

CONFINEMENT IN NONLOCAL INTERACTION EQUATIONS

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ABSTRACT. We investigate some dynamical properties of nonlocal interaction equations. We consider sets of particles interacting pairwise via a potential W , as well as continuum descriptions of such systems. The typical potentials we consider are repulsive at short distances, but attractive at long distances. The main question we consider is whether an initially localized configuration remains localized for all times, regardless of the number of particles or their arrangement. In particular we find sufficient conditions on the potential W for the above “confinement” property to hold. We use the framework of weak measure solutions developed in [8] to provide unified treatment of both particle and continuum systems.

Keywords: nonlocal interactions, confinement, gradient flows, particle approximation.

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1. INTRODUCTION

We consider a mass distribution of particles, represented by a measure $\mu \geq 0$, under the effect of an even *interaction potential*, W . The case of finitely many particles corresponds to purely atomic measures: $\mu = \sum_{i=1}^N m_i \delta_{x_i}$. A measure μ which is absolutely continuous with respect to the Lebesgue measure represents a continuum distribution of particles. The velocity field is given by $v = -\nabla W * \mu$, which represents the combined contributions, at a given particle, of the interaction with all other particles. More precisely, we consider the continuity equation

$$\frac{\partial \mu}{\partial t} = \operatorname{div} [(\nabla W * \mu)\mu] \quad x \in \mathbb{R}^d, t > 0. \quad (1.1)$$

The equation is typically coupled with an initial datum $\mu(0) = \mu_0$.

It is known (cf. [1, 8]) that the equation (1.1) has the structure of a gradient flow of the *interaction energy* functional

$$\mathcal{W}[\mu] := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) d\mu(x) d\mu(y)$$

with respect to Wasserstein metric. The gradient flow structure is usually displayed via the formula

$$\frac{\partial \mu}{\partial t} = \operatorname{div} \left[\left(\nabla \frac{\delta \mathcal{W}}{\delta \mu} \right) \mu \right]$$

in which the symbol $\frac{\delta \mathcal{W}}{\delta \mu}$ represents the formal functional derivative of \mathcal{W} .

We shall be working with solutions for (1.1) in the sense introduced in [8] and briefly recapped in Subsection 3.1 below. Such a concept of solution allows μ to be an atomic

measure $\mu = \sum_{i=1}^N m_i \delta_{x_i(t)}$, representing N particles with masses $m_i > 0$ at location $x_i(t)$. In this case, the PDE is equivalent to the ODE system

$$\dot{x}_i = - \sum_{j \neq i} m_j \nabla W(x_i - x_j), \quad i = 1, \dots, N, \quad (1.2)$$

on any time interval for which C^1 solutions of the ODE exist. Particles collision in finite time is not ruled out for potentials W which are attractive and ‘singular’ at the origin. In this case the ODE system needs to be considered in a generalized sense as discussed later on.

Equations of the form (1.1) arise in several physical and biological contexts, with the choice of W depending on the phenomenon studied [2, 3, 4, 5, 6, 7, 9, 10, 15, 17, 18, 20, 23, 24, 25]. For instance, in population dynamics or collective motion of animals one is interested in the description of the evolution of a density of individuals. Very often the social interaction between two individuals only depends on the distance between them, which suggests a choice of W as a *radial* function, i.e.

$$W(x) = w(|x|).$$

The potential W is also often chosen to be attractive in the long range (modeling the fact that individuals want to remain in a cohesive group) and repulsive in the short range (modeling the fact that individuals repel each other when they are too close in order to avoid collision) [19, 20]. In the simplest situation, this means that

$$\begin{aligned} w'(r) &\leq 0 \text{ if } r < R_a \\ w'(r) &\geq 0 \text{ if } r \geq R_a \end{aligned}$$

for some threshold distance R_a . A recent numerical study of equation (1.1) in \mathbb{R}^2 has shown that such repulsive-attractive potentials lead to the emergence of surprisingly rich geometric structures [16]. Some of these patterns are reminiscent from patterns observed in experiments with bacterial colonies growing on agar plates. Many swarming systems with repulsive-attractive potentials have been studied. Some of these models are discrete, some other are continuous. Specific phase transitions, as well as, the shape of the patterns and the geometry of the steady states have been studied [19, 18, 11, 12, 9, 16, 21, 13, 14].

