

MIMICKING AN ITÔ PROCESS BY A SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION

BY GERARD BRUNICK¹ AND STEVEN SHREVE²

University of California and Carnegie Mellon University

Given a multi-dimensional Itô process whose drift and diffusion terms are adapted processes, we construct a weak solution to a stochastic differential equation that matches the distribution of the Itô process at each fixed time. Moreover, we show how to match the distributions at each fixed time of functionals of the Itô process, including the running maximum and running average of one of the components of the process. A consequence of this result is that a wide variety of exotic derivative securities have the same prices when the underlying asset price is modeled by the original Itô process or the mimicking process that solves the stochastic differential equation.

1. Introduction. We construct a process that mimics certain properties of a given Itô process, but is simpler in the sense that the mimicking process solves a stochastic differential equation (SDE), while the Itô process may have drift and diffusion terms that are themselves stochastic processes. This work is motivated by the problem of model calibration in finance. The financial engineer would like to identify a class of models for an underlying asset price that is flexible enough to allow for calibration to a wide range of possible market prices of derivative securities on that asset. The result of this paper shows the extent to which sophisticated models are no more powerful for calibration purposes than an SDE for the underlying asset price.

Our results are closely related to Krylov [25] and Gyöngy [18]. Krylov [25] calls the measure that records the average amount of time that an Itô process X spends in each Borel set before being killed at the first jump of an independent Poisson process with intensity λ the *Green λ -measure* of X . Given an Itô process with bounded drift and bounded, uniformly positive-definite covariance, Krylov [25] constructs a process with the same Green λ -measure which solves a time-independent diffusion equation. Krylov further asserts that it is possible to construct a process that solves a time-dependent diffusion equation and matches the one-dimensional marginal distributions of such an Itô process. Gyöngy [18] provides a proof of Krylov's assertion and shows that the drift and covariance in the

Received October 2011; revised May 2012.

¹Supported in part by the NSF Grant DMS-04-04682.

²Supported in part by the NSF Grants DMS-04-04682 and DMS-09-03475.

MSC2010 subject classifications. 60G99, 60H10, 91G20.

Key words and phrases. Itô process, stochastic differential equation, derivative security pricing, stochastic volatility models.

diffusion equation solved by the mimicking process may be interpreted as the expected value of the Itô process's instantaneous drift and covariance conditioned on its level. See also Klebaner [24] for a related argument based on semimartingale local time.

Gyöngy [18] was rediscovered by the mathematical finance community in the context of local volatility models. Dupire [12] studies the European option prices generated by a model in which the risk-neutral dynamics of the price process satisfy a time-dependent diffusion equation (see also Derman and Kani [10] for a discrete-time treatment of this topic). These models are now known as *local volatility models*, and the diffusion coefficient of the log-price process is known as the *local volatility surface*. Dupire [12] shows that it is possible to construct a local volatility model that is consistent with a given set of European option prices when that set of prices is sufficiently smooth as a function of maturity and strike, and he shows how the local volatility surface may be implied directly from the call prices. Local volatility models have proven popular with practitioners because they allow for calibration to a wide range of European option prices. Dupire [12] does not find the dynamics of a local volatility model to be particularly plausible; however, he asserts that “the market prices European options *as if* the process was this diffusion.” In effect, the local volatility model mimics the European option prices of some more complicated market process, and this is equivalent to matching the one-dimensional marginal distributions of that process under the equivalent martingale probability measure (also call the *risk-neutral measure*) used for pricing.

In [13], Dupire extends [12] to study the local volatility surface that is implied not by market prices of options but by prices generated from a stochastic volatility model. Using infinitesimal calendar and butterfly spreads, he presents a financial argument that the square of the local volatility function is the expected value of the instantaneous squared stochastic volatility conditioned on the level of the underlying asset price, essentially recovering Gyöngy's result, albeit in a nonrigorous fashion. Following this development, the Gyöngy–Dupire formula has found several applications in finance. For example, Gatheral [17] uses it to compare the properties of a number of stochastic volatility models, and Antonov and Misirpashaev [2] and Piterbarg [30–32] combine it with parameter averaging techniques to produce pricing approximations based on approximations of the second conditional expectation appearing in (3.10) below, a special case of (3.7) in our main result.

Brigo and Mercurio [4, 5] use a related methodology to construct a scalar diffusion whose one-dimensional marginal distributions are given as a mixture of known densities. Bentata and Cont [3] recently announced an extension of Gyöngy's result to jump diffusions under a continuity assumption on the coefficients in the mimicking process and a nondegeneracy assumption on the covariance or the jump measure of the mimicking process.

