

AVERAGE-DISTANCE PROBLEM FOR PARAMETERIZED CURVES

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ABSTRACT. We consider approximating a measure by a parameterized curve subject to length penalization. That is for a given finite compactly supported measure μ , with $\mu(\mathbb{R}^d) > 0$ for $p \geq 1$ and $\lambda > 0$ we consider the functional

$$E(\gamma) = \int_{\mathbb{R}^d} d(x, \Gamma_\gamma)^p d\mu(x) + \lambda \text{Length}(\gamma)$$

where $\gamma : I \rightarrow \mathbb{R}^d$, I is an interval in \mathbb{R} , $\Gamma_\gamma = \gamma(I)$, and $d(x, \Gamma_\gamma)$ is the distance of x to Γ_γ .

The problem is closely related to the average-distance problem, where the admissible class are the connected sets of finite Hausdorff measure \mathcal{H}^1 , and to (regularized) principal curves studied in statistics. We obtain regularity of minimizers in the form of estimates on the total curvature of the minimizers. We prove that for measures μ supported in two dimensions the minimizing curve is injective if $p \geq 2$ or if μ has bounded density. This establishes that the minimization over parameterized curves is equivalent to minimizing over embedded curves and thus confirms that the problem has a geometric interpretation.

Keywords. average-distance problem, principal curves, nonlocal variational problems

Classification. 49Q20, 49K10, 49Q10, 35B65

1. INTRODUCTION

Approximating measures by one dimensional objects arises in several fields. In the setting of optimization problems connected to network planning (such as for urban transportation network) and irrigation it was introduced by Buttazzo, Oudet and Stepanov [3], and has been extensively studied [4, 5, 6, 7, 8, 9, 16, 18, 21].

In this setting the problem is known as the *average-distance problem*. Given a set $\Sigma \subset \mathbb{R}^d$ let $d(x, \Sigma) = \inf_{y \in \Sigma} |x - y|$. Let \mathcal{M} be the set of finite, compactly supported measures in \mathbb{R}^d for $d \geq 2$, with $\mu(\mathbb{R}^d) > 0$.

Problem 1.1. *Given a measure $\mu \in \mathcal{M}$, and parameters $p \geq 1$, $\lambda > 0$, we consider the average-distance problem in the penalized (as opposed to constrained) form: minimize*

$$G_\mu^{\lambda,p}(\Sigma) := \int_{\mathbb{R}^d} d(x, \Sigma)^p d\mu(x) + \lambda \mathcal{H}^1(\Sigma),$$

with the unknown Σ varying in the family

$$\mathcal{A} := \{\Sigma \subseteq \mathbb{R}^d : \Sigma \text{ compact, path-wise connected and } \mathcal{H}^1(\Sigma) < \infty\}.$$

Another application in which a measure is to be approximated by a one-dimensional object arises in machine learning and statistics where one wishes to obtain the curve that best represents the data given by a (probability) measure μ . The problem in this setting was introduced by

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Hastie [12] and Hastie and Stuetzle [13], and its solution is known as the (regularized) principal curve. A variant of the problem can be formulated as follows: let

$$\mathcal{C} := \{\gamma : [0, a] \rightarrow \mathbb{R}^d : a \geq 0, \gamma \text{ is Lipschitz with } |\gamma'| \leq 1 \text{ a.e.}\}.$$

For given $\gamma \in \mathcal{C}$, we define its length, $L(\gamma)$, as its total variation $\|\gamma\|_{TV([0,a])}$. Furthermore given $\gamma \in \mathcal{C}$ we define $\Gamma_\gamma = \gamma([0, a])$. The problem can be stated as follows:

Problem 1.2. *Given a measure $\mu \in \mathcal{M}$, parameters $\lambda > 0$, $p \geq 1$ find $\gamma \in \mathcal{C}$ minimizing*

$$E_\mu^{\lambda,p}(\gamma) := \int_{\mathbb{R}^d} d(x, \Gamma_\gamma)^p d\mu(x) + \lambda L(\gamma).$$

We remark that in machine learning the problem has been considered most often with $p = 2$, with a variety of regularizations, as well as with length constraint (instead of length penalization) [15, 23, 24]. Regularizing with a length term is the lowest order (in other words the weakest) of regularizations considered. We note that the first term of energy measures the approximation error while the second term penalizes the complexity of the approximating object (curve).

The existence of minimizers of Problem 1.2 in the class of parameterized curves follows from Ascoli-Arzelà theorem and the continuity of the functional with respect to the uniform convergence (Lemma 2.2). However it is not clear if for a general measure μ the minimizing curve is injective, in other words it may have self-intersections and not be an embedded curve. Here we show that in two dimensions if $p \geq 2$ then the minimizer in fact is an injective curve. We also show that if μ has bounded density with respect to Lebesgue measure then the minimizer is an injective curve for all $1 \leq p < \infty$. More precisely the main result of our work is:

Theorem 1.3. *Consider dimension $d = 2$. Let $\mu \in \mathcal{M}$ and let $\lambda > 0$ and $p \geq 1$. If $p < 2$ assume that μ is absolutely continuous with respect to Lebesgue measure and that its density, ρ , is bounded. Let $\gamma : [0, L] \rightarrow \mathbb{R}^2$ be an arc-length-parameterized minimizer of $E_\mu^{\lambda,p}$. Then γ is injective and in particular Γ_γ is a curve embedded in \mathbb{R}^2 .*

The theorem implies that the problem can also be posed as a minimization problem among embedded curves. We note that, as we discuss at the beginning of Section 4, the conclusion of the theorem holds for all $1 \leq p < \infty$ if μ is a discrete measure.

