde{t})\})\\&\leq\psi_t+F_*(\tilde{x},\tilde{t},D\psi,D^2\psi,\{u(\,\cdot\,,\tilde{t})\geq u(\tilde{x},\tilde{t})\})\leq 0\end{aligned}$$

(or $G(\tilde{x},\tilde{t},D\psi)\leq 0$ if $\tilde{x}\in\partial O$).

If $\tilde{u}(\tilde{x},\tilde{t})>u(\tilde{x},\tilde{t})$ then $\tilde{u}(x,t)=\varphi^\delta(x,t)$ near $(\tilde{x},\tilde{t})$. Since $\delta<r^4/2$, $\delta-|x-x_0|^4-|t-t_0|^4<0$ when $(x,t)\notin B((x_0,t_0),r)$. Using that $\tilde{u}(\tilde{x},\tilde{t})>u(\tilde{x},\tilde{t})\geq\varphi(\tilde{x},\tilde{t})$ we conclude that $(\tilde{x},\tilde{t})\in B((x_0,t_0),r)$. If $\tilde{x}\in O$ then using that $\varphi^\delta(\tilde{x},\tilde{t})\leq\delta+M(|(\tilde{x}-x_0,\tilde{t}-t_0)|<r^4+Mr<2Mr$ one obtains

$$\begin{aligned}\psi_t\ &+\ F_*(\tilde{x},\tilde{t},D\psi,D^2\psi,\{\tilde{u}(\,\cdot\,,\tilde{t})\geq\tilde{u}(\tilde{x},\tilde{t})\})\\&\leq\ \varphi_t^\delta(\tilde{x},\tilde{t})+F_*(\tilde{x},\tilde{t},D\varphi^\delta,D^2\varphi^\delta,\{\tilde{u}(\,\cdot\,,\tilde{t})\geq\varphi^\delta(\tilde{x},\tilde{t})\})\\&\leq\ \varphi_t(x_0,t_0)+\varepsilon+F_*(\tilde{x},\tilde{t},D\varphi^\delta,D^2\varphi^\delta,\{u_*(\,\cdot\,,t)>2Mr\})\\&\leq\ \varphi_t(x_0,t_0)+\varepsilon+F_*(x_0,t_0,D\varphi,D^2\varphi,\{u_*(\,\cdot\,,t_0)>0\})+\varepsilon<0.\end{aligned}$$

If $\tilde{x}\in\partial O$ then $D\psi(\tilde{x},\tilde{t})=D\varphi^\delta(\tilde{x},\tilde{t})-\lambda n$ for some $\lambda\geq 0$. Here n is the unit outside normal to ∂O at $\tilde{x}$. Therefore using property **(A3)**

$$G(\tilde{x},\tilde{t},D\psi(\tilde{x},\tilde{t}))\leq G(x_0,t_0,D\varphi(0,t_0))+\varepsilon<0.$$

Also note that $\tilde{u}(\,\cdot\,,0)\leq u_0(\,\cdot\,)$ on $\overline{O}$ since $r<t_0$. Together with inequalities above, that implies that $\tilde{u}$ is a subsolution of (10). Note that since $u_*(x_0,t_0)=0$, there exists a sequence (x_n,t_n) converging to (x_0,t_0) such that $u(x_n,t_n)$ converges to 0 as n goes to infinity. But then for n large enough $\tilde{u}(x_n,t_n)>u(x_n,t_n)$, which contradicts the definition of u, since $\tilde{u}$ is a subsolution.

Therefore u_* is a supersolution. The comparison result in Theorem 1.1 implies that $u\leq u_*$, and hence $u=u_*$. Therefore u is a continuous solution of (10). The uniqueness of the solution follows again from Theorem 1.1. □

2 A generalized definition for nonlocal motion and its connections with the level-set approach

In this section we consider the initial-boundary value problem (10) with F and G satisfying the "assumptions of the level set approach". Our aim is to extend the geometrical approach to the weak motion of hypersurfaces in bounded domains with an angle contact boundary condition introduced in [BDL] to the case of nonlocal normal velocity depending not only on the normal direction and the curvature but also on the volume of the set the fronts enclose. Moreover we show its connections with the level-set approach. We first briefly recall the basic ideas of the level-set approach connected to initial boundary value problem (10).

The level-set approach for problems associated with Neumann type boundary conditions (see e.g. [B, GS, IS]) can be described in a similar way to the $I\!R^N$ case (see e.g. [CGG, ES]). Let $\mathcal{E}$ be the collection of triplets (Γ, D^+, D^-) of mutually disjoint subsets of $\overline{O}$ such that Γ is closed and $D^\pm$ is open and $\overline{O} = \Gamma \cup D^+ \cup D^-$. For any $(\Gamma_0, D_0^+, D_0^-) \in \mathcal{E}$, first choose $u_0 \in C(\overline{O})$ so that

$$D_0^+ = \{x \in \overline{O}\ : u_0(x) > 0\},\ D_0^- = \{x \in \overline{O}\ : u_0(x) < 0\} \ \text{ and } \ \Gamma_0 = \{x \in \overline{O}\ : u_0(x) = 0\},$$

By results of Section 1, for every $T > 0$ and $u_0 \in C(\overline{O})$, there exists a unique viscosity solution u of (10) in $C(\overline{O}\times[0,T])$. If, for all $t > 0$, we define $(\Gamma_t, D_t^+, D_t^-) \in \mathcal{E}$ by

$$\Gamma_t = \{x \in \overline{O}\ : u(x,t) = 0\}\ ,\ D_t^+ = \{x \in \overline{O}\ : u(x,t) > 0\}\ ,\ D_t^- = \{x \in \overline{O}\ : u(x,t) < 0\},$$

then, because of **(A4)**, **(A5)** and since a comparison result holds for (10), the collection $\{(\Gamma_t, D_t^+, D_t^-)\}_{t\geq 0}$ is uniquely determined, independently of the choice of u_0, by the initial triplet (Γ_0, D_0^+, D_0^-).

The properties of the generalized level set evolution have been the object of extended study, at least in $I\!R^N$. One of the most intriguing issues – rather important in the study of the asymptotics of reaction-diffusion equations – is whether the so-called *fattening phenomena* occurs or not, i.e. whether the set Γ_t develops an interior or not. Following the $I\!R^N$–case, we say that the *no-interior condition* for $\{(x,t) :\ u(x,t) = 0\}$ holds if and only if

$$\{(x,t)\ : u(x,t) = 0\} = \partial\{(x,t)\ : u(x,t) > 0\} = \partial\{(x,t)\ : u(x,t) < 0\}. \qquad (25)$$

The question when no-interior condition holds is a difficult one. For some conditions on when it holds and examples when it fails (for fronts in $I\!R^N$) see [BSS], [K] and references therein. The importance of the "no-interior condition" and its connection with more geometrical approaches than the level-set one is explained in the following result, proved in $I\!R^N$ in [BSS] and which can be easily extended to the case of nonlocal equations and with nonlinear Neumann boundary conditions. In this result, if A is a subset of some $I\!R^k$, $1\!\!1_A$ denotes the indicator function of A, i.e., $1\!\!1_A(x) = 1$ if $x \in A$ and $1\!\!1_A(x) = 0$ if $x \in A^c$.

Theorem 2.1 *Under the assumptions of the level-set approach, the functions* $1\!\!1_{D_t^+\cup\Gamma_t} - 1\!\!1_{D_t^-}$ *and* $1\!\!1_{D_t^+} - 1\!\!1_{D_t^-\cup\Gamma_t}$ *are respectively the maximal subsolution (and solution) and the minimal supersolution (and solution) of* (10) *associated respectively with the initial data* $u_0 = 1\!\!1_{D_0^+\cup\Gamma_0} - 1\!\!1_{D_0^-}$ *and* $u_0 = 1\!\!1_{D_0^+} - 1\!\!1_{D_0^-\cup\Gamma_0}$. *Moreover, if* Γ_0 *has an empty* *interior,* $\mathbb{1}_{D_t^+} - \mathbb{1}_{D_t^-}$ *is the unique discontinuous solution of* (10) *associated with the initial data* $u_0 = \mathbb{1}_{D_0^+} - \mathbb{1}_{D_0^-}$ *if and only if the property* (25) *holds .*

In fact the main consequence of Theorem 2.1 is that, if (25) holds, the problem is well-posed geometrically since the evolution of the indicator function is uniquely determined.

Now we turn to the geometrical definition. To do so and to simplify the presentation, we have to introduce some notations.

If A is a subset of some $\mathbb{R}^k$, we denote by $\mathrm{Int}(A)$ the interior of A and if $x \in A$ and $r > 0$, we set $B_A(x,r) := B(x,r) \cap A$ (the open ball in the topology of A), $B_A^c(x,r) := B^c(x,r) \cap A$, $\overline{B}_A(x,r) := \overline{B}(x,r) \cap A$ (the closed ball in the topology of A) and $\partial B_A(x,r) := \partial B(x,r) \cap A$.

In the sequel we denote by $(\Omega_t)_{t \in (0,T)}$ a family of open subsets of $\overline{O}$ and we set $\Gamma_t = \partial\Omega_t$. The signed-distance function $d(x,t)$ from x to Γ_t defined by

$$d(x,t) = \begin{cases} d(x,\Gamma_t) & \text{if } x \in \Omega_t, \\ -d(x,\Gamma_t) & \text{otherwise,} \end{cases}$$

where $d(x,\Gamma_t)$ denotes the usual nonnegative distance from $x \in \mathbb{R}^N$ to Γ_t. If Γ_t is a smooth hypersurface, then d is a smooth function in a neighborhood of Γ_t, and for $x \in \Gamma_t$, $n(x,t) = -Dd(x,t)$ is the unit normal to Γ_t pointing away from Ω_t.

Hereafter we consider "geometric" operators F of the form

$$F(x,t,p,X,K) := \overline{F}(x,t,p,X,\lambda(K)) \tag{26}$$

where $\overline{F}$ is a real-valued locally bounded function on $\overline{O} \times [0,T] \times \mathbb{R}^N \times \mathcal{S}(N) \times \mathbb{R}$ satisfying the assumptions of the level set approach and being decreasing with respect to the last variable.

One of the main examples we have in mind is

$$F(x,t,p,X,K) := -\mathrm{Trace}(I - \hat{p} \otimes \hat{p})X - c_0|p|(\lambda(K) - \lambda(K^c))$$

with $c_0 > 0$.

We remark that all the results of this Section hold also in the case of nonlocal operators depending on any measure which is absolutely continuous with respect to the Lebesgue one.

We premise the following result which motivates the definition of generalized super- and sub-flow with nonlocal normal velocity $-F$ and angle boundary condition G we will give later.