Here we extend Gyöngy [18] in two ways. First, we remove the conditions of nondegeneracy and boundedness on the covariance of the Itô process to be mimicked, requiring only integrability of this process and thereby extending the result

to cover popular stochastic volatility models such as the one due to Heston [20]. Second, we show that the mimicking process can preserve the joint distribution of certain functionals of the Itô process (e.g., running maximum and running average) at each fixed time. Our mimicking process is a weak solution to an SDE, and in the case of preservation of the joint distribution of functionals of the Itô process, the coefficients in this SDE may depend on the values of these functionals as well as the current value of the underlying Itô process.

The conditions that permit our construction are so weak that the solution to the SDE we derive is not necessarily unique. Uniqueness results, such as those found in Stroock and Varadhan [34, 35], require the conditional expectations determined by the Gyöngy–Dupire formulas [see (3.7) in this paper] to be sufficiently regular functions of the conditioning variables. It is difficult to see what conditions one should impose on the data of our model (the processes b and σ and the updating function Φ of Theorem 3.6) to ensure such regularity. Of course, if one is willing to assume that the coefficients in the mimicking equation are sufficiently well-behaved, then it is often possible to conclude that the solution to the mimicking equation is unique.

Finally, we mention an independent body of work devoted to a problem similar to the one considered here. If an Itô process is a submartingale, Kellerer [23] has shown that it can be mimicked by a Markov process. More generally, [23] shows that given any set of marginal densities $p(t, \cdot)$, $t \geq 0$, that have finite first moments and satisfy $\int \varphi(y)p(s, y)dy \leq \int \varphi(y)p(t, y)$ for every $t \geq s \geq 0$ and every non-decreasing convex function φ , there is a Markov submartingale whose density at each time t is $p(t, \cdot)$. Madan and Yor [27] provide constructions of such Markov processes in three specific cases in which the first moments of $p(t, \cdot)$ are independent of t . Cox, Hobson and Oblój [8] and Ekström et al. [14] provide related constructions. Forde [15] studies the problem of matching the joint law of a process and its running maximum at an independent exponential time. Our results address the specific case in which the densities $p(t, \cdot)$ are the marginals of an Itô process. Our mimicking process satisfies an SDE, but because the solution to this equation might not be unique, we are not able to establish the Markov property in all cases. On the other hand, we have the Gyöngy–Dupire formulas for the drift and diffusion coefficients of our mimicking process.

This paper is based on the first author’s Ph.D. dissertation [6]. It is organized as follows. Section 2 presents an intuitive discrete-time example that illustrates the main ideas of our construction. In Section 3 we state our main result, Theorem 3.6, and provide some useful corollaries. To prove Theorem 3.6, we construct a weakly relatively compact sequence of processes that mimic some initial target process. We then extract a limit from this sequence, check that the mimicking property is preserved under weak convergence, and compute the semimartingale characteristics of the limiting process. The tools to implement this strategy are developed in Sections 4–6, and the proof of Theorem 3.6 is given in Section 7.

More specifically, in Section 4 we begin with a probability measure on path space and construct a “concatenated” measure which assigns the same unconditional distribution as the original measure to fragments of paths between concatenation time points but changes the dependency structure across these time points. The new dependency structure corresponds to “partially forgetting” the past at each concatenation time point, and the resulting process possesses a limited Markov-like property. The existence and uniqueness of the concatenated measure are provided by Theorem 4.3, and Section 4.1 is devoted to the statement and proof of that theorem. Although the concatenated measure may not be equivalent to the original measure, certain properties of the process, such as finite variation and absolute continuity, are preserved by the construction. The properties we need are set out in Section 4.2. The most important result of this subsection is Proposition 4.15, which provides conditions that are sufficient to ensure that the semimartingale characteristics of the initial process are not disturbed by the concatenation procedure.

Section 5 sets out conditions under which the conditional expectation of one process conditioned on a second process can be written as a function of time and the second process. This result, Proposition 5.1, is extended to include conditioning on a random time as well in Proposition 5.4.

Finally, in Section 6, we set up the machinery for taking the limit of a sequence of concatenated measures. Proposition 6.1 provides conditions on a sequence of weakly converging processes that guarantee joint convergence of the processes and the integral of a function of the processes. Proposition 6.3 shows how to approximate a process in L_1 by a piecewise constant process constructed from the original process by sampling at random times. Proposition 6.5 shows that if a sequence of discrete-time martingales is constructed by integrating with respect to time a sequence of uniformly integrable processes and sampling these integrals at stopping times, and if the maximum time between successive stopping times approaches zero, then the integrand processes must also approach zero.

2. Guiding example. To motivate the results that follow, we first sketch a mimicking result for discrete-time processes. This setting illustrates the main ideas of our proof methodology without the technical complications of continuous time.