We hypothesize that the range of p in the theorem is sharp:

Conjecture 1.4. *For $1 \leq p < 2$ there exist $\lambda > 0$ and measure $\mu \in \mathcal{M}$ for which the global minimizer is not injective.*

The proof of the Theorem relies on regularity properties of minimizers, which are of interest in their own right. In [22] it was shown that for $p = 1$ minimizers of the Problem 1.1, even for measures with smooth densities, can be embedded curves which have a corner. Since these are also minimizers of Problem 1.2, we conclude that minimizers of Problem 1.2 are not C^1 curves in general. We remark that results of [22] can be extended to $p \geq 1$ to show existence of global minimizers which have a corner.

Consequently we consider regularity of minimizers in the sense of obtaining estimates on the total variation of γ' , where γ is an arc-length-parameterized minimizer. This allows us to consider the curvature as a measure and provides bounds on the total curvature of a segment of

the minimizing curve in terms of the mass projecting on the segment. To do so we use techniques developed in [18].

This paper is structured as follows:

- in Section 2 we present preliminary notions and results, and prove existence of minimizers of Problem 1.2. We furthermore show that the minimizers are contained in the convex hull of the support of the measure μ ;
- in Section 3 we prove the injectivity of minimizers (Theorem 1.3) in the two-dimensional case;
- in Section 4 we extend the regularity estimates of [18] to $p > 1$ and prove them in the setting of parameterized curves. We furthermore provide the version of estimates in \mathbb{R}^2 which roughly speaking bounds how much a minimizer can turn to the left by the mass to the right of the curve. This is a key result needed to prove injectivity.

2. PRELIMINARIES

In this section we provide some preliminary results including the proof of existence of minimizers of Problem 1.2 (Lemma 2.2).

We define the distance between curves in \mathcal{C} as follows: Let $\gamma_1, \gamma_2 \in \mathcal{C}$ with domains $[0, a_1], [0, a_2]$ respectively. We can assume that $a_1 \leq a_2$. Let $\tilde{\gamma}_1 : [0, a_2] \rightarrow \mathbb{R}^d$ be the extension of γ_1 to $[0, a_2]$ as follows

$$(2.1) \quad \tilde{\gamma}_1(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, a_1], \\ \gamma_1(a_1) & \text{if } t \in (a_1, a_2]. \end{cases}$$

Let

$$d_{\mathcal{C}}(\gamma_1, \gamma_2) = \max_{t \in [0, a_2]} |\tilde{\gamma}_1(t) - \gamma_2(t)|.$$

The first issue is the existence of minimizers. A preliminary lemma is required. Given a measure $\mu \in \mathcal{M}$, and $p \geq 1$, let

$$F_{\mu}^p : \mathcal{A} \longrightarrow [0, \infty), \quad F_{\mu}^p(\Sigma) := \int_{\mathbb{R}^d} d(x, \Sigma)^p d\mu(x).$$

Lemma 2.1. *Given a measure $\mu \in \mathcal{M}$, parameters $\lambda > 0, p \geq 1$, then for any minimizing sequence $\{\gamma_n\}$ of Problem 1.2 it holds:*

- *length estimate:*

$$\limsup_{n \rightarrow \infty} L(\gamma_n) \leq \frac{|\mu(\mathbb{R}^d)|}{\lambda} (\text{diam supp}(\mu))^p,$$

- *confinement condition: there exists a compact set $K \subseteq \mathbb{R}^d$ such that $\Gamma_{\gamma_n} \subseteq K$ for all n .*

Proof. Boundedness of the length is obtained by using a singleton as a competitor. Fix an arbitrary point $z \in \text{supp}(\mu)$, and let $\gamma : [0, 0] \rightarrow \{z\}$. Then

$$(2.2) \quad \inf_{\mathcal{C}} E_{\mu}^{\lambda, p} \leq E_{\mu}^{\lambda, p}(\gamma) \leq \int_{\mathbb{R}^d} |x - z|^p d\mu(x) \leq |\mu(\mathbb{R}^d)| (\text{diam supp}(\mu))^p.$$

Since $\{\gamma_n\}$ is a minimizing sequence, (2.2) gives

$$(2.3) \quad (\forall \varepsilon)(\exists n_0)(\forall n \geq n_0) \quad \lambda L(\gamma_n) \leq E_{\mu}^{\lambda, p}(\gamma_n) \leq |\mu(\mathbb{R}^d)| (\text{diam supp}(\mu))^p + \varepsilon.$$

To prove the confinement condition, note that for any $r \geq 0, \gamma \in \mathcal{C}$ it holds

$$\Gamma_\gamma \cap (\text{supp}(\mu))_r = \emptyset \implies E_\mu^{\lambda,p}(\gamma) \geq F_\mu^p(\Gamma_\gamma) \geq |\mu(\mathbb{R}^d)|r^p,$$

where $(\text{supp}(\mu))_r = \{x \in \mathbb{R}^d : \inf_{y \in \text{supp}(\mu)} |x - y| \leq r\}$. Thus (2.2) gives

$$(\forall \varepsilon)(\exists n_0) : (\forall n \geq n_0) \quad F_\mu^p(\Gamma_{\gamma_n}) \leq E_\mu^{\lambda,p}(\gamma_n) \leq |\mu(\mathbb{R}^d)|(\text{diam supp}(\mu))^p + \varepsilon,$$

and combining with length estimate (2.3) and taking $\varepsilon = 1$ gives

$$(\exists n_0)(\forall n \geq n_0) \quad \Gamma_{\gamma_n} \subseteq (\text{supp}(\mu))_{(\text{diam supp}(\mu) + |\mu(\mathbb{R}^d)|(\text{diam supp}(\mu))^{p/\lambda + 1 + 1/\lambda})}$$

concluding the proof. \square

Given a measure $\mu \in \mathcal{M}$ and a curve γ , let π be a finite measure supported on $\mathbb{R}^d \times \Gamma_\gamma$ such that the first marginal of π is μ and that for π -a.e. (x, y) , $|x - y| = \min_{z \in \Gamma_\gamma} |x - z|$. The existence of such a measure is proved in Lemma 2.1 of [18]. Let σ be the second marginal of π . Then σ is supported on Γ_γ and π is an optimal transportation plan between μ and σ for the cost $c(x, y) = |x - y|^q$, for any $q \geq 1$. In other words σ is a projection of μ onto Γ_γ .

