Graph Laplacians and their continuum limit

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Content

- Introduction: calculus on graphs
- Consistency: discrete to nonlocal
- Consistency: nonlocal to local
- Continuum limit of the solution to discrete Laplace's equation[1]

Calculus on graphs

- Let $\mathcal{X}$ be the set of n vertices and W a $n \times n$ symmetric matrix with non-negative components. Let $d(x) = \sum_{y \in \mathcal{X}} w_{xy}$ be the degree of x.
- Let $\ell^2(\mathcal{X})$ denote the space of 'square summable functions' $u : \mathcal{X} \to \mathbb{R}$ with respect to the inner product

$$(u, v)_{\ell^2(\mathcal{X})} = \sum_{x \in \mathcal{X}} u(x)v(x).$$

- vector field is a function $V : \mathcal{X}^2 \to \mathbb{R}$ satisfying $V(x, y) = -V(y, x)$. The inner product between vector fields V_1, V_2 is

$$(V_1, V_2)_{\ell^2(\mathcal{X}^2)} = \frac{1}{2} \sum_{x,y \in \mathcal{X}} w_{xy} V_1(x, y) V_2(x, y).$$

Calculus on graphs

- The gradient of a function $u \in \ell^2(\mathcal{X})$ is

$$\nabla u(x,y) = u(y) - u(x)$$

- The divergence of a vector field is

$$\operatorname{div} V(x) = \sum_{y \in \mathcal{X}} w_{xy} V(x,y).$$

- Then we have the divergence formula

$$(\nabla u, V)_{\ell^2(\mathcal{X}^2)} = -(u, \operatorname{div} V)_{\ell^2(\mathcal{X})}$$

Proof of the divergence formula

$$
\begin{aligned}
(\nabla u, V)_{\ell^2(\mathcal{X}^2)} &= \frac{1}{2} \sum_{x,y \in \mathcal{X}} w_{xy}(u(y) - u(x))V(x,y) \\
&= \frac{1}{2} \sum_{x,y \in \mathcal{X}} w_{xy} u(y) V(x,y) - \frac{1}{2} \sum_{x,y \in \mathcal{X}} W_{xy} u(x) V(x,y) \\
&= \frac{1}{2} \sum_{x,y \in \mathcal{X}} w_{xy} u(x) V(y,x) - \frac{1}{2} \sum_{x,y \in \mathcal{X}} W_{xy} u(x) V(x,y) \\
&= -\sum_{x,y \in \mathcal{X}} w_{xy} u(x) V(x,y) = -(u, \operatorname{div} V)_{\ell^2(\mathcal{X})}
\end{aligned}
$$

Graph Laplacian

We define the graph Laplacian $\mathcal{L}$ by

$$\mathcal{L}u(x) = \mathrm{div}(\nabla u) = \sum_{y \in \mathcal{X}} w_{xy}(u(y) - u(x)).$$

Note the connection to the mean value property by

$$\mathcal{L}u(x) = \sum_{y \in \mathcal{X}} w_{xy}(u(y) - u(x)) = \sum_{y \in \mathcal{X}} w_{xy} u(y) - d(x)u(x).$$

By the divergence formula we see that $\mathcal{L}$ is self-adjoint

$$(\mathcal{L}u, v)_{\ell^2(\mathcal{X})} = -(\nabla u, \nabla v)_{\ell^2(\mathcal{X}^2)} = (u, \mathcal{L}v)_{\ell^2(\mathcal{X})}$$

and that it is connected to the graph Dirichlet energy

$$(-\mathcal{L}u, u)_{\ell^2(\mathcal{X})} = (\nabla u, \nabla u)_{\ell^2(\mathcal{X}^2)} =: \mathcal{E}(u) \geq 0.$$

The maximum principle

Lemma 1

Let $u \in \ell^2(\mathcal{X})$ satisfying $\mathcal{L}u(x) \geq 0$ for all $x \in \mathcal{X} \setminus \Gamma$. If the graph $(\mathcal{X}, W)$ is connected to Γ, then

$$\max_{x \in \mathcal{X}} u(x) = \max_{x \in \mathcal{X} \cap \Gamma} u(x).$$

- If maximum occurs at x_0, mean value property implies

$$u(x_0) \leq \frac{1}{d(x_0)} \sum_{y \in \mathcal{X}} w_{x_0 y} u(y) \leq \frac{1}{d(x_0)} \sum_{y \in \mathcal{X}} w_{x_0 y} u(x_0) = u(x_0).$$

- Thus if $w_{x_0 y} > 0$ then

$$u(x_0) = u(y).$$

- By connectedness, we can find a connected 'path' reaching the boundary.