Let \mathbb{N}_0 denote the set of nonnegative integers, let $\mathcal{B}(\mathbb{R})$ denote the Borel σ -field on \mathbb{R} and let $(X_n)_{n \in \mathbb{N}_0}$ denote a (not necessarily Markov) stochastic process in discrete time that takes values in \mathbb{R} . For each $n \in \mathbb{N}_0$, we may construct a measurable transition kernel $p_n : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ with the property that $p_n(X_n; A)$ is a version of $\mathbb{P}[X_{n+1} - X_n \in A | \sigma(X_n)]$ for each $A \in \mathcal{B}(\mathbb{R})$ and $A \mapsto p_n(x; A)$ is a probability measure for each $x \in \mathbb{R}$.

After moving to a suitable extension of our probability space if necessary, we may construct a process Y such that $Y_0 = X_0$; $Y_{n+1} - Y_n$ is conditionally independent of \mathcal{F}_n given Y_n ; and $p_n(Y_n; A)$ is a version of $\mathbb{P}(Y_{n+1} - Y_n \in A | \mathcal{F}_n)$ for each $n \in \mathbb{N}_0$ and $A \in \mathcal{B}(\mathbb{R})$. It follows from these properties that

$$\mathbb{E}[f(Y_{n+1}) | \mathcal{F}_n] = \int f(Y_n + x) p_n(Y_n; dx),$$

so Y is a Markov process. We also have

$$\mathbb{E}[f(X_{n+1})] = \mathbb{E}[\mathbb{E}[f(X_{n+1})|\sigma(X_n)]] = \mathbb{E}\left[\int f(X_n + x)p_n(X_n; dx)\right]$$

and $Y_0 = X_0$, so an inductive argument shows that Y_n has the same law as X_n for each n . This is essentially the construction given by Derman and Kani [11].

Given a discrete-time process X , we now let $\bar{X}_n = \max_{0 \leq i \leq n} X_i$ denote the running maximum of the process X . Although the law of the random variable Y_n constructed above agrees with the law of X_n for each fixed n , the law of the process Y may certainly differ from the law of the process X . In particular, the law of the pair (X_n, \bar{X}_n) may not agree with the law of the pair (Y_n, \bar{Y}_n) when $n \geq 1$. Nevertheless, one can construct a second process Z such that the two-dimensional process (Z, \bar{Z}) is Markov and the joint law of the pair (Z_n, \bar{Z}_n) agrees with the joint law of the pair (X_n, \bar{X}_n) for each n , as we now show.

We let $q_n: \mathbb{R}^2 \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ denote a transition kernel with the property that $p_n(X_n, \bar{X}_n; A)$ is a version of $\mathbb{P}[X_{n+1} - X_n \in A | \sigma(X_n, \bar{X}_n)]$ for each $A \in \mathcal{B}(\mathbb{R})$. Moving to another extension of our probability space, we may construct a process Z such that $Z_0 = X_0$; $Z_{n+1} - Z_n$ is conditionally independent of \mathcal{F}_n given (Z_n, \bar{Z}_n) ; and $p_n(Z_n, \bar{Z}_n; A)$ is a version of $\mathbb{P}(Z_{n+1} - Z_n \in A | \mathcal{F}_n)$ for each $n \in \mathbb{N}_0$.

We define $\Phi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $\Phi(e_1, e_2; x) = (e_1 + x, e_2 \vee (e_1 + x))$, so that $(Z_{n+1}, \bar{Z}_{n+1}) = \Phi(Z_n, \bar{Z}_n; Z_{n+1} - Z_n)$. We may use the function Φ and the increments of the process Z to update the state of the process (Z, \bar{Z}) . One immediate consequence of this structure is that

$$\mathbb{E}[f(Z_{n+1}, \bar{Z}_{n+1}) | \mathcal{F}_n] = \int f \circ \Phi(Z_n, \bar{Z}_n; y)q_n(Z_n, \bar{Z}_n; dy),$$

so (Z, \bar{Z}) is a Markov process. We also have

$$\mathbb{E}[f(X_{n+1}, \bar{X}_{n+1})] = \mathbb{E}\left[\int f \circ \Phi(X_n, \bar{X}_n; y)q_n(X_n, \bar{X}_n; dy)\right],$$

so another inductive argument shows that the law of the pair (Z_n, \bar{Z}_n) agrees with the law of the pair (X_n, \bar{X}_n) for each n . This paper extends this construction to continuous time.