When considering models where individuals both attract and repulse one another, it is fundamental to understand whether or not the group will remain in a fixed bounded domain for all time or if it will expand and occupy larger and larger domains. This is the question of *confinement*. A potential W is said to be *confining* if given any compact domain there exists a ball of radius $R > 0$ such that for all initial data supported in the domain the solution of (1.1) remains supported in the ball of radius R for all time. In this paper we derive sufficient condition for a potential W to be confining. Loosely stated, our main result is the following: if there exists $C_W > 0$ such that

$$w'(r) \geq -C_W \text{ for all } r \leq R_a \quad (1.3)$$

$$\lim_{r \rightarrow +\infty} w'(r) \sqrt{r} = +\infty \quad (1.4)$$

then the potential $W(x) = w(|x|)$ is confining. Inequality (1.3) means that the ‘repulsion strength’ between two particles is always bounded above by C_W . Inequality (1.4) means that the ‘attraction strength’ between two particles does not decay too fast as

these particles get further and further apart, being $r^{-1/2}$ the critical decay rate in our proof. This specific balance between the “attraction strength” in the long range and the “repulsion strength” in the short range allows us to prove confinement.

Note that (1.4) does not prevent the “attraction strength” to go to zero at infinity. In this case we say that the potential is *weakly attractive* at infinity. On the other hand, if the attraction strength $w'(r)$ is greater than $4C_W$ for r large enough we say that the potential is *strongly attractive at infinity*. For potential which are strongly attractive at infinity we are able to derive a better a priori bound on the final size of the support of the solution than the one obtain with potentials that are only weakly attractive at infinity.

In [8], we developed a theory of well-posedness for measure solutions to (1.1). This theory allows to treat particle and continuum solutions at the same level. Moreover, the explicit bounds on the continuous dependence with respect to the initial data allow to approximate continuum solutions by particle or atomic measure solutions. Therefore, the strategy that we follow in this work is the following: in Section 2 we derive confinement results for the particle system (1.2) independent of the number of particles, and then, in Section 3, we use the existence and stability theory of [8] to pass to the continuum limit these confinement results.

Let us emphasize that (1.3)-(1.4) are the key hypotheses to obtain confinement. On the other hand, to obtain well-posedness of measure solutions, the exact set of hypotheses used in [8] is the following:

$$W(x) = W(-x) \text{ for all } x \in \mathbb{R}^d \text{ with } W \in C^1(\mathbb{R}^d \setminus \{0\}), \quad (1.5)$$

$$W \text{ is } \lambda\text{-convex for some } \lambda \leq 0, \text{ i.e. } W(x) - \frac{\lambda}{2}|x|^2 \text{ is convex}, \quad (1.6)$$

$$\text{There exists a constant } C > 0 \text{ such that } W(z) \leq C(1 + |z|^2), \text{ for all } z \in \mathbb{R}^d. \quad (1.7)$$

Note that (1.6) implies that the potential is Lipschitz at the origin, which has to be a local minimum if the potential W is not C^1 .

2. CONFINEMENT FOR PARTICLES

In this section we derive sufficient condition on the potential W so that the particle system (1.2) remains confined for all times. We warn the reader that additional conditions, (1.5)-(1.7), on W will be needed in next section to extrapolate these confinement results to the continuum model. Consider the system of particles $x_1, \dots, x_N \in \mathbb{R}^d$ the dynamics of which are described by

$$\dot{x}_i = - \sum_{j=1}^N m_j \partial^0 W(x_i - x_j) \quad i = 1, \dots, N \quad (2.1)$$

with m_i 's positive and satisfying $\sum_{i=1}^N m_i = 1$. The notation $\partial^0 W$ stands for

$$\partial^0 W(x) := \begin{cases} \nabla W(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We will assume in this section that the potential W is radially symmetric and continuously differentiable away from the origin:

$$W(x) = w(|x|) \quad \text{and} \quad w \in C^1(\mathbb{R} \setminus \{0\}). \quad (2.2)$$