Lemma 2

Let $u \in \ell^2(\mathcal{X})$ such that $\mathcal{L}u(x) > 0$ for all $x \in \mathcal{X} \setminus \Gamma$. Then

$$\max_{x \in \mathcal{X}} u(x) = \max_{x \in \mathcal{X} \cap \Gamma} u(x).$$

- Suppose $x_0 \in \mathcal{X}$ such that $u(x_0) = \max_{x \in \mathcal{X}} u(x)$.
- Then $u(x_0) \geq u(y)$ for each $y \in \mathcal{X}$ thus

$$\mathcal{L}u(x_0) = \sum_{y \in \mathcal{X}} w_{xy}(u(y) - u(x_0)) \leq 0$$

which means $x_0 \in \Gamma$.

Setting: Random geometric graphs

- $U \subset \mathbb{R}^d$ is an open connected bounded domain
- $\rho : U \to [0, \infty)$ a continuous probability density function, satisfying

$$\alpha \leq \rho(x) \leq \alpha^{-1}$$

  for some $\alpha > 0$.
- $X_1, \cdots, X_n$ a sequence of $i.i.d$ r.v distributed according to ρ.
- $\mathcal{X} = \{X_1, \cdots, X_n\}$ the set of vertices
- $\eta : [0, \infty) \to [0, \infty)$ smooth and nonincreasing, supported on [0,1].
- $\eta_\varepsilon(t) = \frac{1}{\varepsilon^d}\eta(\frac{t}{\eta})$.
- $\sigma_\eta = \int_{\mathbb{R}^d} |z_1|^2 \eta(|z|)\, dz$ could be interpreted as surface tension

Setting: Random geometric graphs

- The weight is given by $w_{xy} = \eta_\varepsilon(|x-y|)$.
- The inner product between functions is given by

$$(u, v)_{\ell^2(\mathcal{X})} = \frac{1}{n} \sum_{x \in \mathcal{X}} u(x)v(x)$$

- The inner product between vector fields is given by

$$(V, W)_{\ell^2(\mathcal{X}^2)} = \frac{1}{\sigma_\eta n^2} \sum_{x,y \in \mathcal{X}} \eta_\varepsilon(|x-y|)V(x,y)W(x,y).$$

- The gradient $\nabla_{n,\varepsilon} u$ and divergence $\operatorname{div}_{n,\varepsilon} V$ are

$$\nabla_{n,\varepsilon} u = \frac{1}{\varepsilon}(u(y) - u(x)), \quad \operatorname{div}_{n,\varepsilon} V = \frac{2}{\sigma_\eta n \varepsilon} \sum_{y \in \mathcal{X}} \eta_\varepsilon(|x-y|)V(x,y).$$

Unnormalized graph Laplacian

Definition 3 (Unnormalized graph Laplacian)

The unnormalized graph Laplacian is

$$\mathcal{L}_{n,\varepsilon} u(x) = \frac{2}{\sigma_\eta n \varepsilon^2} \sum_{y \in \mathcal{X}} \eta_\varepsilon(|x-y|)(u(y)-u(x)).$$

Its corresponding nonlocal operator is

$$L_\varepsilon u(x) = \frac{2}{\sigma_\eta \varepsilon^2} \int_U \eta_\varepsilon(|x-y|)(u(y)-u(x))\rho(y)\,dy$$

and the corresponding local operator is

$$\Delta_\rho u = \rho^{-1} \operatorname{div}(\rho^2 \nabla u) = \rho \Delta u + 2\nabla\rho \cdot \nabla u.$$

Symmetric normalized graph Laplacian

Definition 4 (Symmetric normalized graph Laplacian)

Let D be the diagonal matrix with entries $D_{ii} = d(x_i)$. We define the symmetric normalized graph Laplacian by

$$\mathcal{N} = \mathcal{D}^{-\frac{1}{2}} \mathcal{L}_{n,\varepsilon} \mathcal{D}^{-\frac{1}{2}}.$$