3. Main result. In order to precisely state our main result, we need some notation. The symbol \mathcal{E} will always denote a closed subset of a complete separable metric space, that is, a Polish space. Let $C^\mathcal{E}$ be the space of continuous functions from $[0, \infty)$ to \mathcal{E} , endowed with the topology of uniform convergence on compact subsets of $[0, \infty)$. We define the *shift operator* $\Theta: C^\mathcal{E} \times \mathbb{R} \rightarrow C^\mathcal{E}$ by

$$\Theta(x, t) \triangleq x((t + \cdot)^+),$$

the *stopping operator* $\nabla : C^\mathcal{E} \times [0, \infty) \rightarrow C^\mathcal{E}$ by

$$\nabla(x, t) \triangleq x(\cdot \wedge t)$$

and, if \mathcal{E} is a vector space, the *difference operator* $\Delta : C^\mathcal{E} \times [0, \infty) \rightarrow C^\mathcal{E}$ by

$$\Delta(x, t) \triangleq x(t + \cdot) - x(t).$$

In contrast to usual practice, here the shift operator can shift paths to the right because t can be negative, and in this case, the shifted path takes the value $x(0)$ on $[0, -t]$. The difference operator actually maps into $C_0^\mathcal{E}$, the space of continuous functions from $[0, \infty)$ to \mathcal{E} with initial condition zero. If $\mathcal{E} = \mathbb{R}^d$ for some integer d , we write C^d and C_0^d rather than $C^{\mathbb{R}^d}$ and $C_0^{\mathbb{R}^d}$.

Fix a Polish space \mathcal{E} , fix a positive integer d and define $\Omega^{\mathcal{E},d} \triangleq \mathcal{E} \times C_0^d$. We endow $\Omega^{\mathcal{E},d}$ with the product topology. We denote a generic element of $\Omega^{\mathcal{E},d}$ by $\omega = (e, x)$ and define the random variable $E(e, x) = e$ and the \mathbb{R}^d -valued process $X(e, x) = x$. For a random time T , we use the notation X^T to denote the process X stopped at T , that is,

$$(3.1) \quad X_t^T(\omega) = X_{t \wedge T(\omega)}(\omega) = \nabla_t(X(\omega), T(\omega)), \quad t \geq 0.$$

DEFINITION 3.1. We say that $\Phi : \Omega^{\mathcal{E},d} \rightarrow C^\mathcal{E}$ is an *updating function* provided

$$(3.2) \quad \Phi_0(e, x) = e, \quad e \in \mathcal{E},$$

$$(3.3) \quad \Phi^t(e, x) = \Phi^t(e, \nabla(x, t)), \quad t \geq 0, e \in \mathcal{E}, x \in C_0^d,$$

$$(3.4) \quad \Theta(\Phi(e, x), t) = \Phi(\Phi_t(e, x), \Delta(x, t)), \quad t \geq 0, e \in \mathcal{E}, x \in C_0^d.$$

In other words, Φ takes an initial condition in \mathcal{E} [see (3.2)] and a path in C_0^d and generates a path in $C^\mathcal{E}$. Property (3.3) says that the path $\Phi(e, x)$ stopped at t depends only on the initial condition e and the path of x stopped at t . This is a nonanticipative property. Property (3.4) is a type of Markov property, but on a path-by-path basis without the presence of a probability measure. It implies that the path of $\Phi(e, x)$ from time t onward depends only on the value of the path at time t and the increments of x from time t onward. Using the characterization of the Markov property as independence of the future and past given the present, it is easily verified that if ξ is a continuous \mathbb{R}^d -valued Markov process, and if for each t the value of ξ_t can be deduced from the value of $\Phi_t(\xi_0, \xi - \xi_0)$, then $\Phi(\xi_0, \xi - \xi_0)$ is also Markov.

EXAMPLE 3.2 (Process itself). A trivial case of an updating function is obtained if we let $\mathcal{E} = \mathbb{R}^d$, $\Omega^{\mathcal{E},d} = \mathcal{E} \times C_0^d$ and $\Phi(e, x) = e + x$ for $e \in \mathbb{R}^d$ and $x \in C_0^d$. If ξ is a continuous \mathbb{R}^d -valued Markov process and we represent ξ as $(\xi_0, \xi - \xi_0) \in \mathcal{E} \times C_0^d$, then $\Phi_t(\xi_0, \xi - \xi_0) = \xi_t$ and $\Phi(\xi_0, \xi - \xi_0) = \xi$ is Markov.

EXAMPLE 3.3 (Integral-to-date). Let $\mathcal{E} = \mathbb{R}^2$ and $\Omega^{\mathcal{E},1} = \mathcal{E} \times C_0^1$. We interpret a point $(e_1, e_2; x) \in \Omega^{\mathcal{E},1}$ as a path $e_1 + x$ with initial condition $e_1 + x(0) = e_1$ and the initial value of a running integral given by e_2 . It is then easy to check that

$$\Phi_t(e_1, e_2; x) = \left(e_1 + x(t), e_2 + \int_0^t (e_1 + x(s)) ds \right)$$

is an updating function.