We remark that in [19] it has been proven that for any $\Sigma \in \mathcal{A}$, the ridge

$$\mathfrak{R}_\Sigma := \{x : \text{there exist } p, q \in \Sigma, p \neq q, |x - p| = |x - q| = d(x, \Sigma)\}$$

is \mathcal{H}^1 -rectifiable. Thus for any $\Sigma \in \mathcal{A}$ the (point-valued) ‘‘projection’’ map

$$(2.4) \quad \Pi_\Sigma : \mathbb{R}^d \longrightarrow \Sigma, \quad \Pi_\Sigma(x) := \text{the point of } \Sigma \text{ such that } |x - \pi(x)| = d(x, \Sigma)$$

is well defined \mathcal{L}^2 -a.e. Consequently if μ is absolutely continuous with respect to Lebesgue measure the measures π and σ above are uniquely defined and furthermore $\sigma = \Pi_{\Gamma_\gamma} \# \mu$.

Lemma 2.2. *Consider a positive measure μ and parameters $\lambda > 0, p \geq 1$. Problem 1.2 has a minimizer $\gamma \in \mathcal{C}$. Furthermore for any minimizer Γ_γ is contained in $\text{Conv}(\mu)$, the convex hull of the support of μ .*

We note that since the energy is invariant under reparameterizing the curve it follows that the problem has a minimizer $\gamma \in \mathcal{C}$ which is arc-length parameterized.

Proof. Consider a minimizing sequence $\{\gamma_n\}$ in \mathcal{C} . Since a reparameterization does not change the value of the functional we can assume that γ_n are arc-length parameterized for all n . Lemma 2.1 proves that $\{\gamma_n\}$ are uniformly bounded and have uniformly bounded lengths. Let L be the supremum of the lengths and let $\tilde{\gamma}_n$ be the extensions of the curves as in (2.1) to interval $[0, L]$. The curves $\{\tilde{\gamma}_n\}$ satisfy the conditions of Arzelà-Ascoli Theorem. Thus, along a subsequence (which we assume to be the whole sequence) they converge uniformly (and thus in \mathcal{C}) to a curve $\gamma : [0, L] \rightarrow \mathbb{R}^d$. Since all of the curves are 1-Lipschitz, so is γ and thus it belongs to \mathcal{C} .

Since $\varphi \mapsto F_\mu^p(\Gamma_\varphi)$ is continuous and $\varphi \mapsto L(\varphi)$ is lower-semicontinuous with respect to the convergence in \mathcal{C} , it follows $\liminf_{n \rightarrow \infty} E_\mu^{\lambda,p}(\tilde{\gamma}_n) \geq E_\mu^{\lambda,p}(\gamma)$. Since $\{\tilde{\gamma}_n\}_n$ is also a minimizing sequence, γ is a minimizer of $E_\mu^{\lambda,p}$.

Now we prove that any minimizer is contained in the convex hull of $\text{supp}(\mu)$. The argument relies on fact that the projection to a convex set decreases length, which we state in Lemma 2.3 below. Let $\gamma \in \mathcal{C}$ be a minimizer of $E_\mu^{\lambda,p}$. Assume it is not contained in the convex hull, $K = \text{Conv}(\mu)$, of the support of μ . Then there exists $T \in [0, a]$ such that $\gamma(T) \notin K$. Let $[t_1, t_2]$ be the maximal interval such that $\gamma((t_1, t_2)) \cap K = \emptyset$. We claim that $\sigma(\gamma(t_1, t_2)) = 0$. Otherwise

consider $\tilde{\gamma}$ be the projection of γ onto K . The distances between $\tilde{\gamma}(t)$ and points in K are strictly less than the distances between $\gamma(t)$ and the points in K and thus $F_\mu^p(\Gamma_{\tilde{\gamma}}) < F_\mu^p(\Gamma_\gamma)$. By Lemma 2.3 the length of $\tilde{\gamma}$ is less than or equal to the length of γ . Consequently $E_\mu^{\lambda,p}(\tilde{\gamma}) < E_\mu^{\lambda,p}(\gamma)$, which contradicts the assumption that γ is a minimizer. Thus $\sigma(\gamma(t_1, t_2)) = 0$.

If $\gamma(t_1)$ and $\gamma(t_2)$ belong to K then consider γ_2 obtained by replacing the segment $\gamma|_{[t_1, t_2]}$ of γ by a straight line segment. Note that the length of γ_2 is less than the length of γ (since otherwise $\gamma|_{(t_1, t_2)}$ would have to be a line segment which contradicts the fact that it is outside of K). Also note that $F_\mu^p(\Gamma_{\gamma_2}) = F_\mu^p(\Gamma_\gamma)$ and thus $E_\mu^{\lambda,p}(\gamma_2) < E_\mu^{\lambda,p}(\gamma)$, which contradicts the assumption that γ is a minimizer.

If $\gamma(t_1) \notin K$ then $t_1 = 0$. Noting that $\gamma_2 = \gamma|_{[t_2, a]}$ has lower energy than γ contradicts the minimality of γ . The case $\gamma(t_2) \notin K$ is analogous. \square

Lemma 2.3. *Given a convex set Ω , let $\Pi_\Omega : \mathbb{R}^d \rightarrow \bar{\Omega}$ be the projection onto $\bar{\Omega}$ defined by $\Pi_\Omega(x) = \operatorname{argmin}_{z \in \bar{\Omega}} |x - z|$. Let $\gamma : [a, b] \rightarrow \mathbb{R}^d$ be a rectifiable curve. Then*

$$\mathcal{H}^1(\Pi_\Omega(\gamma(I))) \leq \mathcal{H}^1(\gamma(I)).$$

Proof. It is well known that projection to a convex set is a 1-Lipschitz mapping, see Proposition 5.3 in the book by Brezis [1]. That is for all $x, y \in \mathbb{R}^d$

$$|\Pi_\Omega(x) - \Pi_\Omega(y)| \leq |x - y|.$$

Furthermore equality holds only if $x - \Pi_\Omega(x) = y - \Pi_\Omega(y)$. The claim of the lemma readily follows. \square