Its continuum (local) counterpart is the operator

$$u \mapsto \frac{1}{\rho^{\frac{3}{2}}} \operatorname{div} \left(\rho^2 \nabla \left(\frac{u}{\sqrt{\rho}} \right) \right).$$

This follows from the unnormalized case as $\mathcal{N} u(x) = \frac{1}{\sqrt{d(x)}} \mathcal{L}_{n,\varepsilon} \left(\frac{u}{\sqrt{d(x)}} \right)$[3] and as $\varepsilon \to 0$

$$\mathbb{E} d(x) = \int_U \eta_\varepsilon(|x-y|)\rho(y)\, dy \to \rho(x) \int \eta(|z|)\, dz.$$

Basic concentration inequalities

Theorem 5 (Bernstein's Inequality)

Let $X_1, \cdots, X_n$ be a sequence of i.i.d. real valued r.v. with $\mathbb{E}[X_1] = \mu < \infty$, and $\sigma^2 = Var(X_i) < \infty$. Suppose $|X - \mu| \leq b$ almost surely. Letting $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, we have for each $t > 0$

$$\mathbb{P}(S_n - \mu \geq t) \leq \exp\left(-\frac{nt^2}{2(\sigma^2 + \frac{1}{3}bt)}\right)$$

- Under the same assumptions, Hoeffding's inequality states

$$\mathbb{P}(S_n - \mu \geq t) \leq \exp\left(-\frac{nt^2}{2b^2}\right)$$

- For Graph Laplacians, the variance is usually small, so Bernstein's inequality is preferred

Lemma 6 (Discrete to nonlocal)

Let $u : U \to \mathbb{R}$ be Lipschitz continuous, and $\varepsilon > 0$. Then for any $0 < \lambda \leq \varepsilon^{-1}$

$$\max_{x \in X} |\mathcal{L}_{n,\varepsilon} u(x) - \mathcal{L}_\varepsilon u(x)| \leq C Lip(u) \lambda$$

with probability at least $1 - C \exp(-cn\varepsilon^{d+2}\lambda^2 + \log(n))$.

Recall

$$\mathcal{L}_{n,\varepsilon} u(x) = \frac{2}{\sigma_\eta n \varepsilon^2} \sum_{y \in \mathcal{X}} \eta_\varepsilon(|x-y|)(u(y) - u(x))$$

and

$$\mathcal{L}_\varepsilon u(x) = \frac{2}{\sigma_\eta \varepsilon^2} \int_U \eta_\varepsilon(|x-y|)(u(y) - u(x))\rho(y)\, dy.$$

Consistency: discrete to nonlocal

- The proof is a direct application of Bernstein's inequality. We first compute our μ, b, σ^2
- Fix $x \in U$ and let $Y_i = \eta_\varepsilon(|X_i - x|)(u(X_i) - u(x))$ such that

$$\mathcal{L}_{n,\varepsilon} u(x) = \frac{2}{\sigma_\eta \varepsilon^2} \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

- Then our $\mu = \mathbb{E}[Y_i]$ is

$$\int_U \eta_\varepsilon(|y - x|)(u(y) - u(x))\rho(y) dy$$

Consistency: discrete to nonlocal

- Our b is

$$|Y_i| = \eta_\varepsilon(|X_i - x|)|u(X_i) - u(x)| \le \varepsilon^{-d}\|\eta\|_\infty Lip(u)\varepsilon = C\varepsilon^{1-d} Lip(u)$$

- Finally our σ^2 satisfies

$$\begin{aligned} Var(Y_i) \le \mathbb{E}[Y_i^2] &= \int_{B(x,\varepsilon)\cap U} \eta_\varepsilon(|x-y|)^2 (u(y)-u(x))^2 \rho(y)\, dy \\ &\le \|\rho\|_\infty Lip(u)^2 \varepsilon^2 \int_{B(x,\varepsilon)\cap U} \varepsilon^{-2d} \eta(\varepsilon^{-1}|x-y|)^2\, dy \\ &\le C\, Lip(u)^2 \varepsilon^2 \varepsilon^{-d} \int_{B(0,1)} \eta(|z|)^2\, dz \le C\, Lip(u)^2 \varepsilon^{2-d} \end{aligned}$$