Theorem 2.2 *Suppose that the assumptions of the level set approach hold.*

(i) Let $(\Omega_t)_{t\in(0,T)}$ be a family of open subsets of $\overline{O}$ such that the set $\Omega := \cup_{t>0}\Omega_t \times \{t\}$ is open in $\overline{O}\times(0,T)$. Then the function $\chi = 1\!\!1_{\Omega} - 1\!\!1_{\Omega^c}$ is a viscosity supersolution of $(10)(i)-(ii)$ iff for all smooth functions $\theta\colon [0,T]\to I\!\!R$ such that $\theta(t)\le\lambda(\Omega_t)$, for all $t\in(0,T)$ and for all $\alpha>0$, χ is a supersolution of

$$\begin{cases} (i) & u_t + \overline{F}(x,t,Du,D^2u,\theta(t)-\alpha) = 0 \quad \text{in } O\times(0,T), \\ (ii) & G(x,t,Du) = 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \text{in } \partial O\times(0,T), \end{cases} \tag{27}$$

(ii) Let $(\mathcal{F}_t)_{t\in(0,T)}$ be a family of closed subsets of $\overline{O}$ such that the set $\mathcal{F} := \cup_{t>0}\mathcal{F}_t\times\{t\}$ is closed in $\overline{O}\times(0,T)$. Then $\overline{\chi} = 1\!\!1_{\mathcal{F}} - 1\!\!1_{\mathcal{F}^c}$ is a viscosity subsolution of $(10)(i)-(ii)$ iff for all smooth functions $\theta\colon[0,T]\to I\!\!R$ such that $\lambda(\mathcal{F}_t)\le\theta(t)$ for all $t\in(0,T)$ and for all $\alpha>0$, $\overline{\chi}$ is a subsolution of

$$\begin{cases} (i) & u_t + \overline{F}(x,t,Du,D^2u,\theta(t)+\alpha) = 0 \quad \text{in } O\times(0,T), \\ (ii) & G(x,t,Du) = 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \text{in } \partial O\times(0,T). \end{cases} \tag{28}$$

Remark 2.1 One can show that for any family $(\Omega_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) of open (resp. closed) subsets of $\overline{O}$, the function $\chi = 1\!\!1_{\cup_t\Omega_t\times\{t\}} - 1\!\!1_{(\cup_t\Omega_t\times\{t\})^c}$ (resp. $\overline{\chi} = 1\!\!1_{\cup_t\mathcal{F}_t\times\{t\}} - 1\!\!1_{(\cup_t\mathcal{F}_t\times\{t\})^c}$) is lower (resp. upper) semicontinuous if and only if

$$\Omega_t \subseteq \bigcup_{\varepsilon>0}\liminf_{s\to t}\,(\Omega_s-\varepsilon) \qquad (\text{resp. } \bigcap_{\varepsilon>0}\limsup_{s\to t}\,(\mathcal{F}_s+\varepsilon)\subseteq\mathcal{F}_t)$$

where

$$\liminf_{s\to t}\Omega_s := \bigcup_{\delta>0}\bigcap_{0<|s-t|\le\delta}\Omega_s \qquad (\text{resp. } \limsup_{s\to t}\mathcal{F}_s := \bigcap_{\delta>0}\bigcup_{0<|s-t|\le\delta}\mathcal{F}_s).$$

Thus if $(\Omega_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) satisfies the hypotheses of Theorem 2.2 then the map $t\to\lambda(\Omega_t)$ (resp. $t\to\lambda(\mathcal{F}_t)$) is lower semicontinuous (resp. upper semicontinuous).

Proof of Theorem 2.2. We only prove (i), the case (ii) being analogous. We first assume that χ is a viscosity supersolution of (10)(i)-(ii). Let $(x_0,t_0)\in\overline{O}\times(0,T)$ be a strict global minimum point of $\chi-\phi$ where $\phi\in C^\infty(\overline{O}\times[0,T])$. We have to show that for all smooth functions $\theta\colon[0,T]\to I\!\!R$, such that $\theta(t)\le\lambda(\Omega_t)$ for every $t\in(0,T)$ and for all $\alpha>0$ we have

$$\frac{\partial\phi}{\partial t}(x_0,t_0) + F^*(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0),\theta(t_0)-\alpha)\ge 0.$$

This inequality is certainly true if (x_0,t_0) is in the interior of either the set $\{\chi=1\}$ or the set $\{\chi=-1\}$ since in these two cases χ is constant in a neighborhood of (x_0,t_0).

Hence $\dfrac{\partial\phi}{\partial t}(x_0,t_0)=0$, $D\phi(x_0,t_0)=0$, $D^2\phi(x_0,t_0)\le 0$ and $F^*(x_0,t_0,0,0,\theta(t_0)-\alpha)=0$. Assume that $(x_0,t_0)\in\partial\{\chi=1\}\cap\partial\{\chi=-1\})$. The lower semicontinuity of χ yields $\chi(x_0,t_0)=-1$. In this case the inequality follows directly from **(A2)**.

Conversely suppose that for all smooth functions $\theta\colon[0,T]\to I\!\!R$, such that $\theta(t)\le\lambda(\Omega_t)$ for every $t\in(0,T)$ and for all $\alpha>0$, χ is a supersolution of (27). Let $(x_0,t_0)\in\overline{O}\times(0,T)$ be a strict global minimum point of $\chi-\phi$ where $\phi\in C^\infty(\overline{O}\times[0,T])$. We consider again only the case $(x_0,t_0)\in O\times(0,T)$. We have to show the inequality

$$\frac{\partial\phi}{\partial t}(x_0,t_0)+F^*(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0),\{y:\ \chi(y,t_0)>\chi(x,t_0)\})\ge 0.$$

This inequality is obvious if (x_0,t_0) is in the interior of either the set $\{\chi=1\}$ or the set $\{\chi=-1\}$ for the reasons above. Assume that $(x_0,t_0)\in\partial\{\chi=1\}\cap\partial\{\chi=-1\})$ and suppose by contradiction that, for some $\gamma>0$, we have

$$\frac{\partial\phi}{\partial t}(x_0,t_0)+F^*(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0),\{z:\ \chi(z,t_0)>-1\})<-\gamma$$

and hence

$$\frac{\partial\phi}{\partial t}(x_0,t_0)+\overline{F}^*(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0),\lambda(\Omega_{t_0}))<-\gamma.$$

Since the function $t\mapsto\lambda(\Omega_t)$ is lower semicontinuous (see Remark 2.1) it is the supremum of a family of smooth functions. Therefore there exists a smooth function $\theta(\cdot)$ satisfying $\theta(t)\le\lambda(\Omega_t)$ and a small positive constant α such that $\lambda(\Omega_{t_0})-\theta(t_0)-\alpha$ is so small that (using the upper semicontinuity of $\overline{F}^*$)

$$\frac{\partial\phi}{\partial t}(x_0,t_0)+\overline{F}^*(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0),\theta(t)-\alpha)<-\frac{3\gamma}{4}.$$

But this contradicts the assumption that χ is a supersolution of (27), which concludes the proof. □

Now we give the definition of generalized super- and sub-flow in bounded domains with the nonlocal normal velocity $-F$ and angle boundary condition G.

Definition 2.1 A family $(\Omega_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) of open (resp. closed) subsets of $\overline{O}$ is called a *generalized super-flow* (resp. *sub-flow*) with normal velocity $-F(x,t,Dd,D^2d,\{d(y,t)>d(x,t)\})$ and angle condition $G(x,t,Dd)$ iff for all smooth functions $\theta\colon[0,T]\to I\!\!R$ such that for all $t\in(0,T)$

$$\theta(t)\le\lambda(\Omega_t)\qquad(\text{resp. }\ \lambda(\mathcal{F}_t)\le\theta(t))$$

and for all $\alpha>0$, $(\Omega_t)_t$ is a*generalized super-flow* (resp. *sub-flow*) in the sense of Definition 1.1 in [BDL] with normal velocity $-\overline{F}(x,t,Dd,D^2d,\theta(t)-\alpha)$ (resp. $-\overline{F}(x,t,Dd,D^2d,\theta(t)+\alpha)$) and angle condition $G(x,t,Dd)$.

We denote

$$\overline{F}_-(x,t,Dd,D^2d) := \overline{F}(x,t,Dd,D^2d,\theta(t)-\alpha)$$

and

$$\overline{F}_+(x,t,Dd,D^2d) := \overline{F}(x,t,Dd,D^2d,\theta(t)+\alpha).$$

We recall the definition in [BDL] of a generalized super and sub-flow with normal velocity given respectively by $-\overline{F}_-$ and $-\overline{F}_+$ and angle boundary condition G.

Definition 2.2 A family $(\Omega_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) of open (resp. closed) subsets of $\overline{O}$ is called a *generalized super-flow* (resp. *sub-flow*) with normal velocity $-\overline{F}_-(x,t,Dd,D^2d)$ (resp. $-\overline{F}_+(x,t,Dd,D^2d)$) and angle condition $G(x,t,Dd)$ if and only if, for any $x_0 \in \overline{O}$, $t \in (0,T)$, $r>0$, $h>0$ and for any smooth function $\phi : \overline{O}\times[0,T] \to I\!R$ such that

(i) $\dfrac{\partial\phi}{\partial t} + (\overline{F}_-)^*(y,s,D\phi,D^2\phi) < 0$, (resp. $\dfrac{\partial\phi}{\partial t} + (\overline{F}_+)_*(y,s,D\phi,D^2\phi) > 0$) in $\overline{B}_{\overline{O}}(x_0,r)\times[t,t+h]$,

(ii) $G(y,s,D\phi)<0$ (resp. $G(y,s,D\phi)>0$) in $\partial O \cap \overline{B}(x_0,r)\times[t,t+h]$,

(iii) For any $s\in[t,t+h]$, $\{y \in B(x_0,r)\ :\ \phi(y,s)=0\} \neq \emptyset$ and

$$|D\phi(y,s)| \neq 0 \quad \text{on } \{(y,s)\in \overline{B}_{\overline{O}}(x_0,r)\times[t,t+h]\ :\ \phi(y,s)=0\},$$

(iv) $\{y\in\overline{B}_{\overline{O}}(x_0,r)\ :\ \phi(y,t)\geq 0\} \subset \Omega_t$ (resp. $\{y\in\overline{B}_{\overline{O}}(x_0,r)\ :\ \phi(y,t)\leq 0\} \subset \mathcal{F}_t^c$),

(v) for all $s\in[t,t+h]$

$$\{y\in\partial B_{\overline{O}}(x_0,r)\ :\ \phi(y,s)\geq 0\}\subset\Omega_s$$

(resp.

$$\{y\in\partial B_{\overline{O}}(x_0,r)\ :\ \phi(y,s)\leq 0\}\subset\mathcal{F}_s^c),$$

then we have

$$\{y\in\overline{B}_{\overline{O}}(x_0,r)\ :\ \phi(y,t+h)>0\}\subset\Omega_{t+h}$$

(resp.

$$\{y\in\overline{B}_{\overline{O}}(x_0,r)\ :\ \phi(y,t+h)<0\}\subset\mathcal{F}_{t+h}^c).$$

The next result gives the relationship between the notion of generalized super- and sub-flow and the level-set evolutions related to (10). Since it is a straightforward consequence of Theorem 2.2 and Theorem 1.2 in [BDL] about the equivalence of Definition 2.2 with the evolutions related to (28) or (27), we omit its proof.

Theorem 2.3 *Suppose that the assumptions of the level set approach hold.*
(i) Let $(\Omega_t)_{t\in(0,T)}$ *be a family of open subsets of* $\overline{O}$ *such that the set* $\Omega := \cup_{t>0}\Omega_t \times \{t\}$ *is open in* $\overline{O}\times[0,T]$*. Then* $(\Omega_t)_{t\in(0,T)}$ *is a generalized super-flow with normal velocity* $-F$ *and angle boundary condition* G *if and only if the function* $\chi = 1\!\!1_\Omega - 1\!\!1_{\Omega^c}$ *is a viscosity supersolution of* $(10)(i)-(ii)$.
(ii) Let $(\mathcal{F}_t)_{t\in(0,T)}$ *be a family of closed subsets of* $\overline{O}$ *such that the set* $\mathcal{F} := \cup_{t>0}\mathcal{F}_t\times\{t\}$ *is closed in* $\overline{O}\times[0,T]$*. Then* $(\mathcal{F}_t)_{t\in(0,T)}$ *is a generalized sub-flow with normal velocity* $-F$ *and angle boundary condition* G *if and only if the function* $\overline{\chi} = 1\!\!1_{\mathcal{F}} - 1\!\!1_{\mathcal{F}^c}$ *is a viscosity subsolution of* $(10)(i)-(ii)$.