3. INJECTIVITY

The main aim of this section is to prove injectivity for minimizers of Problem 1.2 in two dimensions. We say that $P \in \Gamma_\gamma$ is a *double point* if $\gamma^{-1}(P)$ has at least two elements. Our goal is to show that there are no double points. Note that if Γ_γ is a simple curve, then it admits an injective parameterization, which is shorter than any noninjective parameterization. Thus in the following we will consider only minimizers containing points with order at least 3, that is points P such that for $r > 0$ small enough $(\Gamma_\gamma \cap B(P, r)) \setminus \{P\}$ has at least three connected components.

Lemma 3.1. *Let $\mu \in \mathcal{M}$ and let $\lambda > 0$ and $p \geq 1$. Let $\gamma : [0, L] \rightarrow \mathbb{R}^d$ be an arc-length-parameterized minimizer of $E_\mu^{\lambda,p}$. Assume there exist times $0 < t < s < L$ such that $\gamma(t) = \gamma(s)$. Then γ is differentiable at t and at s .*

Furthermore $\gamma'(t) = \gamma'(s)$ or $\gamma'(t) = -\gamma'(s)$.

Proof. Assume the claim does not hold. Without a loss of generality we can assume that γ is not differentiable at s . By Remark 4.3 the one sided derivatives $\gamma'(s-)$ and $\gamma'(s+)$ exist and have unit lengths. Since γ is not differentiable at s , $\gamma'(s-) \neq \gamma'(s+)$. Given that $\gamma'(s-)$ and $\gamma'(s+)$ unit vectors it follows that they are not positive multiples of each other. Consequently there exist sequences $\{s_n^-\} \searrow s$, $\{s_n^+\} \nearrow s$, such that $\angle \gamma(s_n^-) \gamma(s) \gamma(s_n^+) \rightarrow \alpha < \pi$ as $n \rightarrow \infty$. Note that by Lemma 2.1 for all $z \in \operatorname{supp}(\mu)$ and all $y \in \Gamma_\gamma$, $|z - y| < \operatorname{diam}(\operatorname{supp}(\mu))$.

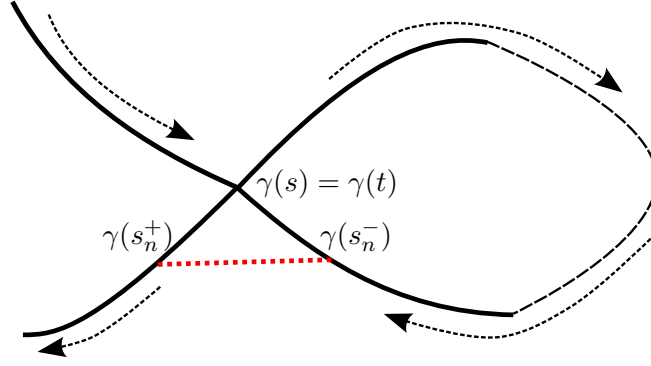


FIGURE 1. This is a schematic representation of the variation. The black lines belong to the (graph of) γ , while the red dotted line belongs to the (graph of) competitor $\tilde{\gamma}_n$. Time increases along the direction of the arrows.

Consider the competitors $\tilde{\gamma}_n$ constructed in the following way: let

$$\begin{aligned} \xi_n^* : [0, 1] &\longrightarrow \mathbb{R}^d, & \xi_n^*(u) &:= (1-u)\gamma(s_n^-) + u\gamma(s_n^+), \\ \xi_n : [0, |\gamma(s_n^-) - \gamma(s_n^+)|] &\longrightarrow \mathbb{R}^d, & \xi_n(u) &:= \xi_n^*(u/|\gamma(s_n^-) - \gamma(s_n^+)|). \end{aligned}$$

Let $\tilde{\gamma}_n : [0, L(\gamma) - (s_n^+ - s_n^-) + |\gamma(s_n^-) - \gamma(s_n^+)|]$, where

$$\tilde{\gamma}_n(u) := \begin{cases} \gamma(u) & \text{if } u \leq s_n^-, \\ \xi_n(u - s_n^-) & \text{if } s_n^- \leq u \leq s_n^- + |\gamma(s_n^-) - \gamma(s_n^+)|, \\ \gamma(u - s_n^- - |\gamma(s_n^-) - \gamma(s_n^+)| + s_n^+) & \text{if } u \geq s_n^- + |\gamma(s_n^-) - \gamma(s_n^+)|. \end{cases}$$

Since by hypothesis $\angle \gamma(s_n^-)\gamma(s)\gamma(s_n^+) \rightarrow \alpha \neq 0$, it follows (for any sufficiently large n)

$$|s_n^+ - s_n^-| - |\gamma(s_n^-) - \gamma(s_n^+)| \geq c|s_n^+ - s_n^-|,$$

for some constant $c > 0$ independent of n . Hence

$$(3.1) \quad L(\gamma) \geq L(\tilde{\gamma}_n) + c|s_n^+ - s_n^-|.$$

By taking n large we can assume that $|s_n^+ - s_n^-| < 1$.

We claim that

$$(3.2) \quad F_\mu^p(\tilde{\gamma}_n) - F_\mu^p(\gamma) \leq \mu \left(\left\{ z : \operatorname{argmin}_{y \in \Gamma_\gamma} |z - y| \subseteq \gamma((s_n^-, s_n^+) \setminus \{s\}) \right\} \right) pD^{p-1}|s_n^+ - s_n^-|,$$

where $D := 1 + \operatorname{diam} \operatorname{supp}(\mu)$. Note that if a point z satisfies

$$d(z, \Gamma_\gamma) < d(z, \Gamma_{\tilde{\gamma}_n})$$

then $\operatorname{argmin}_{y \in \Gamma_\gamma} |z - y| \subseteq \gamma((s_n^-, s_n^+) \setminus \{s\})$.