- Thus $bt \le C\sigma^2$ when $t \le C\, Lip(u)\varepsilon$

Consistency: discrete to nonlocal

- By Bernstein's inequality, if $bt \leq C\sigma^2$

$$\mathbb{P}\left(|\frac{1}{n}\sum_{i=1}^{n} Y_i - \mathbb{E}[Y_1]| \geq t\right) = \mathbb{P}\left(\left|\frac{2}{\sigma_\eta \varepsilon^2}\frac{1}{n}\sum_{i=1}^{n} Y_i - \frac{2\mathbb{E}[Y_1]}{\sigma_\eta \varepsilon^2}\right| \geq \frac{2t}{\sigma_\eta \varepsilon^2}\right)$$
$$\leq 2\exp\left(-c\, Lip(u)^{-2} n\varepsilon^{d-2} t^2\right)$$

- Choose $t = \frac{\sigma_\eta}{2} Lip(u)\varepsilon^2\lambda$ for $\lambda \leq \varepsilon^{-1}$
- Finally, condition on $X_i = x$ then the union bound gives

$$\mathbb{P}\left(\max_{x\in\mathcal{X}} |\mathcal{L}_{n,\varepsilon}u(x) - \mathcal{L}_\varepsilon u(x)| \geq C\, Lip(u)\lambda\right) \leq 2\exp(-cn\varepsilon^{d+2}\lambda^2 + \log n)$$

- The above bound is meaningful when $\varepsilon^{d+2} \gg n^{-1}$

Consistency: nonlocal to local

Lemma 7 (Consistency: nonlocal to local)

Let $\rho \in C^2(\overline{U})$*. Then there exists* $C > 0$ *such that for all* $u \in C^3(\overline{U})$ *and* $x \in U$ *with* $\operatorname{dist}(x, \partial U) \geq \varepsilon$ *we have*

$$|\mathcal{L}_\varepsilon u(x) - \Delta_\rho u(x)| \leq C\|u\|_{C^3(U)}\varepsilon.$$

- Extension of the consistency up to the boundary can be found in Calder, Slepcěv, and Thorpe'20.[2]

Consistency: nonlocal to local

- $B(x,\varepsilon) \cap U = B(x,\varepsilon)$ as $\operatorname{dist}(x, \partial U) \geq \varepsilon$. Changing variable to $z = \frac{x-y}{\varepsilon}$ gives

$$\mathcal{L}_\varepsilon u(x) = \frac{2}{\sigma_\eta \varepsilon^2} \int_{B(0,1)} \eta(|z|)(u(x+z\varepsilon) - u(x))\rho(x+z\varepsilon)\, dz$$

- Let $\beta := \|u\|_{C^3(\overline{U})}$. Taylor expanding, we have

$$u(x+z\varepsilon) - u(x) = \nabla u(x) \cdot z\varepsilon + \frac{\varepsilon^2}{2} z^T \nabla^2 u(x) z + O(\beta\varepsilon^3)$$
$$\rho(x+z\varepsilon) = \rho(x) + \nabla \rho(x) + O(\varepsilon^2)$$

- Plugging in, we have

$$\mathcal{L}_\varepsilon u(x) = \frac{2}{\sigma_\eta} \int_{B(0,1)} \eta(|z|) \Big(\rho(x) \nabla u(x) \cdot z\varepsilon^{-1} + \frac{1}{2} \rho(x) z^T \nabla^2 u(x) z$$
$$+ (\nabla u(x) \cdot z)(\nabla \rho(x) \cdot z)\, dz \Big) + O(\beta\varepsilon).$$

Consistency: nonlocal to local

- $z \mapsto \eta(|z|)\rho(x)\nabla u(x) \cdot z\varepsilon^{-1}$ is odd, so its integral vanishes. Hence

$$\mathcal{L}_\varepsilon u(x) = \frac{2}{\sigma_\eta} \int_{B(0,1)} \eta(|z|) \left(\frac{1}{2}\rho(x) z^T \nabla^2 u(x) z \right.$$
$$\left. + (\nabla u(x) \cdot z)(\nabla \rho(x) \cdot z)\, dz \right) + O(\beta\varepsilon) =: I + J + O(\beta\varepsilon)$$