EXAMPLE 3.4 (Maximum-to-date). Let $\mathcal{E} = \{(e_1, e_2) \in \mathbb{R}^2 : e_1 \leq e_2\}$ and $\Omega^{\mathcal{E},1} = \mathcal{E} \times C_0^1$. We regard the generic element $(e_1, e_2; x) \in \Omega^{\mathcal{E},1}$ as a path $e_1 + x$ with initial condition $e_1 + x(0) = e_1$ and the time-zero maximum-to-date e_2 . Given such a triple, the value of the path at a later time t and the maximum-to-date at that time t are $e_1 + x(t)$ and $e_2 \vee \max_{0 \leq s \leq t} (e_1 + x(s))$, respectively. We thus define

$$\Phi_t(e_1, e_2; x) = \left(e_1 + x(t), e_2 \vee \max_{0 \leq s \leq t} (e_1 + x(s)) \right).$$

It is straightforward to verify that Φ is an updating function. If ξ is a continuous real-valued Markov process, then $\Phi_t(\xi_0, M_0; \xi - \xi_0) = (\xi_t, M_0 \vee \max_{0 \leq s \leq t} \xi_s)$ is also Markov, where M_0 is any random variable satisfying $M_0 \geq \xi_0$ almost surely.

As a final extremal example, we give an updating function that records the entire history of the path.

EXAMPLE 3.5 (Path-to-date). Define $\mathcal{E} = \{(s, x) \in [0, \infty) \times C^d; x \text{ is constant on } [s, \infty)\}$, define $\Omega^{\mathcal{E},d} = \mathcal{E} \times C_0^d$, and set

$$\begin{aligned} \Phi_t(s, x; y) &= (s + t, \nabla(\nabla(x, s) + \Theta(y, -s), s + t)), \\ & \qquad \qquad \qquad x \in C^d, s \in [0, \infty), y \in C_0^d. \end{aligned}$$

Given paths $x \in C^d$ and $y \in C_0^d$ and a time $s \geq 0$, $\nabla(x, s) + \Theta(y, -s)$ is the path that follows x on $[0, s]$ with y appended after time s . The second component of Φ_t is this path stopped at time $s + t$. The first component of Φ_t is the time $s + t$ at which this path is stopped. As t marches forward, the second component of the operator Φ applied to $(s, x; y)$ appends more and more of the path y to the path x , always appending at time s . It is tedious but straightforward to check that Φ is an updating function. For any continuous \mathbb{R}^d -valued process ξ , we have

$$(3.5) \qquad \qquad \Phi_t(0, \xi^0; \xi - \xi_0) = (t, \xi^t), \qquad t \geq 0,$$

where we recall from (3.1) that ξ^t is the process ξ stopped at t .

THEOREM 3.6 (Main result). *Suppose an \mathbb{R}^d -valued process Y is given by*

$$(3.6) \qquad \qquad Y_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \qquad t \geq 0,$$

where W is an \mathbb{R}^r -valued Brownian motion under some probability measure \mathbb{P} , b is an \mathbb{R}^d -valued process adapted to a filtration under which W is a Brownian motion and σ is a $d \times r$ matrix-valued process adapted to the same filtration as b . Let \mathcal{E} be a Polish space, define $\Omega^{\mathcal{E},d} \triangleq \mathcal{E} \times C_0^d$, let $\Phi: \Omega^{\mathcal{E},d} \rightarrow C^{\mathcal{E}}$ be a continuous updating function, let Z_0 be an \mathcal{E} -valued random variable and set $Z = \Phi(Z_0, Y)$, which is a continuous \mathcal{E} -valued process. Finally, assume that $\mathbb{E} \int_0^t (\|b_s\| + \|\sigma_s \sigma_s^{tr}\|) ds < \infty$ for $t \geq 0$. Then there exists an \mathbb{R}^d -valued measurable function \widehat{b} and a $d \times d$ matrix-valued measurable function $\widehat{\sigma}$, both defined on $[0, \infty) \times \mathcal{E}$, and there exists a Lebesgue-null set $N \subset [0, \infty)$, so that³

$$(3.7) \quad \begin{aligned} \widehat{b}(t, Z_t) &= \mathbb{E}[b_t | Z_t], \\ \widehat{\sigma}(t, Z_t) \widehat{\sigma}^{tr}(t, Z_t) &= \mathbb{E}[\sigma_t \sigma_t^{tr} | Z_t], \quad \mathbb{P}\text{-a.s.}, t \in N^c. \end{aligned}$$