3 Applications to the asymptotics of nonlocal reaction-diffusion equations

3.1 The abstract method

In this section, we describe the abstract method to study the asymptotics of solutions to nonlocal semilinear reaction-diffusion equations set in bounded domains with Neumann-type boundary conditions. In the asymptotic problems we have in mind, we are given a family $(u_\varepsilon)_\varepsilon$ of bounded functions on $\overline{O}\times[0,T]$, typically the solutions of reaction-diffusion equations with Neumann type boundary condition and with a small parameter $\varepsilon>0$. The aim is to show that there exists a generalized flow $(\Omega_t)_{t\in[0,T]}$ on $\overline{O}$ with a certain nonlocal normal velocity $-F(x,t,Dd,D^2d,\Omega_t)$ and angle boundary $G(x,t,Dd)$ on ∂O such that, as $\varepsilon\to 0$,

$$u_\varepsilon(x,t)\to b(x,t) \quad \text{if } (x,t)\in\Omega := \bigcup_{t\in(0,T)}\Omega_t\times\{t\},$$

and

$$u_\varepsilon(x,t)\to a(x,t) \quad \text{if } (x,t)\in\Omega^c$$

where, for all (x,t), $a(x,t), b(x,t)\in I\!R$ can be interpreted as local equilibria of this system. In order to be more specific and to present the main steps of the method, we introduce the sets

$$\Omega^1 = \mathrm{Int}\{(x,s)\in\overline{O}\times[0,T] \ : \ \liminf_* \, [u_\varepsilon - b]\,(x,s)\geq 0\}\ , \tag{29}$$

and

$$\Omega^2 = \mathrm{Int}\{(x,s)\in\overline{O}\times[0,T] \ : \ \limsup{}^* \, [u_\varepsilon - a]\,(x,s)\leq 0\}\ . \tag{30}$$

Then we are going to consider the families $(\Omega^1_t)_t$ and $(\Omega^2_t)_t$ defined by

$$\Omega^1_t = \Omega^1\cap(\overline{O}\times\{t\}) \quad \Omega^2_t = \Omega^2\cap(\overline{O}\times\{t\})\ . \tag{31}$$

For simplicity of notations, for $i=1,2$, we identify Ω^i_t and $(\Omega^i_t)^c$ with their projections in $\overline{O}$.

It is worth noticing that Ω^1, Ω^2 are defined as subsets of $\overline{O}\times(0,T]$, they are open by definition and disjoint. In particular we remark that, by construction the functions $\chi = 1\!\!1_{\Omega^1} - 1\!\!1_{(\Omega^1)^c}$ and $\overline{\chi} = 1\!\!1_{(\Omega^2)^c} - 1\!\!1_{\Omega^2}$ are respectively lower and upper semicontinuous functions in $\overline{O}\times(0,T]$ where, in fact, Ω^1 has to be read here as $\bigcup_{t\in(0,T]} \Omega^1_t \times\{t\}$ and Ω^2 as $\bigcup_{t\in(0,T]} \Omega^2_t\times\{t\}$. We finally point out that χ, $\overline{\chi}$ can be extended either by lower semicontinuity or by upper semicontinuity to $\overline{O}\times[0,T]$ and we keep below the same notations for these extensions.

As in [BDL], our method can be described in three steps.
1. *Initialization* : we have to determine the traces Ω^1_0 and Ω^2_0 of Ω^1 and Ω^2 for $t=0$. A convenient way to define these traces are through the function χ and $\overline{\chi}$

$$\Omega^1_0 = \{x\in\overline{O}:\ \chi(x,0)=1\} \quad\text{and}\quad \Omega^2_0 = \{x\in\overline{O}:\ \overline{\chi}(x,0)=-1\}\,. \tag{32}$$

2. *Propagation* : we have to show that $(\Omega^1_t)_t$ and $((\Omega^2_t)^c)_t$ are respectively super- and sub-flow with normal velocity $-F$ and angle condition G.
3. *Conclusion* : we use the following corollary whose proof is a straightforward consequence of Theorem 2.3 and therefore we omit it.

Corollary 3.1 *Assume that the assumptions of the level-set approach hold and that the above $(\Omega^1_t)_t$ and $((\Omega^2_t)^c)_t$ are respectively super and sub-flow with normal velocity $-F$ and angle boundary condition G and suppose there exists $(\partial\Omega^+_0,\Omega^+_0,\Omega^-_0)\in\mathcal{E}$ such that $\Omega^+_0\subseteq\Omega^1_0$ and $\Omega^-_0\subseteq\Omega^2_0$. Then if $(\Gamma_t,\Omega^+_t,\Omega^-_t)$ is the level-set evolution of $(\partial\Omega^+_0,\Omega^+_0,\Omega^-_0)$ we have*
(i) for all $t>0$,

$$\Omega^+_t\subset\Omega^1_t\subset\Omega^+_t\cup\Gamma_t\ ,\ \Omega^-_t\subset\Omega^2_t\subset\Omega^-_t\cup\Gamma_t\ .$$

(ii) If $\cup_t\Gamma_t\times\{t\}$ satisfies the no interior condition, then for all $t>0$, we have

$$\Omega^+_t=\Omega^1_t \quad\textit{and}\quad \Omega^-_t=\Omega^2_t.$$

We turn to comment the first two steps of our method. We first point out that the main difficulty to prove these steps comes from the nonlocal feature of the velocity and our strategy consists in replacing in a suitable way the volume of Ω^1_t and Ω^2_t with smooth functions . More precisely for every smooth function $\theta\colon[0,T]\to I\!\!R$ such that $\theta(t)\le\lambda(\Omega^1_t)$ (resp. $\lambda((\Omega^2_t)^c)\le\theta(t)$) and all constants $\alpha>0$ we introduce the follwing sets

$$\widetilde{\Omega}^1(\alpha,\theta) = \mathrm{Int}\{(x,s)\in\overline{O}\times[0,T]\ :\ \liminf{}_*\left[\frac{u_\varepsilon - b^{\varepsilon,\alpha}}{\varepsilon}\right](x,s)\ge 0\}\ , \tag{33}$$

$$\widetilde{\Omega}^2(\alpha,\theta) = \mathrm{Int}\{(x,s)\in\overline{O}\times[0,T] \ : \ \limsup{}^* \left[\frac{u_\varepsilon - a^{\varepsilon,\alpha}}{\varepsilon}\right](x,s)\leq 0\} \qquad (34)$$

where $(a^{\varepsilon,\alpha})_\varepsilon$ and $(b^{\varepsilon,\alpha})_\varepsilon$ are suitable sequences of real-valued functions defined in $\overline{O}\times[0,T]$ such that $a^{\varepsilon,\alpha}\to a,\ b^{\varepsilon,\alpha}\to b$, as $\varepsilon\to 0$, uniformly in $\overline{O}\times[0,T]$, and for all $\alpha>0$ small enough.

In the Subsection 2.3 of [BDL] the set (33) (resp. (34)) is shown to be a generalized super-flow (resp. sub-flow) in the sense of Definition 2.2 with respect to $-\overline{F}(x,t,Dd,D^2d,\theta(t)-\alpha)$ (resp. $-\overline{F}(x,t,Dd,D^2d,\theta(t)+\alpha)$) and angle boundary condition $G(x,t,Dd)$. Thus our main aim is to show that $\Omega^1 = \widetilde{\Omega}^1(\alpha,\theta)$ and $\Omega^2 = \widetilde{\Omega}^2(\alpha,\theta)$. Indeed if the previous equalities hold we automatically get both the initialization and the propagation of the front with the nonlocal normal velocity $-F(x,t,Dd,D^2d,\{d(y,t)>d(x,t)\})$ and angle boundary condition $G(x,t,Dd)$ as a consequence of the results in [BDL].

3.2 The asymptotics of nonlocal Allen-Cahn equation with monotonicity

This section is devoted to the study of the model case of nonlocal Allen-Cahn Equation set in a bounded domain with a Neumann type boundary conditions, which will be also the occasion of giving the reader a more precise idea of how the abstract method works. More precisely we will focus our attention to the following initial boundary value problem

$$\begin{cases} (i)\ u_{\varepsilon,t} - \Delta u_\varepsilon + b(x)\cdot Du_\varepsilon + \varepsilon^{-2} f(u_\varepsilon, \varepsilon\int_O u_\varepsilon) = 0 & \text{in } O\times(0,T), \\ (ii)\ G(x,t,Du_\varepsilon) = 0 & \text{on } \partial O\times(0,T), \\ (iii)\ u_\varepsilon = g & \text{on } \overline{O}\times\{0\}, \end{cases} \qquad (35)$$

where g is a real-valued continuous function in $\overline{O}$, G satisfies the conditions for the level-set approach to hold, in particular **(A4)**, and **(A5)**, and $b\colon\overline{O}\to I\!R^N$ is a Lipschitz continuous vector field. As far as the reaction term $f\colon I\!R^2\to I\!R$ goes, throughout the paper, we assume that $f\in C^2(I\!R^2, I\!R)$ and $f_0(u):=f(u,0)$ satisfies

$$\begin{cases} f_0 \text{ has exactly three zeroes } m_- < m_0 < m_+, \\ f_0(s) > 0 \text{ in } (m_-,m_0) \text{ and } f_0(s)<0 \text{ in } (m_0,m_+), \\ f_0'(m_\pm)>0,\ f_0''(m_-)<0 \text{ and } f_0''(m_+)>0. \end{cases} \qquad (36)$$

We observe that, for sufficiently small v there exist $h_-(v) < h_0(v) < h_+(v)$ such that

$$f(h_-(v),v) = f(h_0(v),v) = f(h_+(v),v) = 0.$$
$$f(r,v) > 0 \text{ on } (h_-(v),h_0(v)) \text{ and on } (h_+(v),+\infty)\ ,$$
$$f(r,v) < 0 \text{ on } (h_0(v),h_+(v))\text{and on}(-\infty,h_-(v))$$
$$f_u(r,v) \geq \gamma > 0; \quad \text{ on } (-\infty,h_-(v)+\gamma] \cup [h_+(v)-\gamma,+\infty)\ , \tag{37}$$

for some $\gamma > 0$ independent of v.

We assume that for all $(r,v) \in I\!R \times I\!R$

$$f_v(r,v) \leq 0 \tag{38}$$

We note that as a consequence of the above assumptions on f we have:

$$h_\pm(v) \to m_\pm, \quad h_0(v) \to m_0 \ \text{ as } v \to 0 \tag{39}$$

Since, for fixed v, the function $u \mapsto f(u,v)$ satisfies the hypotheses of Aronson and Weinberger [AW] and Fife and McLeod [FM], there exists a unique pair $(q(r,v),c(v))$ such that

$$q_{rr}(r,v) + c(v)q_r(r,v) = f(q(r,v),v) \tag{40}$$

and

$$\lim_{r\pm\infty} q(r,v) = h_\pm(v) \ \text{ and } \ q(0,v) = h_0(v). \tag{41}$$

We continue listing some technical assumptions that we will be making on $(q(\cdot,v),c(v))$:

$$q(\cdot,v) \text{ and } c(v) \ \text{ depend smoothly on } v \tag{42}$$

and referred

$$\begin{aligned}
&(i) \quad \lim_{v\to 0}\sup_r[|v|(|q_v| + |q_{vv}|) + |q_{rv}|] = 0,\\
&(ii) \quad |r|^{-2}|q_{rr}| + |r|^{-1}|q_r| \leq Ke^{-K\delta}, \quad \text{for all } |r| \geq \delta\\
&(iii) \quad q(r,v) \to h_\pm(v) \text{ exponentially fast as } r \to \pm\infty\\
&(iv) \quad q_r(r,v) > 0\\
&(v) \quad q_v(r,v) = O(1), \quad \text{as } v \to 0 \text{ locally uniformly wrt } r
\end{aligned} \tag{43}$$

Finally we impose

$$c(0) = 0 \ \text{ and } \quad -\frac{c(v)}{v} \to c_0 > 0. \tag{44}$$

One example we have in mind is the one considered in [CHL] where

$$f(u,v) = 2u(u^2-1) + vh(u) \tag{45}$$

with $h < 0$. In particular if $h(u) \equiv -C < 0$ then by analogous computations in [BSS] one can show that $c_0 = \frac{3}{2}C$.