- By symmetry, $\int_{B(0,1)} \eta(|z|) z_i z_j \, dz = 0$ for $i \neq j$. Thus

$$\begin{aligned} I &= \frac{1}{\sigma_\eta}\rho(x) \sum_{i,j=1}^{d} u_{x_i x_j}(x) \int_{B(0,1)} \eta(|z|) z_i z_j \, dz \\ &= \frac{1}{\sigma_\eta}\rho(x) \sum_{i=1}^{d} u_{x_i x_i}(x) \int_{B(0,1)} \eta(|z|) z_i^2 \, dz = \rho(x)\Delta u(x). \end{aligned}$$

Consistency: nonlocal to local

- Similarly

$$J = \frac{2}{\sigma_\eta}\rho(x)\sum_{i=1}^{d} u_{x_i}(x)\rho_{x_i}(x)\int_{B(0,1)} \eta(|z|)z_i^2\,dz = 2\nabla\rho(x)\cdot\nabla u(x).$$

- Finally, by product rule

$$\begin{aligned}\mathcal{L}_\varepsilon u(x) &= \rho(x)\Delta u(x) + 2\nabla\rho(x)\cdot\nabla u(x) + O(\beta\varepsilon)\\ &= \rho^{-1}\operatorname{div}(\rho^2(x)\nabla u(x)) + O(\varepsilon\|u\|_{C^3(\overline{U})})\end{aligned}$$

Theorem 8 (Consistency of graph Laplacian)

Let $u \in C^3(\overline{U})$, $\rho \in C^2(\overline{U})$. Then there exists $C > 0$ such that

$$\max_{x\in\mathcal{X}} |\mathcal{L}_{n,\varepsilon}u(x) - \Delta_\rho u(x)| \le C(\lambda+\varepsilon)\|u\|_{C^3(\overline{U})}$$

with probability at least $1 - C\exp(-cn\varepsilon^{d+2}\lambda^2 + \log(n))$.

Continuum limit of the solution to (DBP)

Let $g : U \to \mathbb{R}$ be some Lipschitz function. We define the discrete boundary value problem

- (DBP)

$$\begin{cases} \mathcal{L}_{n,\varepsilon} u(x) = 0 & \text{if } x \in \mathcal{X} \setminus \partial_\varepsilon U \\ u(x) = g(x) & \text{if } x \in \mathcal{X} \cap \partial_\varepsilon U \end{cases} \tag{1}$$

- (CBP)

$$\begin{cases} \Delta_\rho u(x) = 0 & \text{if } x \in U \\ u(x) = g(x) & \text{if } x \in \partial U \end{cases} \tag{2}$$

Continuum limit of solutions to (DBP)

Theorem 9 (Continuum limit of solutions to (DBP))

Let $\varepsilon \in (0,1)$. *Suppose* $u_{n,\varepsilon} \in \ell^2(\mathcal{X}), u \in C^3(\overline{U})$ *are solutions to (DBP) and (CBP) respectively. Then for any* $\lambda \in (0,1]$

$$\max_{x \in \mathcal{X}} |u_{n,\varepsilon}(x) - u(x)| \leq C \|u\|_{C^3(\overline{U})}(\lambda + \varepsilon) \tag{3}$$

with probability at least $1 - C\exp(-cn\varepsilon^{d+2}\lambda^2 + \log(n))$.

Continuum limit of solutions to (DBP)

Let $\phi \in C^3(\overline{U})$ be the solution to

$$\begin{cases} -\Delta_\rho \phi = 1 \text{ in } U \\ \phi = 0 \text{ on } \partial U \end{cases} \tag{4}$$

- $\phi \geq 0$, as at its minimum $\Delta_\rho \phi(x) = \rho^{-1} \operatorname{div}(\rho^2 \nabla \phi(x)) \geq 0$.
- Goal is to show that $|u_{n,\varepsilon}(x) - u(x)| \leq K\phi(x)$ for

$$K = 3C_1 \|u\|_{C^3(\overline{U})}(\lambda + \varepsilon)$$

Continuum limit of the solution to (DBP)

Recall from the consistency results, that there exists some C_1 such that

$$\max_{x \in \mathcal{X} \setminus \partial_\varepsilon U|} |\mathcal{L}_{n,\varepsilon}\phi(x) + 1| \leq C_1(\lambda + \varepsilon) \tag{5}$$

and

$$\max_{x \in \mathcal{X} \setminus \partial_\varepsilon U} |\mathcal{L}_{n,\varepsilon}u(x)| \leq C_1 \|u\|_{C^3(U)}(\lambda + \varepsilon) \tag{6}$$