Furthermore, there exists a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\}_{t \geq 0}, \widehat{\mathbb{P}})$ that supports a continuous \mathbb{R}^d -valued adapted process \widehat{Y} , a continuous \mathcal{E} -valued adapted process \widehat{Z} and a d -dimensional Brownian motion \widehat{W} satisfying

$$(3.8) \quad \widehat{Y}_t = \int_0^t \widehat{b}(s, \widehat{Z}_s) ds + \int_0^t \widehat{\sigma}(s, \widehat{Z}_s) d\widehat{W}_s, \quad \widehat{Z} = \Phi(\widehat{Z}_0, \widehat{Y}), \quad t \geq 0,$$

and such that for each $t \geq 0$, the distribution of \widehat{Z}_t under $\widehat{\mathbb{P}}$ agrees with the distribution of Z_t under \mathbb{P} .

Although both Y in (3.6) and \widehat{Y} in (3.8) are d -dimensional processes, the “state” \widehat{Z} of the system in (3.8) can be of a much lower dimension than the state process needed to describe (3.6). In (3.6) the processes b and σ are typically given by stochastic differential equations driven by additional Brownian motions not mentioned in the statement of the theorem. The process \widehat{Z} is typically the process \widehat{Y} itself augmented by some functional of the path of \widehat{Y} . We give examples below. Indeed, the remainder of this section illustrates the applications of Theorem 3.6. In this section we also show by example that (3.8) can have multiple solutions and discuss conditions that guarantee uniqueness. The subsequent sections are devoted to the proof of Theorem 3.6.

Note that Y in Theorem 3.6 is a martingale if and only if b_s is zero for Lebesgue almost every s almost surely. In this case, \widehat{b} is also zero, and \widehat{Y} is a local martingale. But since \widehat{Z}_s has the same distribution as Z_s for each s , the integrability condition assumed on $\sigma \sigma^{tr}$ implies the same condition on $\widehat{\sigma} \widehat{\sigma}^{tr}$ and \widehat{Y} is in fact a martingale.

As a first application, we take $Y = X - X_0$ and $Z = X$ in Theorem 3.6 and use the updating function of Example 3.2. We then have the following corollary, which is the result obtained by Gyöngy [18], but here without the boundedness and nondegeneracy assumptions of [18].

³We interpret (3.7) and subsequent similar equations to mean that for each fixed $t \in N^c$, the left-hand side of each equation is a version of the conditional expectation appearing on the right-hand side.

COROLLARY 3.7 (Process itself). *Suppose an \mathbb{R}^d -valued process X is given by*

$$(3.9) \quad X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \quad t \geq 0,$$

where W, \mathbb{P}, b and σ are as in Theorem 3.6. Then there exists an \mathbb{R}^d -valued measurable function \widehat{b} and a $d \times d$ matrix-valued measurable function $\widehat{\sigma}$, both defined on $[0, \infty) \times \mathbb{R}^d$, and there exists a Lebesgue-null set N , so that

$$(3.10) \quad \begin{aligned} \widehat{b}(t, X_t) &= \mathbb{E}[b_t | X_t], \\ \widehat{\sigma}(t, X_t) \widehat{\sigma}^{tr}(t, X_t) &= \mathbb{E}[\sigma_t \sigma_t^{tr} | X_t], \quad \mathbb{P}\text{-a.s.}, t \in N^c. \end{aligned}$$

Furthermore, there exists a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\}_{t \geq 0}, \widehat{\mathbb{P}})$ that supports a continuous \mathbb{R}^d -valued adapted process \widehat{X} and a d -dimensional Brownian motion \widehat{W} satisfying

$$(3.11) \quad \widehat{X}_t = \widehat{X}_0 + \int_0^t \widehat{b}(s, \widehat{X}_s) ds + \int_0^t \widehat{\sigma}(s, \widehat{X}_s) d\widehat{W}_s, \quad t \geq 0,$$

and such that for each $t \geq 0$, the distribution of \widehat{X}_t under $\widehat{\mathbb{P}}$ agrees with the distribution of X_t under \mathbb{P} .

EXAMPLE 3.8 (Fake Brownian motion). Let G_1 and G_2 be standard normal random variables, let $(B_t)_{t \geq 0}$ be Brownian motion and assume that G_1, G_2 , and W are independent. Define the process

$$X_t = \sqrt{t}(G_1 \cos B_{\ln t} + G_2 \sin B_{\ln t}), \quad t \geq 1,$$

and set $\mathcal{F}_t = \sigma(G_1, G_2, B_s, 0 \leq s \leq \ln t)$ for $t \geq 1$. Then X is a continuous martingale with respect to $\{\mathcal{F}_t\}_{t \geq 1}$ and $\langle X \rangle_t = \int_1^t \sigma_s^2 ds$ for $t \geq 1$, where $\sigma_t = -G_1 \sin B_{\ln t} + G_2 \cos B_{\ln t}$. In particular, we may write X in the form

$$X_t = X_1 + \int_1^t \sigma_s dW_s, \quad t \geq 1,$$

for some Brownian motion $(W_t)_{t \geq 1}$.