We recall that the notion of viscosity solution can be extended to integro-differential equations of the form (35)(i) with f satisfying the monotonicity condition (38) (see, i.e. [AT]).

Below for the reader's convenience we give a sketch of proof of the comparison result between viscosity bounded sub and supersolutions of the problem (35) for fixed $\varepsilon > 0$. Then the comparison result and the Perron's Method yield the existence of a unique continuous viscosity solution of (35). The existence and the uniqueness of a smooth solution of (35) without the assumption (38) was obtained in [CHL] in the case of homogeneous Neumann boundary condition.

Lemma 3.1 *Assume (36) and (38). Let u, v be respectively bounded lower and upper semicontinuous viscosity sub- and supersolution of (35) (i)-(ii) with fixed $\varepsilon = \varepsilon_0 > 0$ and $u(x,0) \le v(x,0)$. Then*

$$u(x,t) \le v(x,t) \quad \textit{for all } (x,t) \in \overline{O} \times [0,T].$$

Proof.

1. Without loss of generality, we can fix $\varepsilon_0 = 1$. For simplicity, we assume that u, v are smooth functions and ∂O is C^2. (For general case, one can combine the arguments in [CIL] and in [AT]).

2. Suppose that $u - v$ has a maximum $\delta_0 > 0$ at (x_0, t_0) in $\bar{O} \times [0,T]$. Fix a positive constant L and let

$$\bar{u}(x,t) = e^{-Lt}u(\cdot,t), \quad \bar{v}(x,t) = e^{-Lt}v(\cdot,t).$$

Then $\bar{u}(x,t) - \bar{v}(x,t)$ has a maximum $\delta(L) > 0$ in $\bar{O} \times [0,T]$. Note that if the maximum is attained at a point (x_L, t_L), then $e^{Lt_L}\delta(L) \le \delta_0$.

3. Consider the signed distance function $d(x, \partial O)$ with $d > 0$ in $\bar{O}$ and extend this function from a neighborhood of ∂O to $\bar{O}$ such that the extended function $d(x)$ is C^2 in $\bar{O}$. Then for $\varepsilon > 0$ the function

$$w(x,t) = \bar{u}(x,t) - \bar{v}(x,t) + \varepsilon d(x)$$

has a maximum at $P_\varepsilon = (x_\varepsilon, t_\varepsilon)$ in $\bar{O} \times [0,T]$, which converges to $(x_L, t_L), t_L > 0$ with $\bar{u} - \bar{v} = \delta(L)$ at (x_L, t_L). Suppose that $x_\varepsilon \in \partial O$. Since at P_ε we have $D\bar{u} = D\bar{v} + \varepsilon n(x_\varepsilon) + \lambda n(x_\varepsilon)$ with $\lambda \ge 0$, by (A3) we have

$$G(x_\varepsilon, t_\varepsilon, D\bar{u}) \ge G(x_\varepsilon, t_\varepsilon, D\bar{v} + \varepsilon n(x_\varepsilon)) > G(x_\varepsilon, t_\varepsilon, D\bar{v}),$$

which contradicts the definition of u, v.

4. Therefore $x_\varepsilon \in O$ and we have the following inequality at P_ε:

$$\begin{aligned} L(\bar{u}-\bar{v}) \quad &\leq \Delta(\bar{u}-\bar{v}) + b(x_\varepsilon)\cdot(D\bar{u}-D\bar{v}) + e^{-Lt_\varepsilon}[f(v, \textstyle\int_O v) - f(u, \int_O u)] \\ &\leq O(\varepsilon) + e^{-Lt_\varepsilon}[f(v, \textstyle\int_O v) - f(u, \int_O u)]. \end{aligned}$$

Since $u \leq v + e^{Lt_\varepsilon}\delta(L)$, $e^{Lt_\varepsilon}\delta(L)$ is bounded and $f(u, \cdot)$ is decreasing, by sending $\varepsilon \to 0$ we get

$$L\delta(L) \leq e^{-Lt_L} K(1+|O|)e^{Lt_L}\delta(L) = M(1+|O|)\delta(L),$$

where M is the global Lipschitz constant of the function f (since $u, e^{Lt_L}\delta(L)$ are bounded and f is C^2, we may assume that f is globally Lipschitz without loss of generality). Thus we get a contradiction if we choose $L > M(1+|O|)$. $\square$

In the following Lemma we show some estimates satisfied by the $(u_\varepsilon)_\varepsilon$ by using the comparison result in Lemma 3.1.

Lemma 3.2 *Under the assumptions of Lemma 3.1 the family $(u_\varepsilon)_\varepsilon$ satisfies*

$$m_- + o(1) \leq u_\varepsilon(x,t) \leq m_+ + o(1) \quad \text{as } \varepsilon \to 0 \tag{46}$$

uniformly in $\overline{O} \times [0,T]$.

Proof. We just sketch the proof, being it quite standard. We first show that family $(u_\varepsilon)_\varepsilon$ is uniformly bounded in $(x,t) \in \overline{O} \times [0,T]$. To this purpose, one observes that the functions $\overline{u} := Ct + M$ and $\underline{u} := -Ct - M$ with $M, C > 0$ large enough are respectively super and subsolution of (35). Thus by comparing u^ε with $\underline{u}$ and $\overline{u}$ we obtain $||u^\varepsilon||_\infty \leq K$ for some $K > 0$.

Now we prove that $u_\varepsilon(x,t) \leq m_+ + o(1)$ as $\varepsilon \to 0$ (the other inequality being proved in a similar way). Since $||u^\varepsilon||_\infty \leq K$ and $f_v \leq 0$ we have that u^ε is a supersolution of

$$\begin{cases} (i)\ u_{\varepsilon,t} - \Delta u_\varepsilon + b(x)\cdot Du_\varepsilon + \varepsilon^{-2} f(u_\varepsilon, \varepsilon K\lambda(O)) = 0 & \text{in } O \times (0,T) \\ (ii)\ G(x,t,Du_\varepsilon) = 0 & \text{on } \partial O \times (0,T), \\ (iii)\ u_\varepsilon = u_0 & \text{on } \overline{O} \times \{0\}, \end{cases} \tag{47}$$

We consider the solution of the ode

$$\begin{cases} \dot{\zeta}(s,\xi) + \varepsilon^{-2} f(\zeta, \varepsilon K\lambda(O)) = 0 \quad s \in (0,\infty) \\ \zeta(0,\xi) = \xi \end{cases} \tag{48}$$

where $\xi > \max(||g||_\infty, |m_+|)$. By using the properties of f one can show that

$$\zeta(s,\xi) \le h_+(\varepsilon K\lambda(O)) + (\xi - h_+(\varepsilon K\lambda(O)))\exp(-\varepsilon^{-2}Ct)$$

for some $C > 0$. Since $h_+(\varepsilon K\lambda(O)) \to m_+$ as $\varepsilon \to 0$, we get $\zeta(s,\xi) \le (m_+ + o(1))$ as $\varepsilon \to 0$. Finally having by construction $u^\varepsilon(x,0) \le \xi$ we obtain $u^\varepsilon(x,t) \le m_+ + o(1)$ as $\varepsilon \to 0$ and we conclude. □

Under the current assumptions we expect that the front evolution associated with the asymptotics of (35) is motion by mean curvature and an additional term depending on the volume enclosed by the front. The corresponding geometric pde is

$$\begin{cases} u_t - \mathrm{tr}[(I - \widehat{Du}\otimes\widehat{Du})D^2u] + b(x)\cdot Du - c_0|Du|\mu(x,t,u) = 0 & \text{in } O\times(0,T) \\ G(x,t,Du) = 0 & \text{on } \partial O\times(0,T), \\ u = u_0 & \text{on } \overline{O}\times\{0\}, \end{cases} \tag{49}$$

where $\mu(x,t,u) := m_+\lambda(\Omega^+_{x,t}(u)) + m_-\lambda((\Omega^+_{x,t}(u))^c)$, $\Omega^+_{x,t}(u) := \{u(y,t) > u(x,t)\}$, and $c_0 > 0$ is defined in (44).

We premise the following Lemma where we show a key property of G which is used in the sequel to check the Neumann boundary condition. To formulate it, we use the following notation : for $p \in I\!R^N$ and $x \in \partial O$, $\mathcal{T}(p) := p - p\cdot n(x)n(x)$. $\mathcal{T}(p)$ represents the projection of p on the tangent hyperplane to ∂O at x.

Lemma 3.3 *Assume that* **(A2)** *and* **(A4)** *hold and that, for some $x \in \partial O$, $t \in (0,T)$ and $\tilde{p} \in I\!R^N$, we have $G(x,t,\tilde{p}) \le 0$ (resp. $G(x,t,\tilde{p}) \ge 0$), then there exists a constant $K(T)$ such that, if $p\cdot n(x) \le -K(T)|\mathcal{T}(p)|$, then*

$$G(x,t,\tilde{p}+p) \le 0\ .$$

(resp. if $p\cdot n(x) \ge K(T)|\mathcal{T}(p)|$, then

$$G(x,t,\tilde{p}+p) \ge 0\ .)$$

We refer the reader to [BDL] for the proof of Lemma 3.3, we only remark that, by **(A4)**, $G(x,t,0) \equiv 0$ and therefore the above result holds with $\tilde{p} = 0$.

The main result of this Section is

Theorem 3.1 *Assume (36), (43) and let* u_ε *be the solution of (35) where* $g\ :\overline{O}\to I\!R$ *is a continuous function such that the set* $\Gamma_0=\{x\in\overline{O}\ :g(x)=m_0\}$ *is a nonempty subset of* $\overline{O}$*. Then, as* $\varepsilon\to 0$*,*

$$u_\varepsilon(x,t)\to\left\{\begin{array}{lcl} m_+ & & \{u>0\},\\ & \textit{locally uniformly in} & \\ m_- & & \{u<0\},\end{array}\right.$$

where u *is the unique viscosity solution of (49) with* $u_0=d_0$*, the signed distance to* Γ_0*, which is positive in the set* $\{g>m_0\}$ *and negative in the set* $\{g<m_0\}$*. If, in addition, the no interior condition* (25) *holds, then, as* $\varepsilon\to 0$*,*

$$u_\varepsilon(x,t)\to\left\{\begin{array}{lcl} m_+ & & \{u>0\},\\ & \textit{locally uniformly in} & \\ m_- & & \overline{\{u>0\}}^{c}.\end{array}\right.$$

Proof of Theorem 3.1. We consider the open sets Ω^1 and Ω^2 of sets defined in Section 3.1 by (29), (30) with $b(x,t)\equiv m_+$ and $a(x,t)\equiv m_-$.

By following the abstract method in Section 3.1 we have to show that Ω^1_0 and Ω^2_0 are not empty and the families $(\Omega^1_t)_{t>0}$, and $((\Omega^2_t)^c)_{t>0}$ are respectively a generalized super-flow and sub-flow with normal velocity $-F$ and angle condition G. We will consider the Ω^1-case, the Ω^2-case being treated in a similar way. To this purpose we proceed as follows.