The constant pD^{p-1} is due to the fact that any such point $z \in \operatorname{supp}(\mu)$ satisfies, due to Lemma 2.2 and construction of $\tilde{\gamma}_n$,

$$\max\{d(z, \Gamma_\gamma), d(z, \Gamma_{\tilde{\gamma}_n})\} \leq D.$$

By construction there exists a point $z'_n \in \Gamma_{\tilde{\gamma}_n} \setminus \Gamma_\gamma$ satisfying $|z - z'_n| = d(z, \Gamma_{\tilde{\gamma}_n})$. Denoting by $z' \in \gamma((s_n^-, s_n^+) \setminus \{s\})$ a point satisfying $|z - z'| = d(z, \Gamma_\gamma)$, we conclude

$$||z - z'|^p - |z - z'_n|^p| \leq pD^{p-1}|s_n^- - s_n^+|.$$

Since $\bigcap_n \gamma((s_n^-, s_n^+) \setminus \{s\}) = \emptyset$, it follows that

$$\lim_{n \rightarrow +\infty} \mu \left(\left\{ z : \operatorname{argmin}_{y \in \Gamma_\gamma} |z - y| \subseteq \gamma((s_n^-, s_n^+) \setminus \{s\}) \right\} \right) = 0.$$

Combining with (3.1) gives that the minimality of γ is contradicted by $\tilde{\gamma}_n$ for sufficiently large n .

To show the second claim assume that $\gamma'(t) \neq \gamma'(s)$ and $\gamma'(t) \neq -\gamma'(s)$. Consider the following "reparameterization" of the curve γ . Let $\tilde{\gamma} : [0, L] \rightarrow \mathbb{R}^d$ be defined by

$$\tilde{\gamma}(r) = \begin{cases} \gamma(r) & \text{for } r \in [0, t], \\ \gamma(s - (r - t)) & \text{for } r \in (t, s], \\ \gamma(r) & \text{for } r \in (s, L]. \end{cases}$$

Then $\tilde{\gamma}$ is also a minimizer of $E_\mu^{\lambda, p}$. However $\tilde{\gamma}$ is not differentiable at t (and at s), which contradicts the first part of the lemma. \square

Proof of Theorem 1.3. Let $\gamma : [0, L] \rightarrow \mathbb{R}^2$ be an arc-length-parameterized minimizer of $E_\mu^{\lambda, p}$, and let $\Gamma_\gamma = \gamma([0, L])$. Recall that $P \in \Gamma_\gamma$ is a double point if $\gamma^{-1}(P)$ has at least two elements. Our goal is to show that Γ_γ has no double points.

We claim that there exists $\delta_1 \in (0, 1)$ such that γ is injective on $[0, \delta_1)$. Assume the opposite, i.e. there exist sequences $r_k \rightarrow 0$ and s_k such that $\gamma(s_k) = \gamma(r_k)$. Three cases arise.

- If $\gamma(0)$ is a double point, i.e. there exists $s \neq 0$ with $\gamma(s) = \gamma(0)$, then for sufficiently small δ_1 the competitor $\gamma|_{[\delta_1, L]}$ contradicts the minimality of γ .
- If $\inf_k s_k = 0$, then by considering subsequences we can assume that $r_k < s_{k+1}$ for all k . Then the intervals $[s_k, r_k]$ are all mutually disjoint. Since $\gamma(s_k) = \gamma(r_k)$ we conclude that $\|\gamma'\|_{TV(s_k, r_k)} \geq \pi$ which implies that $\|\gamma'\|_{TV([0, L])}$ is infinite. This contradicts the regularity estimate of Proposition 4.2.
- If $\inf_k s_k > 0$, then by considering a subsequence we can assume that $\lim_{k \rightarrow \infty} k s_k = s$ for some $s \in (0, L]$. The continuity of γ gives that $\gamma(0) = \gamma(s)$ which reduces the argument to the first case.

Likewise for δ_1 small $\gamma((L - \delta_1, L])$ has no double points.

Assume that there are double points on $\gamma([\delta_1, L - \delta_1])$. Let

$$t_2 = \sup\{t < L - \delta_1 : \gamma(t) \text{ is a double point}\}.$$

Note that γ is injective on $(t_2, L]$. We claim that $\gamma(t_2)$ is a double point. Assume it is not. Then there exist increasing sequences $s_k < r_k < t_2$ converging to t_2 such that $\gamma(s_k) = \gamma(r_k)$. By considering their subsequences we can assume that $r_k < s_{k+1}$ for all k . Then the intervals $[s_k, r_k]$ are all mutually disjoint. Since $\gamma(s_k) = \gamma(r_k)$ we conclude that $\|\gamma'\|_{TV(s_k, r_k)} \geq \pi$ which implies that $\|\gamma'\|_{TV([0, L])}$ is infinite. This contradicts the regularity estimate of Proposition 4.2. Thus $\gamma(t_2)$ is a double point. Hence, for sufficiently small δ_1 , there exists $t_1 \in (\delta_1, t_2)$ such that $\gamma(t_1) = \gamma(t_2)$.

We introduce:

$$(3.3) \quad f(x) = \sup_{0 \leq z \leq x} \frac{\alpha'(z)}{|(1, \alpha'(z))|} \quad \text{and} \quad g(x) = \inf_{0 \leq z \leq x} \frac{\beta'(z)}{|(1, \beta'(z))|}.$$

Note that $f(x) - g(x) > 0$ on $(0, \delta)$ and that, using the assumption on total variation of γ' , it follows that for $x \in [0, \delta)$, $|f(x)|$ and $|g(x)|$ are less than $\frac{1}{8}$.