We also assume that there exists a ball of finite radius such that W is attractive outside of this ball. Inside this ball W can be both repulsive and attractive, but its repulsive “strength” is bounded. To be more precise we assume that there exists constants $R_a > 0$ and $C_W > 0$ such that

$$w'(r) > 0 \quad \text{for } r > R_a, \quad (2.3)$$

$$w'(r) > -C_W \quad \text{for } 0 < r < R_a. \quad (2.4)$$

Finally, since the center of mass $\sum_{i=1}^N m_i x_i(t)$ is preserved by (2.1), we assume without loss of generality that

$$\sum_{i=1}^N m_i x_i(t) = 0 \quad \text{for all } t \geq 0$$

and we define $r(t)$ to be the radius of the cloud of particles:

$$r(t) = \max\{|x_1(t)|, \dots, |x_N(t)|\}.$$

Remark 2.1. Under assumptions (2.2), (2.3), (2.4) on W , solutions of (2.1) exist globally in time and are unique forward in time, provided one make a “suitable” choice at collision times. Indeed as long as particles do not collide the velocity field is Lipschitz continuous. Even if it is possible for particles to collide in finite time, since there is a finite number of particles there can only be a finite number of collision times, and therefore the system of ODE is well defined in the time intervals between these collision times. Now, if two particles collide at time t^* , then we assume that they stick together for $t \geq t^*$ (this may not be the only possibility for extending the solution without some assumption on w near the origin; for instance if one assumes (1.6)). Because of collisions, the solution of (2.1) is “unique” (in the sense described above) only forward in time, but not backward in time. Proposition (2.2) stated below guarantees that the radius $r(t)$ of the cloud of particles grows at most linearly with respect to time, therefore particles can not reach infinity in finite time: this gives global existence in time.

We now state the main results of this section, postponing all the proofs to the end of this section.

Proposition 2.2. *Suppose W satisfies (2.2), (2.3) and (2.4). For $r > 2R_a$ define*

$$\sigma(r) := \inf_{2r \geq s \geq r/2} w'(s) \geq 0. \quad (2.5)$$

If $r(t) > 2R_a$ then

$$\frac{d^+}{dt} r(t) := \limsup_{h \rightarrow 0^+} \frac{r(t+h) - r(t)}{h} \leq -\frac{\sigma(r(t))}{6} + \frac{2}{3}C_W.$$

As a corollary we get an explicit bound for the radius of a cloud of particles evolving under the influence of a strongly attractive potential at infinity (i.e. its attractive strength at infinity is at least four times larger than its maximum repulsive strength).

Corollary 2.3. *Suppose W satisfies (2.2), (2.3) and (2.4). If there exists \bar{R} such that*

$$w'(r) \geq 4C_W \quad \text{for all } r \geq \bar{R}, \quad (2.6)$$

then there exists $R^ > 0$ depending only $r(0)$, \bar{R} , R_a , and W such that $r(t) \leq R^*$ for all $t \geq 0$.*

The next result states that potentials which are weakly attractive at infinity (i.e. their attractive strength goes to zero as $r \rightarrow +\infty$) can also be confining, as long as the rate of decay at infinity of the attractive strength is not too rapid. Here $w'(r) \sim r^{-1/2}$ is the threshold decay rate at infinity. In our proof, which is based on energetic arguments, it is essential for the potential W to be bounded on compact sets.

Proposition 2.4. *Suppose that, in addition to (2.2), (2.3) and (2.4), W satisfies*

$$\liminf_{r \rightarrow 0} w(r) > -\infty \quad \text{and} \quad \lim_{r \rightarrow \infty} w'(r)\sqrt{r} = +\infty, \quad (2.7)$$

then there exists $\bar{R} > 0$ depending only $r(0)$ and W such that $r(t) \leq \bar{R}$ for all $t \geq 0$.

Remark 2.5. In Corollary 2.3 an explicit bound for the radius of the support of the cloud of particles is provided. In the proof of Proposition 2.4 we also derive an explicit bound for the radius of the support. However, for strongly attractive potential at infinity which also satisfies $\liminf_{r \rightarrow 0} w(r) > -\infty$, the bound provided by Corollary 2.3 is better than the one provided in the proof of Lemma 2.4.

Remark 2.6. Let us remark that conditions (2.2), (2.3) and (2.4) alone are not enough for confinement. Counterexamples follow from the work by Theil [22].

The rest of the section is devoted to the proofs of all the above results.