1. Let us consider a smooth function $\theta\colon[0,T]\to I\!R$ satisfying for all $t\in(0,T)$, $\theta(t)\le\lambda(\Omega_t)$ and a constant $\alpha>0$. By the definition of Ω^1_t and Lemma 3.2 for small $\varepsilon>0$ we have

$$\begin{aligned}\int_O u_\varepsilon\,dx\ &\ge\ \int_O m_+1\!\!1_{\Omega_t}+m_-1\!\!1_{\Omega_t^c}\,dx-(m_+-m_-)\alpha\\ &\ge\ (m_+-m_-)\theta(t)+m_-\lambda(O)-(m_+-m_-)\alpha \qquad (50)\end{aligned}$$

We denote by

$$\mu(t):=(m_+-m_-)(\theta(t)-\alpha)+m_-\lambda(O) \qquad (51)$$

and we set $f^\varepsilon(u,t):=f(u,\varepsilon\mu(t))$ (for clarity of notations we drop the the dependence on α and θ in μ and f^ε). ¿From (38) it follows that for ε small enough u_ε is a supersolution also of

$$\left\{\begin{array}{lll}(i)\ u_{\varepsilon,t}-\Delta u_\varepsilon+b(x)\cdot Du_\varepsilon+\varepsilon^{-2}f^\varepsilon(u_\varepsilon,t)=0 & \text{in } O\times(0,T) & \\ (ii)\ G(x,t,Du_\varepsilon)=0 & \text{on } \partial O\times(0,T), & (52)\\ (iii)\ u_\varepsilon=u_0 & \text{on } \overline{O}\times\{0\}, & \end{array}\right.$$

Let $m_+^{\varepsilon,\alpha}$ and $m_-^{\varepsilon,\alpha}$ be the greatest and the smallest stable equilibria for $f^\varepsilon(u,t)$. We introduce the following set (depending on α and θ):

$$\widetilde{\Omega}^1(\alpha,\theta) := \mathrm{Int}\{(x,s): \ \liminf{}_* \left[\frac{u_\varepsilon(x,s) - m_+^{\varepsilon,\alpha}(s)}{\varepsilon}\right] \geq 0\} \tag{53}$$

and for all $t \in (0,T]$ we set

$$\widetilde{\Omega}^1_t(\alpha,\theta) := \widetilde{\Omega}^1(\alpha,\theta) \cap (\overline{O} \times \{t\}). \tag{54}$$

2. We show that for all $\alpha > 0$ and $\theta(\cdot)$ as above we have $\widetilde{\Omega}^1(\alpha,\theta) \equiv \Omega^1$ (see Lemma 3.4).

3. In [BDL] it is proved that $\widetilde{\Omega}^1_t(\alpha,\theta) \neq \emptyset$ for $t > 0$ small, and $(\widetilde{\Omega}^1_t(\alpha,\theta))_t$ is a generalized super-flow in the sense of Definition 2.2 with respect to the velocity $-\overline{F}_-(x,t,Dd,D^2d)$ and angle boundary condition $-G(x,t,Dd)$. Thus we automatically obtain that $\Omega^1_0 \neq \emptyset$ (namely the *initialization* of the front) and $(\Omega^1_t)_t$ is a generalized super-flow (namely the *propagation* of the front) with respect to the nonlocal normal velocity $-F(x,t,Dd,D^2d,\Omega^1_t)$ and angle boundary condition $-G(x,t,Dd)$. The conclusion then follows from Corollary 3.1. □

Remark 3.1 We remark that the initialization procedure can be proved also directly without introducing the set (53). More precisely one can reproduce the same arguments of the first step of Theorem 2.3 in [BDL] by constructing globally in $\overline{O}$ sub- and supersolutions of (52) with $f^\varepsilon(u) := f(u,0) \pm \varepsilon K'\lambda(O))$, for a suitable choice of K', associated with radially symmetric moving fronts. We point out that this can be done without the monotonicity condition (38), only the assumption that $(u^\varepsilon)_\varepsilon$ is uniformly bounded being necessary in order to replace in the equation (35) $f(u^\varepsilon, \varepsilon\int_O u^\varepsilon)$ with $f(u,0) \pm \varepsilon K'\lambda(O)$.

In the next Lemma we prove that $\widetilde{\Omega}^1(\alpha,\theta) = \Omega^1$. Such an equality is a consequence of the estimates on u_ε that the initialization step in [BDL] gives and which have been properly adapted to this case.

Lemma 3.4 *For all $\alpha > 0$ and for every smooth function $\theta\colon [0,T] \to I\!R$ such that $\theta(t) \leq \lambda(\Omega^1_t)$ for all $t \in (0,T)$ we have $\Omega^1 = \widetilde{\Omega}^1(\alpha,\theta)$.*

Proof. It is enough to show that "$\Omega^1 \subseteq \widetilde{\Omega}^1(\alpha,\theta)$", the other inclusion being trivially satisfied by the definition of these sets. More precisely we are going to show the following inequality

$$\{(x,s): \ \liminf{}_* \ [u_\varepsilon(x,s) - m_+] \geq 0\} \subseteq \{(x,s): \ \liminf{}_* \left[\frac{u_\varepsilon(x,s) - m_+^{\varepsilon,\alpha}(s)}{\varepsilon}\right] \geq 0\} \tag{55}$$

from which the inclusion "$\Omega^1 \subseteq \widetilde{\Omega}^1(\alpha,\theta)$" follows (by considering the interior of the above sets). To this end we follow the strategy of proof of Proposition 2.1 in [BDL].

Let $(x_0,t_0) \in \overline{O} \times (0,T]$ be such that

$$\liminf_* \; [u_\varepsilon(x_0,t_0) - m_+] \geq 0.$$

Then, by definition of $\liminf_*$, for any $\gamma > 0$ there exist $h>0, r>0, \bar{\varepsilon}>0$ such that for all $\varepsilon < \bar{\varepsilon}$, $x \in \overline{B}_{\overline{O}}(x_0,r)$ and $|t-t_0|<h$ we have $u_\varepsilon(x,t) \geq m_+ - \gamma$.

We suppose that $x_0 \in \partial O$ (the case $x \in O$ being similar and even simpler). By the smoothness of O, if η is small enough and if $\bar{x} := x_0 - \eta n(x_0)$ then $B(\bar{x},\eta) \subseteq O$ and $\overline{B}(\bar{x},\eta) \cap \partial O = \{x_0\}$. Consider the function $\psi_\eta(x) = \eta^2 - |x-\bar{x}|^2$. We observe that $D\psi_\eta(x_0)\cdot n(x_0) = -\eta < 0$. Thus we can find $R > \eta$ and $\bar{\delta} > 0$ such that $B(\bar{x},R) \subseteq B(x_0,r)$ and the function

$$\psi(x) = R^2 - |x-\bar{x}|^2 \tag{56}$$

satisfies $D\psi(x)\cdot n(x) < 0$ on $\{x \in \overline{O} : \; |d(x)| < \bar{\delta}\}$, $d(\cdot)$ being the signed distance function to the set $\{x : \; \psi(x) = 0\}$. By Lemma 3.3, choosing R close enough to η, we may have also $G(x,t,D\psi(x)) < 0$ on $\{x \in \partial O : \; |d(x)| < \bar{\delta}\}$ for, say, any $t \leq 1$.

We introduce the function $\Psi \colon \overline{O} \times [0,T] \to I\!R$ given by

$$\Psi(x,s) = \psi(x) - C(s - t_0 + h), \tag{57}$$

with $C>0$ which will be chosen later and denote by $d(\cdot,s)$ the signed distance to the set $\{\Psi(\cdot,s) = 0\}$ which is normalized to have the same signs of Ψ in $\overline{O}\times[0,T]$. Here

$$d(x,s) = [(R^2 - C(s-t_0+h))^+]^{1/2} - |x-\bar{x}| \;.$$

By the choice of R, there is $0 < \delta' < \dfrac{(m_+ - m_0)}{2} \wedge \bar{\delta}$ such that for all $0<\delta<\delta'$ and $t \in (t_0-h, t_0+h)$ we have

$$u_\varepsilon(x,t) \geq (m_+ - \gamma)1\!\!1_{\overline{B}_{\overline{O}}(x_0,r)} - K 1\!\!1_{(\overline{B}_{\overline{O}}(x_0,r))^c} \geq (m_0 + 2\delta) 1\!\!1_{\{\Psi(\cdot,t)>0\}} - K 1\!\!1_{\{\Psi(\cdot,t)\leq 0\}}$$

where K is an upper bound of $||u^\varepsilon||_\infty$.

Now we need the following two Lemmae whose proof is postponed in the Appendix **??**

Lemma 3.5 *Under the assumptions of Theorem 3.1, for any $\beta>0$, there are constants $\tau > 0$ and $\bar{\varepsilon}$ (depending on β) such that for all $0<\varepsilon\leq\bar{\varepsilon}$ and for all $t \in (t_0-h,t_0+h)$ we have*

$$u_\varepsilon(x,t+t_\varepsilon) \geq (m_+^{\varepsilon,\alpha}(t+t_\varepsilon) - \beta\varepsilon) 1\!\!1_{\{d(x,t+t_\varepsilon)\geq\beta\}} + (m_-^{\varepsilon,\alpha}(t+t_\varepsilon) - \beta\varepsilon) 1\!\!1_{\{d(x,t+t_\varepsilon)<\beta\}} \quad \text{on } \overline{O}\,.$$

where $t_\varepsilon = \tau\varepsilon^2|\log\varepsilon|$.

Lemma 3.6 *There exist $\bar{h} < h$, $\bar{\beta} > 0$, depending only on ψ defined in (56) such that, if $\beta \leq \bar{\beta}(\psi)$ and $\varepsilon \leq \bar{\varepsilon}(\beta, \psi)$, then there exits a subsolution $w^{\varepsilon,\beta}$ of (52) in $\overline{O} \times (t_0 - t_\varepsilon, t_0 + \bar{h})$ such that,*

$$w^{\varepsilon,\beta}(x, t_0 - t_\varepsilon) \leq [m_+^\varepsilon(t_0 - t_\varepsilon) - \beta\varepsilon]\mathbb{1}_{\{d(\cdot, t_0 - t_\varepsilon) \geq \beta\}} + [m_-^\varepsilon(t_0 - t_\varepsilon) - \beta\varepsilon]\mathbb{1}_{\{d(\cdot, t_0 - t_\varepsilon) < \beta\}} \quad \text{in } \overline{O}.$$

Moreover, if $(x, t) \in \overline{O} \times (t_0 - t_\varepsilon, t_0 + \bar{h})$ satisfies $d(x, t) > 2\beta$, then

$$\liminf{}_* \left[\frac{w^{\varepsilon,\beta}(x, t) - m_+^{\varepsilon,\alpha}(t)}{\varepsilon} \right] \geq -2\beta \ .$$

We first observe that if in Lemma 3.5 we choose $t = t_0 - 2t_\varepsilon$ then for ε small enough we have

$$u_\varepsilon(x, t_0 - t_\varepsilon) \geq (m_+^{\varepsilon,\alpha}(t_0 - t_\varepsilon) - \beta\varepsilon)\mathbb{1}_{\{d(x, t - t_\varepsilon) \geq \beta\}} + (m_-^{\varepsilon,\alpha}(t - t_\varepsilon) - \beta\varepsilon)\mathbb{1}_{\{d(\cdot, t - t_\varepsilon) < \beta\}} \quad \text{in } \overline{O} \ .$$

Then Lemma 3.6 yields a subsolution $w^{\varepsilon,\beta}$ of (52)(i)-(ii) such that

$$w^{\varepsilon,\beta}(x, t_0 - t_\varepsilon) \leq u_\varepsilon(x, t_0 - t_\varepsilon) \quad \text{in } \overline{O}$$

thus by the maximum principle we have

$$w^{\varepsilon,\beta}(x, t) \leq u_\varepsilon(x, t) \quad \text{in } O \times (t_0 - t_\varepsilon, t_0 + h).$$

Morever from (77) it follows that if $t \in (t_0 - t_\varepsilon, t_0 + \bar{h})$, $x \in \overline{O} \cap \overline{B}_{\overline{O}}(x_0, r)$ and $d(x, t) > 2\beta$ we have

$$\liminf{}_* \left[\frac{u_\varepsilon(x, t) - m_+^{\varepsilon,\alpha}(t)}{\varepsilon} \right] \geq -2\beta \ .$$

Since β is arbitrary and does not depend on $\bar{h}$ and $d(x_0, t_0) > 0$ (by construction), we have

$$\liminf{}_* \left[\frac{u_\varepsilon(x_0, t_0) - m_+^{\varepsilon,\alpha}(t_0)}{\varepsilon} \right] \geq 0.$$

Thus we have proved the inequality (55) and we conclude. $\square$

Remark 3.2 The asymptotic result of this subsection continues to hold also in the case of nonlinearities f depending on (x, t, u, v) provided the monotonicity condition (38) is satisfied for all (x, t, u, v).