Let $\bar{\ell}$ be the line passing through $Q = (x, \alpha(x))$ whose slope is $\frac{1}{8}$. It stays above the graph of α on (x, δ) . Let Z be the intersection point of $\bar{\ell}$ and $\ell_\beta^+(x)$. We note that $\theta < \arctan(\frac{1}{8})$ and thus $[QZ]$ is the shortest side of triangle ΔPZQ . Therefore

$$|PZ| < 2|QP| = 2(\alpha(x) - \beta(x)) \leq 2x(f(x) - g(x)).$$

Since $f(x) - g(x)$ is a nondecreasing function we conclude that

$$\sup_{z \in A_\beta(x)} d(z, \Gamma_\gamma) \leq 2x(f(x) - g(x)).$$

Likewise

$$\sup_{z \in U_\alpha(x)} d(z, \Gamma_\gamma) \leq 2x(f(x) - g(x)).$$

Lemma 4.4 implies that

$$(3.4) \quad f(x) \leq \frac{p}{\lambda}(2x(f(x) - g(x))^{p-1} \mu(U_\alpha(x))) \quad \text{and} \quad g(x) \geq \frac{p}{\lambda}(2x(f(x) - g(x))^{p-1} \mu(A_\beta(x))).$$

We first focus on $p \geq 2$. From the above inequalities it follows that for some constant $c > 0$

$$(f(x) - g(x))^{2-p} \leq cx^{p-1}.$$

Since as $x \rightarrow 0^+$ the left-hand side has uniform positive lower bound while the right-hand side converges to zero we obtain a contradiction, as desired.

We now consider the more delicate case: $1 \leq p < 2$. Recall that we now assume that μ has bounded density ρ . To obtain a bound on $\mu(A_\beta(x))$ we estimate the area of $A_\beta(x)$. The area of $A_\beta(x)$ is bounded from above by the sum of the areas of the region between the curves to the left of line segment $[PQ]$ and the area of triangle PZQ on Figure 2.

We note that $\theta < \arctan(\frac{1}{8})$ and the angle $\angle ZQP$ is $\frac{\pi}{2} + \arctan(\frac{1}{8})$. Therefore $\xi = \angle QZP > \frac{\pi}{6}$. Using the law of sines and $\alpha(x) - \beta(x) \leq x(f(x) - g(x))$ we obtain

$$\frac{1}{2x(f(x) - g(x))} < \frac{\sin \xi}{\alpha(x) - \beta(x)} = \frac{\sin \theta}{|QZ|} < \frac{1}{8|QZ|}.$$

Therefore

$$\text{Area}(\Delta PZQ) \leq \frac{1}{2}|QP| \cdot |QZ| \leq \frac{1}{8}x^2(f(x) - g(x))^2.$$

Consequently

$$\text{Area}(A_\beta(x)) \leq x^2(f(x) - g(x)) + x^2(f(x) - g(x))^2 \leq 2x^2(f(x) - g(x)).$$

Same upper bound holds for $\text{Area}(U_\alpha(x))$. Therefore

$$\max\{\mu(U_\alpha(x)), \mu(A_\beta(x))\} \leq 2\|\rho\|_{L^\infty} x^2(f(x) - g(x)).$$

Combining with estimate (3.4), (choosing δ_2 sufficiently small) and using that $2x(f(x) - g(x)) < 1$ gives that for a.e $x \in [0, \delta)$

$$0 \leq f(x) - g(x) \leq 4 \frac{p}{\lambda} \|\rho\|_{L^\infty} x^2 (f(x) - g(x)).$$

This implies that for a.e. $x > 0$ small enough $f(x) = 0$ and $g(x) = 0$, which means that the curves coincide. Contradiction. \square

4. CURVATURE OF MINIMIZERS

In [22, 18] we studied the average-distance problem considered over the family of connected 1-dimensional sets. Here we study the problem among a more restrictive set of objects, namely parameterized curves. The conditions for stationarity and regularity estimates of [22, 18] still apply in this setting. Here we state the estimates for general $p \geq 1$, while we previously considered only $p = 1$. The extension is straightforward.

We start by stating the conditions for the case that μ is a discrete measure, $\mu = \sum_{i=1}^n m_i \delta_{x_i}$ where $m_i > 0$ for all i and $\sum_{i=1}^n m_i = 1$. Arguing as in Lemma 7 of [22] we conclude that any minimizer of Problem 1.2 is a solution of a euclidean traveling salesman (for Problem 1.1 the minimizers were Steiner trees) and is thus a piecewise linear curve with no self-intersections (i.e. γ is injective). Such γ can be described as a graph as follows. Let V , the set of vertices, be the collection of all minimizers over Γ_γ of distance to each of the point in $X = \{x_1, \dots, x_n\}$. That is let

$$(4.1) \quad V = \bigcup_{i=1}^n \operatorname{argmin}_{z \in \Gamma_\gamma} |z - x_i|.$$

We can write $V = \{v_1, \dots, v_m\}$ where v are ordered as they appear along Γ_γ (in increasing order with respect to parameter of γ). Then Γ_γ is the piecewise linear curve $[v_1, \dots, v_m]$.

For $j = 1, \dots, m$ let I_j be the set of indices of points in X for which v_j is the closest point in V

$$(4.2) \quad \begin{aligned} I_j &= \{i \in \{1, \dots, n\} : (\forall k = 1, \dots, m) \quad d(x_i, v_j) \leq d(x_i, v_k)\} \\ &= \{i \in \{1, \dots, n\} : (\forall y \in \Gamma_\gamma) \quad d(x_i, v_j) \leq d(x_i, y)\}. \end{aligned}$$

If $i \in I_j$ then we say that x_i *talks to* v_j . We say that a vertex v_j is *tied down* if for some i , $v_j = x_i$. We then say that v_j is *tied to* x_i . A vertex which is not tied down is called *free*. As shown in [22], if x_i talks to v_j and v_j is free then x_i cannot talk to any other vertex.