Proof of Proposition 2.2 and Corollary 2.3. Since for any $t \geq 0$ (even a collision time) there exists Δt such that on $[t, t + \Delta t)$ the ODE system has a C^1 solution, it suffices to provide the proof at $t = 0$, under the assumption that $r_0 := r(0) > 2R_a$. From the definition of $r(t)$ we easily get

$$\frac{d^+}{dt} r^2(0) = \max_{\{i: |x_i|=r_0\}} -2 \sum_{j \neq i} m_j \frac{(x_i - x_j) \cdot x_i}{|x_i - x_j|} w'(|x_i - x_j|). \quad (2.8)$$

From now on, let us drop the time dependence on the particles. We can assume that the maximum is achieved for $i = 1$ and that $x_1 = |x_1| e_1 = r_0 e_1$. Let J_R be the set of indices of particles that are within the radius R_a of x_1 , and thus may be pushing x_1 away. Let $J_A = \{j : x_j \cdot e_1 < \frac{r_0}{2}\}$ be the indices of particles that are “strongly attracting” x_1 . Let $J_{rest} = \{1, \dots, N\} \setminus (J_R \cup J_A)$.

Since the center of mass is 0, $\sum_{j=1}^N m_j x_j \cdot e_1 = 0$, then we deduce that

$$r_0 \sum_{j \in J_A} m_j \geq - \sum_{j \in J_A} m_j x_j \cdot e_1 = \sum_{j \in J_R \cup J_{rest}} m_j x_j \cdot e_1 \geq \frac{r_0}{2} \sum_{j \in J_R \cup J_{rest}} m_j.$$

Let $m_A = \sum_{j \in J_A} m_j$ and $m_R = \sum_{j \in J_R} m_j$. It follows that

$$m_A \geq \frac{1}{2} \sum_{j \in J_R \cup J_{rest}} m_j = \frac{1}{2} (1 - m_A). \quad (2.9)$$

Hence $m_A \geq \frac{1}{3}$ and $m_R \leq \frac{2}{3}$.

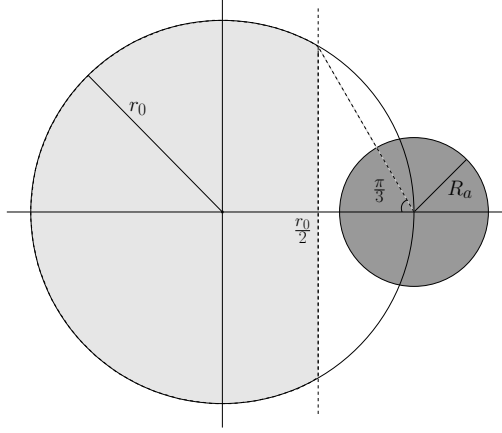


FIGURE 1. Sketch of the geometrical distribution of the location of particles.

From (2.8), (2.4), (2.3) and (2.5), and since $(x_i - x_j) \cdot x_i \geq 0$, it follows that at $t = 0$

$$\begin{aligned} \frac{1}{2} \frac{d^+}{dt} r^2(0) &\leq - \sum_{j \in J_R \cup J_A} m_j \frac{(x_1 - x_j) \cdot x_1}{|x_1 - x_j|} w'(|x_1 - x_j|) \\ &\leq \sum_{j \in J_R} m_j r_0 C_W - \sum_{j \in J_A} m_j \cos\left(\frac{\pi}{3}\right) r_0 \sigma(r_0) \\ &\leq \frac{2}{3} r_0 C_W - \frac{1}{6} r_0 \sigma(r_0), \end{aligned}$$

where in the last two steps we have used that the maximum angle between $x_1 - x_j$ for $j \in J_A$ and x_1 is $\pi/3$ as depicted in Figure 1. Dividing by r_0 establishes the claim of Proposition 2.2 at $t = 0$, which, as we remarked before, implies the claim for arbitrary $t \geq 0$.

Now, let us show Corollary 2.3. Let \bar{r} be a solution of

$$\begin{cases} \frac{d\bar{r}}{dt} = -\frac{\sigma(\bar{r})}{6} + \frac{2}{3} C_W \\ \bar{r}(0) = r_0. \end{cases}$$

Since σ is a continuous function a solution exists. If the solution is nonunique, we choose the maximal solution.