- Trivial case: if $C_1(\lambda + \varepsilon) \geq \frac{1}{2}$ then by the maximum principle

$$|u_{n,\varepsilon}(x) - u(x)| \leq 2\|g\|_\infty \leq 4C_1(\lambda + \varepsilon)\|u\|_{C^3(\overline{U})}$$

Continuum limit of the solution to (DBP)

- Main case: suppose $C_1(\lambda+\varepsilon) \le \frac{1}{2}$. Then $\mathcal{L}_{n,\varepsilon}\phi \le -\frac{1}{2}$.
- Let $w = u - u_{n,\varepsilon} - K\phi$. Then on $\mathcal{X} \setminus \partial_\varepsilon U$

$$\begin{aligned}\mathcal{L}_{n,\varepsilon} w(x) &= \mathcal{L}_{n,\varepsilon} u(x) - \mathcal{L}_{n,\varepsilon} u_{n,\varepsilon}(x) - K\mathcal{L}_{n,\varepsilon}\phi(x) \\ &\ge \frac{K}{2} - C_1\|u\|_{C^3(U)}(\lambda+\varepsilon) > 0\end{aligned}$$

  if

$$K = 3C_1\|u\|_{C^3(\overline{U})}(\lambda+\varepsilon).$$

- Observe that for $x \in \mathcal{X} \cap \partial_\varepsilon U$

$$|u_{n,\varepsilon}(x) - u(x)| \le Lip(g)\varepsilon \le \|u\|_{C^3(\overline{U})}\varepsilon$$

  So, on the boundary $w \le -K\phi + \|u\|_{C^3(\overline{U})}\varepsilon \le \|u\|_{C^3(\overline{U})}\varepsilon$

Continuum limit of the solution to (DBP)

- The maximum principle applied to w we have $w(x) \leq \|u\|_{C^3(\overline{U})}\varepsilon$ on $\mathcal{X}$. Thus $\exists C > 0$ such that

$$\begin{aligned} u(x) - u_{n,\varepsilon}(x) &\leq \|u\|_{C^3(\overline{U})}\varepsilon + K\phi(x) \\ &= \|u\|_{C^3(\overline{U})}\varepsilon + 2C_1\|\phi\|_\infty\|u\|_{C^3(\overline{U})}(\lambda + \varepsilon) \\ &=: C\|u\|_{C^3(\overline{U})}(\lambda + \varepsilon) \end{aligned}$$

- The same argument could be applied to $w = u_{n,\varepsilon} - u - K\phi$ to get

$$u_{n,\varepsilon}(x) - u(x) \leq C\|u\|_{C^3(\overline{U})}(\lambda + \varepsilon)$$

Connection to the graph Dirichlet energy

Theorem 10

Let $\mathcal{E}(u) = (\nabla u, \nabla u)_{\ell^2(\mathcal{X}^2)}$ and $\mathcal{A} = \{u \in \ell^2(\mathcal{X}) : u(x) = g(x) \text{ on } \Gamma\}$. Suppose $u \in \mathcal{A}$ satisfies

$$\mathcal{E}(u) = \min_{w \in \mathcal{A}} \mathcal{E}(w).$$

Then u solves (DBP).

Let $u \in \mathcal{A}$ and $v \in \ell^2(\mathcal{X})$ such that $v = 0$ on Γ. Then

$$\begin{aligned}
0 = \frac{d}{dt}\bigg|_{t=0} \mathcal{E}(u+tv) &= \frac{d}{dt}\bigg|_{t=0} (-\mathcal{L}(u+tv), u+tv)_{\ell^2(\mathcal{X})} \\
&= \frac{d}{dt}\bigg|_{t=0} \left((-\mathcal{L}u, u) + t(-\mathcal{L}u, v) + t(u, 0\mathcal{L}v) + t^2(-\mathcal{L}v, v)\right) \\
&= (-\mathcal{L}u, v) + (u, -\mathcal{L}v) = 2(-\mathcal{L}u, v).
\end{aligned}$$

Conclusion

Thank you!

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