As in [22] we consider the optimal transportation plan between μ and its projection onto Γ_γ . That is, consider an n by m matrix T such that

$$(4.3) \quad T_{ij} \geq 0, \quad \sum_{j=1}^m T_{ij} = m_i, \quad \text{and } T_{ij} > 0 \text{ implies } i \in I_j$$

Note that $\mu = \sum_{i=1}^n \sum_{j=1}^m T_{ij} \delta_{x_i}$. Furthermore observe that if v_j is tied to x_i then $i \in I_j$ and $T_{ij} = m_i$. Let $\pi = \sum_{i,j} m_i \delta_{x_i} \otimes \delta_{v_j}$. We note that the first marginal of π is μ and that it describes an optimal transportation plan between μ and a measure supported on $V \subset \Gamma_\gamma$. We define σ to

be the second marginal of π as before (above Lemma 2.2). Then σ is a projection of μ onto the set Γ_γ in that the mass of μ is transported to a closest point on Γ_γ . More precisely

$$(4.4) \quad \sigma = \sum_{j=1}^m \sum_{i=1}^n T_{ij} \delta_{v_j}.$$

We note that the matrix T describes an optimal transportation plan between μ and σ with respect to any of the transportation costs $c(x, y) = |x - y|^q$, for $q \geq 1$.

We note that in this discrete setting

$$(4.5) \quad \begin{aligned} E_\mu^{\lambda,p}(\gamma) &= \sum_{i=1}^n m_i d^p(x_i, \Gamma_\gamma) + \lambda \sum_{i=1}^{m-1} |v_{i+1} - v_i| \\ &= \sum_{j=1}^m \sum_{i \in I_j} T_{ij} |x_i - v_j|^p + \lambda \sum_{i=1}^{m-1} |v_{i+1} - v_i| \end{aligned}$$

Taking the first variation in v_j provides an extension to $p > 1$ of conditions for stationarity of Lemma 9 in [22]:

Lemma 4.1. *Assume that γ minimizes $E_\mu^{\lambda,p}$ for discrete $\mu = \sum_{i=1}^n m_i \delta_{x_i}$. Let V be the set of vertices as defined in (4.1) and T be any matrix (transportation plan) satisfying (4.3). Then*

- For endpoints $j = 1$ and $j = m$ let $w = v_2$ if $j = 1$ and $w = v_{m-1}$ otherwise.
If $p > 1$ or v_j is free then

$$(4.6) \quad \sum_{i \in I_j} p T_{ij} (x_i - v_j) |x_i - v_j|^{p-2} + \lambda \frac{w - v_j}{|w - v_j|} = 0$$

If v_j is tied to x_k and $p = 1$ then

$$(4.7) \quad \left| \sum_{i \in I_j, i \neq k} T_{ij} \frac{x_i - v_j}{|x_i - v_j|} + \lambda \frac{w - v_j}{|w - v_j|} \right| \leq m_k$$

- If $j = 2, \dots, m-1$ then if $p > 1$ or v_j is free

$$(4.8) \quad \sum_{i \in I_j} p T_{ij} (x_i - v_j) |x_i - v_j|^{p-2} + \lambda \left(\frac{v_{j-1} - v_j}{|v_{j-1} - v_j|} + \frac{v_{j+1} - v_j}{|v_{j+1} - v_j|} \right) = 0$$

If v_j is tied to x_k and $p = 1$ then

$$(4.9) \quad \left| \sum_{i \in I_j, i \neq k} T_{ij} \frac{x_i - v_j}{|x_i - v_j|} + \lambda \left(\frac{v_{j-1} - v_j}{|v_{j-1} - v_j|} + \frac{v_{j+1} - v_j}{|v_{j+1} - v_j|} \right) \right| \leq m_k.$$

The proof of the lemma is analogous to one in [22].

Note that the condition at a corner provides an upper bound on the turning angle in terms of the $p-1$ -st moment of the mass that talks to the corner. These conditions can be used as in [18] to obtain estimates on the curvature (in the sense of a measure) of minimizers γ of $E_\mu^{\lambda,p}$ for general compactly supported measures μ . In particular adapting the proof of Theorem 5.1 and Lemma 5.2 of [18] implies:

Proposition 4.2. *Let $\mu \in \mathcal{M}$ and let $\lambda > 0$ and $p \geq 1$. If $\gamma : [0, L] \rightarrow \mathbb{R}^d$ is an arc-length-parameterized minimizer of $E_\mu^{\lambda,p}$ then $\gamma' \in BV([0, L], \mathbb{R}^d)$ and*

$$(4.10) \quad \|\gamma'\|_{TV([0,L])} \leq \frac{p}{\lambda} \text{diam}(\text{supp}(\mu))^{p-1} \mu(\mathbb{R}^d).$$

Remark 4.3. *A particular consequence of this estimate is that for all $T \in [0, L]$, $v = \lim_{t \rightarrow T+} \gamma'(t)$ exists (see Corollary 2.23 of [17]). Furthermore since $|\gamma'(t)| = 1$ a.e. it follows that $|v| = 1$. It is then straightforward to prove, using the definition of a one-sided derivative, that γ has a right derivative $\gamma'(T+)$ and that $\gamma'(T+) = v$. Analogous statements hold for the left derivative.*

We note that the estimate holds if we consider the same problem on the class of curves with fixed endpoints: $\gamma : [0, L] \rightarrow \mathbb{R}^d$ with $\gamma(0) = P$ and $\gamma(L) = Q$ with $P, Q \in \text{Conv}(\mu)$. The proof is essentially the same.