Consequently for all $t \geq 0$, $r(t) \leq \max\{\bar{r}(t), 2R_a\}$. Now, since by assumption $\sigma(r) \geq 4C_W$ for $r \geq \bar{R}$, we get $\frac{d\bar{r}}{dt} \leq 0$ whenever $\bar{r} \geq \bar{R}$. This implies $\bar{r}(t) \leq \max\{\bar{r}(0), \bar{R}\}$, so $r(t) \leq \max\{r(0), \bar{R}, 2R_a\}$ for $t \geq 0$, as desired. \square

We now turn to the proof of Proposition 2.4. Compared to the previous proof, here we will make use of the fact that the system of ODE (2.1) is a gradient flow of the interaction energy

$$\mathcal{W}[x_1, \dots, x_N] = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N m_j m_k W(x_j - x_k). \quad (2.10)$$

The assumption that $W(x)$ remain bounded from below as $x \rightarrow 0$ guarantees that the interaction energy is finite for all time, even if collisions take place. The idea of the proof is as follow: Note that there are no direct energetic obstacles to prevent the support of the solution from becoming large. That is the boundedness of the interaction energy does not prevent a particle from traveling far from the origin, as long as its mass is small. However it turns out that for even a small particle to go far from the center of mass, there must exist significant mass nearby. That is for the small particle to go far, there must be particles of relatively large total mass which are “pushing” it out. However the existence of a “large” mass far from the center of mass does violate the fact that the energy is bounded.

Proof of Proposition 2.4. Since $w(r)$ is increasing for r large enough, and since it does not diverge as $r \rightarrow 0$, it is bounded from below and we can assume without loss of generality that $w(r) \geq 0$ for all $r > 0$ by adding a suitable constant to w . Let

$$\theta(r) := \inf_{s \geq r} w'(s) \sqrt{s}.$$

Let $r_0 := r(0)$ and let \bar{R} be any number such that

$$\bar{R} \geq 6R_a, \bar{R} \geq r_0, \text{ and } \theta\left(\frac{\bar{R}}{6}\right) > 6^{5/4} \sqrt{C_W \|W\|_{L^\infty(B(0, 2r_0))}}. \quad (2.11)$$

Note that it is possible to choose such an \bar{R} because of (2.7).

Let us observe that for any $r > 2R_a$

$$w(r) \geq \frac{\sqrt{r}}{2} \theta\left(\frac{r}{2}\right). \quad (2.12)$$

This follows by noting that $w'(s) \geq \theta(r/2)/\sqrt{s} \geq \theta(r/2)/\sqrt{r}$ for all $s \in (r/2, r)$, integrating from $r/2$ to r , and using that $w(r/2) \geq 0$.

Assume that the statement of the proposition does not hold. Let t_1 be the first time at which a particle reaches the distance \bar{R} from the origin. Consider the ODE system (2.1) in which this particle is identified as $x_1(t_1)$ and assume without loss of generality that $x_1(t_1) = |x_1(t_1)|e_1$.

We can also assume without loss of generality that there are no collisions at time t_1 , that is that the ODE system has C^1 solutions on the time interval $(t_1 - \Delta t, t_1 + \Delta t)$, for Δt small enough. Indeed if there is a collision at time t_1 , we can always replace \bar{R} by $\bar{R} + \Delta R$. Since we have assumed that the claim of the Lemma doesn't hold the radius of the support will eventually reach $\bar{R} + \Delta R$, and since there are finitely many collisions (if any) one can choose ΔR so that there are no collisions at the first time the support reaches $\bar{R} + \Delta R$. By the choice of t_1 , we have that $|x_1(t_1)| = \bar{R}$ and

$$\frac{1}{2} \frac{d^+}{dt} r^2(t_1) = \dot{x}_1(t_1) \cdot x_1(t_1) \geq 0.$$

Therefore, we deduce

$$-\sum_{j \geq 2} m_j \nabla W(x_1(t_1) - x_j(t_1)) \cdot x_1(t_1) \geq 0. \quad (2.13)$$