3.3 Some remarks on the asymptotics of nonlocal Allen-Cahn equation without monotonicity

In this subsection we extend the asymptotic result of the subsection 3.2 under a suitable relaxation of the monotonicity assumption (38).

As we have already pointed out in the subsection 3.2 the monotonicity condition on f allows us to have a rather simple proof of the asymptotics by replacing the nonlocal term with some suitable smooth functions of the time. On the other hand, as we observed in Remark 3.1, the initialization step can be proved without the monotonicity condition (38), moreover the proof of Lemma 3.6 can be readily extended also to the case when u^ε is a supersolution of

$$u_{\varepsilon,t} - \Delta u_\varepsilon + b(x) \cdot Du_\varepsilon + \varepsilon^{-2} f(u_\varepsilon, \varepsilon\mu(t)) = o(1) \ \text{ in } O \times (0,T), \tag{58}$$

where $\mu(t)$ is the function defined in (51) and $o(1) \to 0$ as $\varepsilon \to 0$ locally uniformly in (x,t) (that is trivially satisfied in the monotone case). Indeed if we examine the proof of Lemma 3.6, we build a function $w^{\varepsilon,\beta}$ satisfying for some positive constant ν (independent on ε)

$$w_t^{\varepsilon,\beta} - \Delta w^{\varepsilon,\beta} + b(x) \cdot Dw^{\varepsilon,\beta} + \varepsilon^{-2} f(w^{\varepsilon,\beta}, \varepsilon\mu(t)) \le -\varepsilon^{-1}\nu + O(1), \quad \text{as } \varepsilon \to 0. \tag{59}$$

locally in the space and in suitable small intervals (see (76)). Thus one can choose $\delta > 0$ such that for $\varepsilon > 0$ small enough u_ε and $w^{\varepsilon,\beta}$ are respectively strict sub and supersolutions of

$$u_{\varepsilon,t} - \Delta u_\varepsilon + b(x) \cdot Du_\varepsilon + \varepsilon^{-2} f(u_\varepsilon, \varepsilon\mu(t)) = -\delta \tag{60}$$

and hence we can compare them.

Now we give some sufficient conditions on f implying (59). Typically we have in mind the case when the monotonicity condition (38) holds between the two stable equilibria of W such as

$$f(u,v) = 2(u + \theta(x,t)v)(u - m_-)(u - m_+)$$
$$f(u,v) = 2(u + h(u)v)(u - m_-)(u - m_+)$$

with $\theta(x,t), h(u) > 0$.

In this subsection we assume the hypotheses of subsection 3.2 except (38) and we add the following assumptions on f:

(H1) $f(m_\pm, v) = 0$ for all $v \in \mathbb{R}$.

(H2) There exists $C > 0$ such that for every $\delta \in (0, m_+ - m_-)$ $f_v(u,v) \le C\delta$ for all $u \in [m_- - \delta, m_+ + \delta]$ and for all $v \in \mathbb{R}$.

(H3) $f_v(u,v) \le 0$ for all $u \in [m_-, m_+]$ and for all $v \in I\!R$.

(H4) For all ε there exists a smooth solution u_ε of (35) and the family u^ε is uniformly bounded in $\overline{O} \times [0,T]$.

Proposition 3.1 *Under the current assumptions the family $(u^\varepsilon)_\varepsilon$ satisfies*

$$m_- - L\exp(-\varepsilon^{-2}Kt) \le u^\varepsilon(x,t) \le m_+ + L\exp(-\varepsilon^{-2}Kt)$$

for suitable $L, K > 0$ and for $(x,t) \in \overline{O} \times (0,T)$.

Proof. We prove the inequality

$$u^\varepsilon(x,t) \le m_+ + L\exp(-\varepsilon^{-2}Kt)$$

the other one being proved in an analogous way. Let us introduce the function $M_+^\varepsilon(t) := \sup_{\overline{O}}(u^\varepsilon(x,t) - m_+)^+$ and we claim that it is a viscosity subsolution of the following variational inequality:

$$\min(\zeta(t), \dot{\zeta}(t) + \varepsilon^{-2}\gamma(\zeta(t))) = 0 \quad \text{for all } t \in (0,T), \tag{61}$$

where $\gamma > 0$ is the constant appearing in (37). To this purpose, let ψ be a smooth function of the time t such that $M_+^\varepsilon(t) - \psi(t)$ has a global maximum at $\bar{t} \in (0,T)$, and $M_+^\varepsilon(\bar{t}) = \psi(\bar{t})$. If $M_+^\varepsilon(\bar{t}) = 0$ then (61) is trivially satisfied. Otherwise $M_+^\varepsilon(\bar{t}) = (u^\varepsilon(\bar{x},\bar{t}) - m_+)^+$ with $\bar{x} \in \overline{O}$. We first assume that $M_+^\varepsilon(\bar{t}) = u^\varepsilon(\bar{x},\bar{t}) - m_+$ with $\bar{x} \in O$. We note that $u^\varepsilon - \psi$ has a maximum at $(\bar{x},\bar{t})$ as well. Thus $\psi_t(\bar{t}) = u_t^\varepsilon(\bar{x},\bar{t})$ and $Du^\varepsilon(\bar{x},\bar{t}) = D\psi(\bar{t}) = 0$. Since u_ε is a smooth solution of (35) we have

$$\psi_t(\bar{t}) + \varepsilon^{-2} f(u^\varepsilon(\bar{x},\bar{t}), \varepsilon \int_O u^\varepsilon(x,\bar{t})\,dx) = 0.$$

On another hand since $u^\varepsilon(\bar{x},\bar{t}) > m_+$ and (37), (**H1**) hold, we have

$$\begin{aligned} f(u^\varepsilon(\bar{x},\bar{t}), \varepsilon \int_O u^\varepsilon(x,\bar{t})\,dx) &\ge f_u(\xi, \varepsilon \int_O u^\varepsilon(x,\bar{t})\,dx)(u^\varepsilon(\bar{x},\bar{t}) - m_+) \\ &\ge \gamma(u^\varepsilon(\bar{x},\bar{t}) - m_+) \end{aligned}$$

where $m_+ \le \xi \le u^\varepsilon(\bar{x},\bar{t})$. Therefore we have

$$\min(\psi_t(\bar{t}), \psi_t(\bar{t}) + \varepsilon^{-2}\gamma(M_+^\varepsilon(\bar{t}))) \le 0. \tag{62}$$

Now we assume that $M_+^\varepsilon(\bar{t}) = u^\varepsilon(\bar{x},\bar{t}) - m_+$ with $\bar{x} \in \partial O$. We introduce the function

$$\Psi(x,t) = u^\varepsilon(x,t) - \psi(t) + \alpha d(x),$$

where $\alpha > 0$ and $d(\cdot)$ is the signed distance function from ∂O. Let (x_α, t_α) be a sequence of maximum points of Ψ. By the same arguments of Lemma 3.1 one can show that $x_\alpha \in O$ and $u^\varepsilon(x_\alpha, t_\alpha) - m_+$ converges to $M_+^\varepsilon(\bar{t})$ as $\alpha \to 0$. We may also assume that $u^\varepsilon(x_\alpha, t_\alpha) - m_+ \geq 0$. Thus we have

$$\begin{aligned} 0 &= \psi_t(t_\alpha) + \varepsilon^{-2} f(u^\varepsilon(x_\alpha, t_\alpha), \varepsilon \int_O u^\varepsilon(x, t_\alpha)\, dx) \\ &\geq \psi_t(t_\alpha) + \varepsilon^{-2}\gamma(u^\varepsilon(x_\alpha, t_\alpha) - m_+) \end{aligned}$$

By letting $\alpha \to 0$ we get (62). Hence we prove the claim.

Now let us consider the solution of the ode:

$$\begin{cases} \dot{\zeta}(t,\xi) + \varepsilon^{-2}\gamma\zeta(t,\xi) = 0 & \text{in } (0,T) \\ \zeta(0,\xi) = \xi \end{cases}$$

with $\xi \geq ||u^\varepsilon||_\infty$. Easy computations show that

$$\zeta(t,\xi) \leq L\exp(-\varepsilon^{-2}Kt) \quad \text{for all } t \in (0,T)$$

for some positive constants L, K independent on ε. We note that ζ is a supersolution of $\min(\zeta, \dot{\zeta}(t,\xi) + \varepsilon^{-2}\gamma\zeta(t,\xi)) = 0$. Since $M^\varepsilon(0) \leq \zeta(0)$ then by maximum principle we have $M^\varepsilon(t) \leq \zeta(t)$ for all $t > 0$. In particular we get $u^\varepsilon(x,t) \leq m_+ + L\exp(-\varepsilon^{-2}Kt)$ for all $(x,t) \in \overline{O} \times (0,T)$ and we conclude. $\square$

To conclude this subsection, we observe that Proposition 3.1, the hypothesis **(H2)** and the fact that $||u_\varepsilon||_\infty \leq M$ imply

$$f(u^\varepsilon, \varepsilon \int_O u^\varepsilon(x,t)\, dx) - f(u^\varepsilon, \varepsilon\mu(t)) \leq \varepsilon C \exp(-\varepsilon^{-2}Kt) \tag{63}$$

for some $C > 0$ (depending on $||u^\varepsilon||_\infty$, $m_\pm$) and for all $t \in (0,T)$. Hence for all $t_0 \in (0,T)$ and for all $h > 0$, u^ε is a supersolution also of

$$u_{\varepsilon,t} - \Delta u_\varepsilon + b(x) \cdot Du_\varepsilon + \varepsilon^{-2} f^\varepsilon(u_\varepsilon, t) = o(\varepsilon) \ \text{ in } O \times (t_0 - h, t_0 + h). \tag{64}$$

Thus the proofs of Lemma 3.5 and 3.6 can be adapted under the current assumptions and consequently Lemma 3.4 continues to hold.

Appendix A

The additional assumptions on O, F, and G are

(A6) For all $x \in \overline{O}$, $t \in [0,T]$ and $K, L \in \mathcal{B}$,

$$-\infty < F^*(x,t,0,0,K) = F_*(x,t,0,0,L) < \infty. \tag{65}$$

and either the additional conditions that Ishii and Sato imposed or the conditions that Barles imposed hold:

(A6a) (Ishii and Sato)

- $G \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N) \cap C^{1,1}(\mathbb{R}^N \times [0,T] \times \mathbb{R}\backslash\{0\})$.
- There exists a function $\omega : [0,\infty) \to [0,\infty)$ continuous at 0 satisfying $\omega(0) = 0$ such that if for $X, Y \in \mathcal{S}^N$ and $\mu_1, \mu_2 \in [0,\infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$F(y,t,q,Y,K) - F(x,t,p,X,K) \le \\ \omega\Big(\mu_1(|x-y|^2 + (\tfrac{|p-q|}{|p|\wedge|q|} \wedge 1)^2) + \mu_2 + |p-q| + |x-y|(|p| \vee |q| + 1)\Big)$$

for all $t \in (0,T]$, $x, y \in \overline{O}$, $p, q \in \mathbb{R}^N\backslash\{0\}$ and $K \in \mathcal{B}$.