Conditioned on the value of $B_{\ln t}$, the random variables X_t/\sqrt{t} and σ_t are independent and standard normal, so they are unconditionally independent and standard normal. Consequently,

$$\mathbb{E}[\sigma_t^2 | X_t] = \mathbb{E}[\sigma_t^2] = 1, \quad \mathbb{P}\text{-a.s.}, t \geq 1,$$

and we may take $\widehat{b} = 0$ and $\widehat{\sigma} = 1$ in the previous corollary. As X_1 is standard normal, the previous corollary, adapted to the time interval $[1, \infty)$, asserts that the process X has the same one-dimensional marginal distributions as a Brownian motion on $[1, \infty)$. This is not hard to check directly in this example.

This construction is due to Oleszkiewicz [28] who was interested in producing a *fake Brownian motion* (see also [1, 19]). A fake Brownian motion is a continuous martingale that has the same one-dimensional marginal distributions as a Brownian motion but is not itself a Brownian motion. Oleszkiewicz shows that the process X constructed above can be extended to produce a fake Brownian motion on the time interval $[0, \infty)$. The argument given in this example can be extended to show that the process which mimics Oleszkiewicz’s fake Brownian motion in the sense of Corollary 3.7 is simply Brownian motion.

Taking $Y = X - X_0$ and $Z_t = (X_t, A_t)$, and using the updating function in Example 3.3, we obtain the following corollary about the distribution of a process and its running integral.

COROLLARY 3.9 (Integral-to-date). *Suppose a real-valued process X is given by (3.9) where W, \mathbb{P}, b and σ are as in Theorem 3.6 with $d = r = 1$. Let A be a continuous process such that*

$$A_t = A_0 + \int_0^t X_s ds, \quad t \geq 0.$$

Then there exists a real-valued measurable function \widehat{b} and a $[0, \infty)$ -valued measurable function $\widehat{\sigma}$, both defined on $[0, \infty) \times \mathbb{R}^2$, and there exists a Lebesgue-null set N , such that

$$\begin{aligned} \widehat{b}(t, X_t, A_t) &= \mathbb{E}[b_t | X_t, A_t], \\ \widehat{\sigma}^2(t, X_t, A_t) &= \mathbb{E}[\sigma_t^2 | X_t, A_t], \quad \mathbb{P}\text{-a.s.}, t \in N^c. \end{aligned}$$

Furthermore, there exists a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\}_{t \geq 0}, \widehat{\mathbb{P}})$ that supports continuous real-valued adapted processes \widehat{X} and \widehat{A} and a real-valued Brownian motion \widehat{W} satisfying

$$\begin{aligned} \widehat{X}_t &= \widehat{X}_0 + \int_0^t \widehat{b}(s, \widehat{X}_s, \widehat{A}_s) ds + \int_0^t \widehat{\sigma}(s, \widehat{X}_s, \widehat{A}_s) d\widehat{W}_s, \quad t \geq 0, \\ \widehat{A}_t &= \widehat{A}_0 + \int_0^t \widehat{X}_s ds, \quad t \geq 0, \end{aligned} \tag{3.12}$$

and such that for each $t \geq 0$, the distribution of the pair $(\widehat{X}_t, \widehat{A}_t)$ under $\widehat{\mathbb{P}}$ agrees with the distribution of the pair (X_t, A_t) under \mathbb{P} .

Taking $Y = X - X_0$ and $Z_t = (X_t, M_t)$, and using the updating function in Example 3.4, we obtain the following corollary about the distribution of a process and its running maximum.

COROLLARY 3.10 (Maximum-to-date). *Suppose a real-valued process X is given by (3.9) where W, \mathbb{P}, b and σ are as in Theorem 3.6 with $d = r = 1$. Let M_0 be a random variable satisfying $M_0 \geq X_0$ almost surely and define*

$$M_t = M_0 \vee \max_{0 \leq s \leq t} X_s, \quad t \geq 0.$$

Then there exists a real-valued measurable function \widehat{b} and a $[0, \infty)$ -valued measurable function $\widehat{\sigma}$, both defined on $[0, \infty) \times \mathbb{R}^2$, and there exists a Lebesgue-null set N , such that

$$\begin{aligned} \widehat{b}(t, X_t, M_t) &= \mathbb{E}[b_t | X_t, M_t], \\ \widehat{\sigma}^2(t, X_t, M_t) &= \mathbb{E}[\sigma_t^2 | X_t, M_t], \quad \mathbb{P}\text{-a.s.}, t \in N^c. \end{aligned}$$