Let J_R , J_A , and J_{rest} be as in the proof of Lemma 2.2: $J_R = \{j : x_j(t_1) \in B(x_1(t_1), R_a)\}$, $J_A = \{j : x_j(t_1) \cdot e_1 < \frac{\bar{R}}{2}\}$, and $J_{rest} = \{1, \dots, N\} \setminus (J_R \cup J_A)$ with the geometrical

interpretation of Figure 1. Arguing as for (2.9) one obtains $m_A \geq \frac{1}{3}$. Thus from (2.13) and proceeding as in the proof of Proposition 2.2, it follows that

$$\begin{aligned} m_R C_W &\geq \sum_{j \in J_R} -m_j \nabla W(x_1 - x_j) \cdot e_1 \geq \sum_{j \in J_A} m_j \nabla W(x_1 - x_j) \cdot e_1 \\ &\geq \frac{1}{2} \sum_{j \in J_A} m_j w'(|x_1 - x_j|) \geq \frac{1}{6\sqrt{2\bar{R}}} \theta\left(\frac{\bar{R}}{2}\right). \end{aligned} \quad (2.14)$$

To obtain the last inequality we have used the fact that for all $\bar{R}/2 \leq s \leq 2\bar{R}$, $w'(s) \geq \theta(\bar{R}/2)/\sqrt{s} \geq \theta(\bar{R}/2)/\sqrt{2\bar{R}}$. The above computation gives a lower bound on the mass m_R of particles repulsing the particle the furthest away. It shows that, in order for the particle the furthest away to be pushed even further, there must be significant mass nearby.

We now turn toward energetic arguments. Note that at time 0 the interaction energy (2.10) satisfies

$$2\mathcal{W}[x_1, \dots, x_N] \leq \|W\|_{L^\infty(B(0, 2r_0))}.$$

On the other hand at time t_1 , using the positivity of W together with the fact that $\bar{R} > 6R_a$, we get that

$$2\mathcal{W}[x_1, \dots, x_N] \geq \sum_{j \in J_R} \sum_{k \in J_A} m_j m_k W(x_j - x_k) \geq \frac{m_R}{3} \inf_{r \geq \bar{R}/3} w(r) = \frac{m_R}{3} w(\bar{R}/3)$$

where we have used the fact that if $\bar{R} > 6R_a$, then particles in J_A are at least at a distance $\bar{R}/3$ from particles in J_R . Since $\bar{R}/3 \geq 2R_a$ we will be able to use (2.12). Since the interaction energy is decreasing as a function of time, we conclude that

$$\begin{aligned} \|W\|_{L^\infty(B(0, 2r_0))} &\geq \frac{m_R}{3} w(\bar{R}/3) \\ &\geq \frac{1}{3} \left[\frac{1}{6\sqrt{2\bar{R}}} \theta\left(\frac{\bar{R}}{2}\right) \right] \left[\frac{\sqrt{\bar{R}/3}}{2} \theta\left(\frac{\bar{R}}{6}\right) \right] \geq \frac{1}{6^{5/2} C_W} \theta\left(\frac{\bar{R}}{6}\right)^2, \end{aligned}$$

which contradicts (2.11). \square

3. CONFINEMENT FOR GENERAL MEASURE SOLUTIONS

In this section we use the theory developed in [8] in order to pass to the limit the confinement results derived in the previous section for particles. We start with a short summary of the results of [8].

3.1. Weak measure solutions. We shall briefly resume here the weak measure solution theory for the equation (1.1) developed in [1, 8]. We shall work in the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on \mathbb{R}^d , thus normalizing the total mass to 1. This is not restrictive in view of the following invariance property: if $\mu(t)$ is a solution, so is $M\mu(Mt)$ for all $M > 0$. We additionally require our measure solution to belong to the metric space

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 d\mu(x) < +\infty \right\}$$

of probability measures with finite second moment, endowed with the 2-Wasserstein distance d_W (see [1, 26] for further details).

Definition 3.1 (Weak measure solutions). *A locally absolutely continuous curve*

$$\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$$

is a weak measure solution to (1.1) with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 W * \mu$ belongs to $L^1_{loc}([0, +\infty); L^2(\mu(t)))$ and

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial t}(x, t) d\mu(t)(x) dt + \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) = \\ \int_0^{+\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla \varphi(x, t) \cdot \partial^0 W(x - y) d\mu(t)(x) d\mu(t)(y) dt, \end{aligned} \quad (3.1)$$

for all test functions $\varphi \in C_c^\infty([0, +\infty) \times \mathbb{R}^d)$.

The case of a measure $\mu(t)$ given by a finite combination of Dirac deltas centered at $x_i(t)$, $i = 1, \dots, N$ solving (2.1) is included in the notion of solution provided in Definition 3.1 (see [8, Remark 2.10]).