(A6b) (Barles)

- ∂O is $W^{3,\infty}$.
- For every $T > 0$ there exists $\vartheta(T) > 0$ such that for all $t \in (0,T)$, $x, y \in \overline{O}$ and $p, q \in \mathbb{R}^N$ $|G(x,t,p) - G(y,t,q)| \le \vartheta(T)((|p| + |q|)|x-y| + |p-q|)$
- For any $C > 0$ there exists a function $\omega_C : [0,\infty) \to [0,\infty)$ continuous at 0 satisfying $\omega_C(0) = 0$ such that for all $\eta > 0$, for $x, y \in \overline{O}$, $p, q \in \mathbb{R}^N\backslash\{0\}$, and $X, Y \in \mathcal{S}^N$ satisfying

$$\begin{aligned} -\frac{C\eta}{\varepsilon^2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{C\eta}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C\eta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ |p-q| &\le C\varepsilon(|p| \wedge |q|) \\ |x-y| &\le C\eta\varepsilon \end{aligned}$$

the following holds for all $K \in \mathcal{B}$:

$$F(y,t,q,Y,K) - F(x,t,p,X,K) \le \omega_C\Big(\frac{|x-y|^2}{\varepsilon^2} + \eta + |x-y|(|p| \vee |q| + 1)\Big)$$

The main difference between the conditions is that the conditions of Ishii and Sato require less regularity on the domain (that is that ∂O is C^1 which was our basic assumption on O), while condition Barles require less regularity on the boundary condition (that is that G is uniformly continuous, which is a part of **(A3)**).

Appendix B

In this appendix we give the proof Lemmae 3.5 and 3.6.
Proof of Lemma 3.5. We essentially follow the lines of the proof of Step 1 of Theorem 2.3 in [BDL] and we reproduce here some key points for sake of completeness and refer the reader to Subsection 2.3 in [BDL] for all the details.
1. We modify the function f^ε taking in account the t-dependence and, to do so, we do it here in the following way : because of the assumptions on f^ε, there exists a function $r \mapsto f_\delta(u)$ $(0 < \delta < \delta')$ such that, for every $T > 0$, if ε is small enough, $f_\delta(u) \geq f^\varepsilon(u,t) + 2\varepsilon$ for any $u \in I\!R$ and $t \in [0,T]$. Moreover f_δ is a cubic type nonlinearity satisfying (36) with three zeros which are $m_- - \delta$, $m_0 + \delta/2$ and $m_+ - \delta$.

We modify the function f^ε in two steps; we first introduce a smooth cut-off function $\zeta_1 \in C_0^\infty(I\!R)$ such that $0 \leq \zeta_1 \leq 1$ in $I\!R$, $\zeta_1(u) = 1$ in $(m_0 - \delta, m_0 + \delta)$ and $\zeta_1(u) = 0$ for $u \leq m_0 - 2\delta$ and $u \geq m_0 + 2\delta$. We set

$$\tilde{f}_\delta^\varepsilon(u,t) = \zeta_1(u) f_\delta(u) + (1 - \zeta_1(u)) \left[f^\varepsilon(u,t) + \varepsilon\beta) \right] \ , \tag{66}$$

where $0 < \beta \leq 1$. Using the assumptions on f^ε, it is easy to see that, for δ small enough, $\tilde{f}_\delta^\varepsilon$ has the same regularity properties as the f^ε and has exactly three zeros, $m_-^{\varepsilon,\alpha}(t) + O(\beta\varepsilon)$, $m_0 + \delta/2$, $m_+^{\varepsilon,\alpha} + O(\beta\varepsilon)$; moreover $\tilde{f}_\delta^\varepsilon \geq f^\varepsilon$ on $I\!R$ with $\tilde{f}_\delta^\varepsilon(u) = f^\varepsilon(u,t) + \varepsilon\beta$ if $|u - m_0| \geq 2\delta$ and $\tilde{f}_\delta^\varepsilon$ is independent of t for $|u - m_0| \leq \delta$. 2. Then we consider another cut-off function $\zeta_2 \in C_0^\infty(I\!R)$ such that $0 \leq \zeta_2 \leq 1$ in $I\!R$ $\zeta_2(u) = 0$ in $(-\infty, m_0 + \delta/4] \cup [m_0 + \delta, +\infty)$ and $\zeta_2(s) = 1$ in $[m_0 + \delta/3, m_0 + 2\delta/3]$. Finally we consider

$$\bar{f}_\delta^\varepsilon(u,t) = (1 - \zeta_2(u)) \tilde{f}_\delta^\varepsilon(u,t) + \zeta_2(u) \frac{\frac{\delta}{2} + m_0 - u}{|\log \varepsilon|}.$$

We note that, because again of the properties of f^ε, $\bar{f}_\delta^\varepsilon$ has exactly three zeros : $m_-^{\varepsilon,\alpha}(t) + O(\beta\varepsilon)$, $m_0 + \delta/2$, $m_+^{\varepsilon,\alpha}(t) + O(\beta\varepsilon)$; moreover, for ε small enough, $\bar{f}_\delta^\varepsilon \geq f^\varepsilon + \beta\varepsilon$ in $I\!R$ and $\bar{f}_\delta^\varepsilon = f^\varepsilon + \varepsilon\beta$ for $|u - m_0| \geq 2\delta$.
2. We consider the solution $\chi(\xi, \cdot, t)$ of the ode

$$\begin{cases} \dot{\chi} + \bar{f}_\delta^\varepsilon(\chi, t) = 0 \\ \chi(\xi, 0, t) = \xi \in I\!R, \end{cases} \tag{67}$$

3. Tedious computations show that χ satisfies

$$\chi_\xi(\xi, s, t) > 0 \quad \text{in } I\!R \times [0, +\infty) \times [0, +\infty),$$

$$\begin{cases} \text{for all } \beta > 0,\ T > 0, \text{ there exists } a(\beta, \delta, T) > 0 \text{ such that} \\ \\ \chi(\xi, s, t) \geq m_+^\varepsilon(t) - \beta\varepsilon \quad \text{for } s \geq a|\log\varepsilon| \text{ and } \xi \geq \delta + m_0 \\ \\ \qquad\qquad \text{for all } t \in [0, T], \end{cases} \tag{68}$$

and

$$\begin{cases} \text{for every } a, T > 0, \text{ there exists } M(a,T) \in I\!R \text{ such that, for } \varepsilon \text{ small enough,} \\ (\chi_\xi(\xi,s,t))^{-1}|\chi_{\xi\xi}(\xi,s,t)| \le \varepsilon^{-1}M(a,T) \quad \text{for } 0 < s \le a|\log\varepsilon|, \\ \qquad\qquad \text{for all } t \in [0,T]. \end{cases} \tag{69}$$

4. If δ is small enough, for every $a > 0$ and $T > 0$, there exists $\widetilde{M}(a,T) > 0$ such that, for ε small enough and for $0 < s \le a|\log\varepsilon|$, we have

$$|\chi_t(\xi,s,t)| \le \widetilde{M}(a,T)\varepsilon \ .$$

5. Let φ be a smooth function such that

$$-K \le \varphi \le m_0 + 2\delta \ \text{ in } I\!R, \quad \varphi(z) = K \ \text{ in } \{z < 0\} \ \text{ and } \ \varphi(z) = m_0 + 2\delta \ \text{ on } \{z \ge \delta\}. \tag{70}$$

Fix $t \in (t_0 - h, t_0 + h)$ and define $w \colon \overline{O} \times [0,T] \to I\!R$ by

$$w(x,s) = \chi(\varphi(d(x,s)) - \frac{M(s-t)}{\varepsilon}, \frac{s-t}{\varepsilon^2}, s).$$

Then w is a viscosity subsolution of 52(i)-(ii) in $O \times (t, t+t_\varepsilon)$, where $t_\varepsilon = a\varepsilon^2|\log\varepsilon|$.
6. By construction we have

$$u_\varepsilon(x,t) \ge (m_0 + 2\delta)1\!\!1_{\{d(x,t)\ge\delta\}} - K1\!\!1_{\{d(x,t)<\delta\}} \qquad \text{on } \overline{O} \ .$$

and on an other hand,

$$\begin{aligned} w(x,t) &= \chi(\varphi(d(x,t)), 0, x, t) = \varphi(d(x,t)) \\ &\le (m_0 + 2\delta)1\!\!1_{\{d(x,t)\ge\delta\}} + m_- 1\!\!1_{\{d(x,t)<\delta\}}. \end{aligned}$$

Thus, the maximum principle yields that

$$w(x,s) \le u_\varepsilon(x,s) \ \text{ on } \overline{O} \times [t, t + a\varepsilon^2|\log\varepsilon|] \ . \tag{71}$$

Evaluating (71) for $s = t + a\varepsilon^2|\log\varepsilon|$ and for $x \in \overline{O}$ such that $d(x,s) \ge \delta$ we get

$$\chi(m_0 + 2\delta - Ka\varepsilon|\ell n\varepsilon|, a|\log\varepsilon|, x, a\varepsilon^2|\log\varepsilon|) \le u_\varepsilon(x, a\varepsilon^2|\log\varepsilon| + t)$$

But, since for ε small enough

$$m_0 + 2\delta - Ka\varepsilon|\log\varepsilon| \ge m_0 + \delta,$$

it follows from (68) that

$$m_+^{\varepsilon,\alpha}(t+t_\varepsilon)+O(\beta\varepsilon)\le u_\varepsilon(x,t+t_\varepsilon)\quad \text{if } d(x,t+t_\varepsilon)\ge\delta\ .$$

7. Finally, because of the properties of $\bar f^\varepsilon_\delta$, if a is large we have also $\chi(\xi,a|\log\varepsilon|,x,a\varepsilon^2|\log\varepsilon|+t)\ge m_-^{\varepsilon,\alpha}(t+t_\varepsilon)+O(\beta\varepsilon)$ for all bounded ξ. Therefore for all $x\in\overline{O}$ we have

$$m_-^{\varepsilon,\alpha}(t+t_\varepsilon)+O(\beta\varepsilon)\le u_\varepsilon(x,t+t_\varepsilon)\quad \text{for any } x\in\overline{O}$$

and

$$[m_+^{\varepsilon,\alpha}(t+t_\varepsilon)+O(\beta\varepsilon)]1\!\!1_{\{d(x,t+t_\varepsilon)\ge\delta\}}+[m_-^{\varepsilon,\alpha}(x,t+t_\varepsilon)+O(\beta\varepsilon)]1\!\!1_{\{d(x,t+t_\varepsilon)<\delta\}}\le u_\varepsilon(x,t+t_\varepsilon).$$

Then the result holds for $\tau=a$ by taking $\beta<\delta$ small enough in order to replace if necessary $O(\beta\varepsilon)$ by $\beta\varepsilon$. $\square$

Proof of Lemma 3.6.

Also in this case, since the proof is very similar to the ones of Lemma 2.3 and Theorem 2.4 in [BDL], we just outline the kea ideas by refering the reader to [BDL] for the details.