A consequence of this observation is that we can formulate a localized version of the estimate. In particular let γ be the minimizer of $E_\mu^{\lambda,p}$ as in the Proposition 4.2. Let π and σ be as defined above Lemma 2.2. For any interval $I = (t, t + \delta) \subset [0, L]$

$$(4.11) \quad \|\gamma'\|_{TV(I)} \leq p \text{diam}(\text{supp}(\mu))^{p-1} \frac{1}{\lambda} \sigma(\gamma(I)).$$

The estimate bounds how much can the curve γ turn within interval I based on the $p - 1$ -st moment of the mass in μ that projects onto the set $\gamma(I)$. Let μ_I be the measure defined as $\mu_I(U) = \pi(U \times \gamma(I))$, that is the μ measure of the set of points that projects onto $\gamma(I)$. The estimate follows from Proposition 4.2 using the observation that $\gamma|_I$ is a minimizer of $E_{\mu_I}^{\lambda,p}$ among curves which start at $\gamma(t)$ and end at $\gamma(t + \delta)$.

In this work we need finer information. We focus on dimension $d = 2$. We need information not only on how much a curve turns but also on about the direction it turns in.

Lemma 4.4. *Consider dimension $d = 2$. Let $\mu \in \mathcal{M}$, $\lambda > 0$ and $p \geq 1$. Let $\gamma : [0, L] \rightarrow \mathbb{R}^2$ be an arc-length-parameterized minimizer of $E_\mu^{\lambda,p}$. Let $t \in [0, L]$. By rotation and translation we can assume that $\gamma(t) = 0$, $\gamma'(t) = e_1$. Let $I = (t, t + \delta)$ be such that $t + \delta < L$, $\|\gamma'\|_{TV(I)} < \frac{1}{2}$. We define the region underneath the curve segment $\gamma(I)$ to be as depicted on Figure 3. That is let $\ell_1^- = \{s(0, 1) : s \leq 0\}$ and $\ell_2^- = \{\gamma(t + \delta) + s(\gamma'(t + \delta))^{\perp} : s \leq 0\}$. Let U be the connected component of $\mathbb{R}^2 \setminus (\ell_1^- \cup \ell_2^- \cup \gamma(I))$ which contains the point $\gamma(t + \frac{\delta}{2}) - \frac{\delta}{4}(0, 1)$.*

Let D be the maximal distance of a point in $\text{supp}(\mu) \cap U$, which talks to $\gamma(I)$. That is let $D = \sup\{d(x, \gamma(I)) : x \in \text{supp}(\mu) \cap U, \text{argmin}_{z \in \gamma(I)} d(x, z) \cap \gamma(I) \neq \emptyset\}$. Then

$$(4.12) \quad \sup_{s \in I} \gamma'(s) \cdot e_2 \leq \frac{p}{\lambda} D^{p-1} \mu(U).$$

Proof. By approximating as in the proof of Theorem 5.1 in [18] the problem can be reduced to considering discrete measures. Thus we assume that $\mu = \sum_{i=1}^n m_i \delta_{x_i}$.

Let A be the region above the curve segment $\gamma(I)$. That is let $\ell_1^+ = \{s(0, 1) : s \geq 0\}$ and $\ell_2^+ = \{\gamma(t + \delta) + s(\gamma'(t + \delta))^{\perp} : s \geq 0\}$ and let A be the connected component of $\mathbb{R}^2 \setminus (\ell_1^+ \cup \ell_2^+ \cup \gamma(I))$ which contains the point $\gamma(t + \frac{\delta}{2}) + \frac{\delta}{4}(0, 1)$. Note that all of the mass of μ that talks to $\gamma(I)$ is contained in $U \cup A \cup \gamma(I)$.

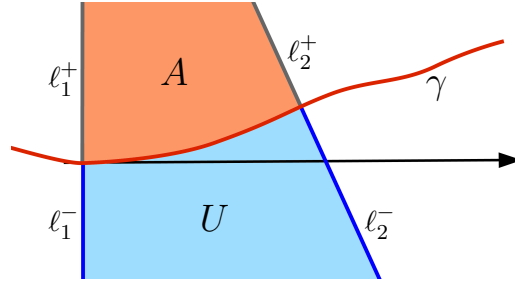


FIGURE 3. The geometry of the configuration near the last double point.

Due to an assumption on I , for all $v_j \in \gamma(I)$ the angle between $v_{j+1} - v_j$ and e_1 is less than $\pi/4$ and so is the angle $v_j - v_{j-1}$ and e_1 . Therefore if $i \in I_j$ and $x_i \in A$ then the directed angle between e_1 and $x_i - v_j$ is between $\pi/4$ and $3\pi/4$. Therefore $(v_j - x_i) \cdot e_2 < 0$.

Let us first consider the case that $p > 1$. Then from (4.8) follows that for j such that $v_j \in \gamma(I)$

$$\begin{aligned} \sum_{i \in I_j, x_i \in U} p T_{ij} |x_i - v_j|^{p-2} (v_j - x_i) \cdot e_2 &\geq \sum_{i \in I_j} p T_{ij} |x_i - v_j|^{p-2} (v_j - x_i) \cdot e_2 \\ &= \lambda \left(\frac{v_{j+1} - v_j}{|v_{j+1} - v_j|} - \frac{v_j - v_{j-1}}{|v_j - v_{j-1}|} \right) \cdot e_2 \end{aligned}$$

Consider $s \in (t, t + \delta)$. Summing up over all j such that $v_j \in \gamma((t, s))$ gives

$$p\mu(U)D^{p-1} \geq \lambda\gamma'(s-) \cdot e_2,$$

which establishes the desired claim.

Consider now $p = 1$. From (4.9) follows that for j such that $v_j \in \gamma(I)$

$$\begin{aligned} \sum_{i \in I_j, x_i \in U} T_{ij} \frac{1}{|x_i - v_j|} (v_j - x_i) \cdot e_2 &\geq \sum_{i \in I_j} T_{ij} \frac{1}{|x_i - v_j|} (v_j - x_i) \cdot e_2 \\ &\geq \lambda \left(\frac{v_{j+1} - v_j}{|v_{j+1} - v_j|} - \frac{v_j - v_{j-1}}{|v_j - v_{j-1}|} \right) \cdot e_2 - \mu(v_j) \end{aligned}$$

Summing over j such that $v_j \in \gamma((t, s))$ again provides the desired claim. \square

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