The following result is a combination of [8, Theorems 2.12 and 2.13]:

Theorem 3.2 (Existence and d_W -Stability). *Let W satisfy the assumptions (1.5), (1.6) and (1.7). Then, there exists a unique weak measure solution to (1.1) in the sense of Definition 3.1. Moreover, given two weak measure solutions $\mu^1(t)$ and $\mu^2(t)$, we have*

$$d_W(\mu^1(t), \mu^2(t)) \leq e^{-\lambda t} d_W(\mu_0^1, \mu_0^2) \quad (3.2)$$

for all $t \geq 0$.

3.2. Confinement. We are now ready to state and prove the two main theorems of this paper. As in the proof of Corollary 2.3, let us consider $\bar{r}(t)$ to be the maximal solution of

$$\begin{cases} \frac{d\bar{r}}{dt} = -\frac{\sigma(\bar{r})}{6} + \frac{2}{3}C_W, \\ \bar{r}(0) = r_0 \end{cases},$$

with the considerations done there.

Theorem 3.3. *Assume W satisfies (1.5)–(1.7) as well as (2.2)–(2.4). Let μ_0 be a compactly supported probability measure with radius of support $r_0 > 0$. Let $\mu(t)$ be the solution to (1.1) and $r(t)$ its radius of support, then $r(t) \leq \max\{\bar{r}(t), 2R_a\}$. Moreover, if W also satisfies (2.6), then $r(t) \leq R^*$ for all $t \geq 0$.*

Remark 3.4. Of course (2.2) implies (1.5). Also, (1.6) implies (2.4). We choose to write it like this in order to separate the hypotheses necessary for well-posedness of measure solutions from the ones necessary for confinement.

Proof. We can assume without the loss of generality that μ_0 has center of mass 0. Since \mathcal{W} is translation invariant, $\mu(t)$ remains centered at 0 for all $t > 0$. Let us remark that the claims of the Theorem hold in the case of an initial data formed by a finite number of atoms due to Proposition 2.2 and Corollary 2.3.

To show the first claim for general initial data, let us consider a sequence of particle approximations $\mu_n(0)$ of $\mu(0)$ satisfying

$$\mu_n(0) = \sum_{i=1}^n m_{n,i} \delta_{x_{n,i}(0)}, \quad m_{n,i} > 0, \quad \sum_{i=1}^n m_{n,i} = 1 \quad (3.3)$$

$$|x_{n,i}(0)| < r_0 \text{ for all } n \text{ and } i \text{ with } \sum_{i=1}^n m_{n,i} x_{n,i} = 0 \quad (3.4)$$

$$\lim_{n \rightarrow \infty} d_w(\mu_n(0), \mu(0)) = 0 \quad (3.5)$$

Then, by stability of solutions given in Theorem 3.2, given any $t > 0$ we have

$$\lim_{n \rightarrow \infty} d_w(\mu_n(t), \mu(t)) = 0. \quad (3.6)$$

Reasoning as in the proof of corollary 2.3 we deduce that the support of $\mu_n(t)$ is contained in $\bar{B}(0, \max\{\bar{r}(t), 2R_a\})$ for all $t \geq 0$. Because of (3.6) this implies that the support of $\mu(t)$ must also be contained in $\bar{B}(0, \max\{\bar{r}(t), 2R_a\})$ for all $t \geq 0$. The second claim follows analogously using Corollary 2.3. \square

Theorem 3.5. *Assume W satisfies (1.5)–(1.7) as well as (2.2)–(2.4) together with (2.7). Then, given a compactly supported probability measure μ_0 with center of mass at x_0 such that $\text{supp } \mu_0 \subset \bar{B}(x_0, r_0)$, there exists $\bar{R} \geq r_0$, depending only on r_0 and W , such that the solution $\mu(t)$ to (1.1) satisfies*

$$\text{supp } \mu(t) \subset \bar{B}(x_0, \bar{R}) \quad \text{for all } t \geq 0.$$

Proof. As before we can assume that μ_0 has center of mass 0, which implies that for all times $\mu(t)$ has center of mass 0 as well. As in the proof of Theorem 3.3 we consider a sequence of particle approximations $\mu_n(0)$ of $\mu(0)$ satisfying (3.3)–(3.5). Because of Lemma 2.4 the claim of the theorem holds for each particle approximation. Therefore, due to (3.6), it also holds for the limit $\mu(t)$. \square

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