We consider the smooth function Ψ given by (**??**) and we observe that for $C>0$ large enough one has for some $\varrho>0$

$$\frac{\partial\Psi}{\partial t}(x,s)+F^*(x,s,D\Psi,D^2\Psi,\mu(s))<-\varrho\ \text{ in } O\times(0,T)\ .$$

On an other hand, we have also

$$G(x,s,D\Psi(x,s))<-\varrho\ ,$$

on a ∂O–neighborhood of $\{\Psi=0\}$ and for small h. Using the smoothness of Ψ and the fact that, for small h, $D\Psi(x,s)\neq0$ if $\Psi(x,s)=0$, there exist $\gamma>0$ and $\bar h\le h$ such that d is smooth in the set $Q_{\gamma,\bar h}=\{(x,s)\ :\ |d(x,s)|\le\gamma,\ t_0-\bar h\le s\le t_0+\bar h\}$ and $|D\Psi|\neq0$ in $Q_{\gamma,\bar h}$. We note that on the set $\cup_{t_0-\bar h\le s\le t_0+\bar h}\{\Psi(x,s)=0\}$, d satisfies

$$\begin{aligned}&d_t+F^*(x,s,Dd,D^2d,\theta(s)-\alpha)=\\&d_t-\Delta d-c_0|Dd|[(m_+-m_-)(\theta(s)-\alpha)+m_-\lambda(O)]\le-\frac{\varrho}{2|D\Psi|}.\end{aligned}$$

Moreover recalling the properties of Ψ on ∂O, we have also

$$G(x,t,Dd)\le-\frac{\varrho}{2|D\Psi|}\ \text{ on } \partial O\cap Q_{\gamma,\bar h}\,. \tag{72}$$

We note that we can choose $C, \bar{h}$ such that $d(x_0, t_0) > 0$ and we may assume that

$$|Dd| = 1 \quad \text{and} \quad D^2 d Dd = 0 \quad \text{in } Q_{\gamma,\bar{h}} \ .$$

Let $(q^\varepsilon(r,s), c^\varepsilon(s))$ be the unique pair satisfying

$$q^\varepsilon_{rr}(r,s) + c^\varepsilon(s) q^\varepsilon_r = f^\varepsilon(q^\varepsilon, s). \tag{73}$$

We consider in $Q_{\gamma,\bar{h}}$ a function of the form

$$v^\varepsilon(x,s) = q^\varepsilon(\varepsilon^{-1}(d(x,s) - 2\beta), s) - 2\beta\varepsilon \tag{74}$$

We verify that v^ε is a viscosity solution of (52)(i)-(ii) in $Q_{\gamma,\bar{h}}$. As far as the boundary condition (52)(ii) is concerned, we first observe that $Dv(x,s) = \varepsilon^{-1} q_r Dd$ and thus $G(x,s,Dv(x,s)) = \varepsilon^{-1} q^\varepsilon_r G(x,s,Dd(x,s)) < 0$ because of (**A4**) and (72).

Moreover we have

$$v^\varepsilon_t - \Delta v^\varepsilon + b(x) \cdot Dv^\varepsilon + \varepsilon^{-2} f^\varepsilon(v^\varepsilon, s) = \varepsilon^{-2} \mathrm{I}_\varepsilon + \varepsilon^{-1} \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon \ , \tag{75}$$

where

$$\begin{aligned} \mathrm{I}_\varepsilon &= -q^\varepsilon_{rr} - c^\varepsilon(s) q^\varepsilon_r + f^\varepsilon(q^\varepsilon, s), \\ \mathrm{II}_\varepsilon &= q^\varepsilon_r(d_t - \Delta d + b(x) \cdot Dd + \varepsilon^{-1} c^\varepsilon(s)) - 2\beta f^\varepsilon_u(q^\varepsilon, s) \\ \mathrm{III}_\varepsilon &= q^\varepsilon_t + O(1). \end{aligned}$$

We note that $\varepsilon^{-1} c^\varepsilon(s) \to -c_0 \mu(s)$ as $\varepsilon \to 0$ locally uniformly in s and for all α.

By analogous computations to the one in [BS], one can see that if β is small enough then v^ε satisfies for some constant $\nu(\varrho, \beta) < 0$

$$v^\varepsilon_t - \Delta v^\varepsilon + b(x) \cdot Dv^\varepsilon + \varepsilon^{-2} f^\varepsilon(v,s) \leq \varepsilon^{-1} \nu(\varrho, \beta) + O(1) \ \text{ as } \varepsilon \to 0 \tag{76}$$

for all $(x,s) \in Q_{\gamma,h}$. Next we extend the subsolution v^ε to $\overline{O} \times (t_0 - t_\varepsilon, t_0 + \bar{h})$ and we do it in two steps.

First we have

Lemma 3.7 *For ε small enough, the functions $g^\varepsilon_\pm$ defined on $[0,T]$ by $g^{\varepsilon,\alpha}_\pm(s) = m^{\varepsilon,\alpha}_\pm(s) - \varepsilon\beta$ are viscosity subsolutions of (52)(i)-(ii).*

We leave the proof of Lemma 3.7 to the reader since it follows rather easily from the properties of f^ε, $m^{\varepsilon,\alpha}_+$ and $m^{\varepsilon,\alpha}_-$.

The next step is to define the function $\bar{v}^\varepsilon : \{(x,s) \in O \times [t_0 - t_\varepsilon, t_0 + \bar{h}] : \ d(x,s) \leq \gamma\} \to I\!R$ by

$$\bar{v}^\varepsilon(x,s) = \begin{cases} \sup(v^\varepsilon(x,s), g^{\varepsilon,\alpha}_-(s)) & \text{if } d(x,s) > -\gamma \\ g^{\varepsilon,\alpha}_-(s) & \text{otherwise} \end{cases}$$

By similar computations of Lemma 4.2 in [BS] and using Lemma 3.7, it is easy to prove that $\bar{v}^\varepsilon$ is a viscosity subsolution of (52)(i)-(ii).

Then we choose a smooth function $\varphi : I\!R \to I\!R$ such that $\varphi' \le 0$ in $I\!R$, $\varphi = 1$ in $(-\infty, \gamma/2)$, $0 < \varphi < 1$ in $(\gamma/2, 3\gamma/4)$, $\varphi = 0$ in $(3\gamma/4, +\infty)$, and, finally, $\varphi'' \le 0$ in a neighborhood of $\gamma/2$.

The function $w^{\varepsilon,\beta} : \overline{O} \times [t_0 - t_\varepsilon, t_0 + \bar{h}] \to I\!R$ defined by

$$w^{\varepsilon,\beta}(x,s) = \begin{cases} \varphi(d(x,s))\bar{v}^\varepsilon(x,s) + (1 - \varphi(d(x,s)))g_+^{\varepsilon,\alpha}(s) & \text{if } d(x,s) < \gamma, \\ \\ g_+^{\varepsilon,\alpha}(s) & \text{otherwise,} \end{cases}$$

is a viscosity subsolution of (52)-(10)(ii) on $O \times (t_0 - t_\varepsilon, t_0 + \bar{h})$, if ε and $\bar{h}$ are sufficiently small. Moreover

$$w^{\varepsilon,\beta}(\cdot, t_0 - t_\varepsilon) \le (m_+^{\varepsilon,\alpha}(t_0 - t_\varepsilon) - \beta\varepsilon)1\!\!1_{\{d(x,t_0-t_\varepsilon)\ge\beta\}} + (m_-^{\varepsilon,\alpha}(t_0 - t_\varepsilon) - \beta\varepsilon)1\!\!1_{\{d(x,t_0-t_\varepsilon)<\beta\}} \quad \text{in } \overline{O}$$

and if $s \in (t_0 - t_\varepsilon, t_0 + \bar{h})$, $x \in \overline{O} \cap B(x_0, r)$ and $d(x,s) > 2\beta$, then

$$\liminf{}_* \left[\frac{w^{\varepsilon,\beta}(x,s) - m_+^{\varepsilon,\alpha}(s)}{\varepsilon} \right] \ge -2\beta\ . \tag{77}$$

Thus we can conclude.

References

[AC] S. M. Allen and J. W. Cahn, A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, *Acta Metal.* **27** (1979), 1085–1095.

[AT] O. Alvarez and A. Tourin, Viscosity solutions of nonlinear integro-differential equations, *Ann. Inst. Henri Poincaré*, **13** (1996) 293-317.

[AW] D. G. Aronson and H. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.* **30** (1978), 33–76.

[B] G. Barles, Nonlinear Neumann boundary conditions for quasilinear degerate elliptic equations and applications, *J. Diff. Eqs.* **54** (1999) 191–224.

[BDL] G. Barles, F. Da Lio, A Geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions. To appear in Interfaces and Free Boundaries.

[BG] G. Barles and C. Georgelin, A simple proof for the convergence of an approximation scheme for computing mean curvature motions, *SIAM J. Num. Anal.* **32** (1995), 484–500.

[BSS] G. Barles, H. M. Soner and P. E. Souganidis, Front propagation and phase field theory, *SIAM J. Control. Opt.* **31** (1993), 439–469.

[BS] G. Barles, P.E. Souganidis, A new approach to front propagation problems : theory and applications, *Arch. Rational Mech. Anal.* **141** (1998), 237–296.

[Ca] P. Cardaliaguet, On front propagation problems with nonlocal terms, *Adv. Differential Equations* **5** (2000), no. 1-3, 213-268.

[C] X. Chen, Generation and propagation of interfaces in reaction-diffusion systems, *J. Diff. Eqs.* **96** (1992), 116–141.

[CHL] X. Chen, D. Hilhorst, E. Logak Asymptotic behavior of solutions of an Allen-Cahn equation with a nonlocal term. *Nonlinear Anal.* **28** (1997), no. 7, 1283–1298.

[CGG] Y.-G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Diff. Geom.* **33** (1991), 749–786.

[CIL] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. AMS* **27** (1992), 1–67.

[ES] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I, *J. Diff. Geom.* **33** (1991), 635–681.

[ESS] L. C. Evans, H. M. Soner and P. E. Souganidis, Phase transitions and generalized motion by mean curvature, *Comm. Pure Appl. Math.* **45** (1992), 1097–1123.

[F] P.C. Fife, Dynamics for Internal Layers and Diffusive Interfaces, in: *CCMS-NSF Regional Conf.Ser. in Appl.Math*, SIAM, Philadelphia,1988.

[FM] P.C Fife and B. McLeod, The approach of solutions of nonlinear differential diffusion equations to traveling solutions, *Arch. Rational Mech. Anal.,* **65** (1977), 335-361.

[FS] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, *Applications of Mathematics,* Springer-Verlag, New York, 1993.

[GS] Y. Giga and M.-H. Sato, Generalized interface evolution with Neumann boundary condition, *Diff.Int.Equ.***6** (1993)1217-1230

[HLS] D. Hilhorst, E. Logak, R. Schtzle, Global existence for a non-local mean curvature flow as a limit of a parabolic-elliptic phase transition model, *Interfaces Free Bound.* 2 (2000), no. 3, 267–282.

[IS] H. Ishii and M.-H. Sato, Nonlinear oblique derivative problems for singular degenerate parabolic equations on general domain, Nonlinear Analysis (to appear).

[K] Y. Koo, A fattening principle for fronts propagating by mean curvature plus a driving force, *Comm. Part. Diff. Equ.*, **24**, (1999) 1035-1053.

[KKR] M. A. Katsoulakis, G. Kossioris and F. Reitich, Generalized motion by mean curvature with Neumann conditions and the Allen-Cahn model for phase transitions, *J. Geom. Anal.*, **5**, No.2, (1995) 255-279.

[KRS] J. B. Keller, J. Rubinstein and P. Sternberg, Fast reaction, slow diffusion and curve shortening, *SIAM J. in Appl. Math.*, **49**, (1989), 116–133.

[OS] S. Osher and Sethian, Fronts moving with curvature dependent speed : algorithms based on Hamilton-Jacobi equations, *J. Comp. Phys.* **79** (1988), 12–49.

[Sl] D. Slepčev, Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions, *Nonlinear Anal.* **52** (2003) 79-115.

[S] H. M. Soner, Motion of a set by the curvature of its boundary, *J. Diff. Eqs.* **101** (1993), 313–372.

[SS] P. Soravia, P. E. Souganidis, Phase-field theory for FitzHugh-Nagumo-type systems, *SIAM J. Math. Anal.* 27 (1996), no. 5, 1341–1359. 35B25 (35B40 35K57)

[Sou1] P. E. Souganidis, Front propagation : Theory and Applications. In "Viscosity solutions and applications" Capuzzo Dolcetta, I. (ed.) et al. Lectures given at the 2nd session of the Centro Internazionale Matematico Estivo (CIME), Montecatini Terme, Italy, June 12-20, 1995. Berlin: Springer. Lect. Notes Math. 1660, (1997) 186-242.

[Sou2] P. E. Souganidis, Interface dynamics in phase transitions, *Proceeding ICM 94*, Birkhäuser Verlag, Basel, (1995), 1133–1144.