Furthermore, there exists a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\}_{t \geq 0}, \widehat{\mathbb{P}})$ that supports continuous real-valued adapted processes \widehat{X} and \widehat{M} and a real-valued Brownian motion \widehat{W} satisfying

$$(3.13) \quad \begin{aligned} \widehat{X}_t &= \widehat{X}_0 + \int_0^t \widehat{b}(s, \widehat{X}_s, \widehat{M}_s) ds + \int_0^t \widehat{\sigma}(s, \widehat{X}_s, \widehat{M}_s) d\widehat{W}_s, \quad t \geq 0, \\ \widehat{M}_t &= \widehat{M}_0 \vee \max_{0 \leq s \leq t} \widehat{X}_s, \quad t \geq 0, \end{aligned}$$

and such that for each $t \geq 0$, the distribution of the pair $(\widehat{X}_t, \widehat{M}_t)$ under $\widehat{\mathbb{P}}$ agrees with the distribution of the pair (X_t, M_t) under \mathbb{P} .

Taking $Y = X - X_0$ and $Z_t = (t, X^t)$, and using the updating function in Example 3.5, we obtain the following corollary, which states that every Itô process with integrable drift and covariance is a weak solution to an SDE with path-dependent coefficients.

COROLLARY 3.11 (Path-to-date). *Suppose a real-valued process X is given by (3.9) where W, \mathbb{P}, b and σ are as in Theorem 3.6. Then there exist path-dependent functionals \widehat{b} and $\widehat{\sigma}$, both defined on $[0, \infty) \times C^d$, with \widehat{b} taking values in \mathbb{R}^d and $\widehat{\sigma}$ taking values in the space of $d \times d$ matrices and a Lebesgue-null set N such that*

$$\begin{aligned} \widehat{b}(t, X^t) &= \mathbb{E}[b_t | X^t], \\ \widehat{\sigma}(t, X^t) \widehat{\sigma}^{tr}(t, X^t) &= \mathbb{E}[\sigma_t \sigma_t^{tr} | X^t], \quad \mathbb{P}\text{-a.s.}, t \in N^c. \end{aligned}$$

Furthermore, there exists a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \{\widehat{\mathcal{F}}_t\}_{t \geq 0}, \widehat{\mathbb{P}})$ that supports a continuous \mathbb{R}^d -valued adapted process \widehat{X} and a d -dimensional Brownian motion \widehat{W} satisfying

$$\widehat{X}_t = \widehat{X}_0 + \int_0^t \widehat{b}(s, \widehat{X}^s) ds + \int_0^t \widehat{\sigma}(s, \widehat{X}^s) d\widehat{W}_s, \quad t \geq 0,$$

and such that \widehat{X} has the same distribution under $\widehat{\mathbb{P}}$ as X has under \mathbb{P} .

We close this section with a brief discussion of the nonuniqueness that can arise in equation (3.8) of Theorem 3.6 and its relationship to the strong Markov property. We first provide a simple example within the context of Corollary 3.7, where X and Z are the same process.

EXAMPLE 3.12 (Nonuniqueness). Let $d = 1$ and $b = 0$ in Corollary 3.7 and let $X_t = \int_0^t \sigma_s dW_s$, where $\sigma_s = \mathbb{I}_{(1,\infty)}(s)\mathbb{I}_{\{W_1>0\}}$. Then $\hat{X}_t = \mathbb{I}_{(1,\infty)}(t) \times \mathbb{I}_{\{W_1>0\}}(W_t - W_1)$. From (3.10) we see that $\hat{\sigma}(t, y) = 0$ for $0 \leq t \leq 1$, and for $t > 1$,

$$\hat{\sigma}^2(t, y) = \mathbb{E}[\sigma_t^2 | X_t = y] = \begin{cases} 1, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases}$$

Both $\hat{X}_t^1 \equiv 0$ and $\hat{X}_t^2 = \mathbb{I}_{(1,\infty)}(t)(W_t - W_1)$ are solutions of (3.11). The weak solution \hat{X} that has the same one-dimensional distributions as X is obtained by an initial randomization that is independent of W and determines whether \hat{X} agrees with \hat{X}^1 or \hat{X}^2 , each of these events having probability $\frac{1}{2}$. This process is Markov, but not strong Markov, as can be seen by considering the stopping time that is the first time after time 2 that zero is reached.

The previous example shows that the mimicking process may not be strong Markov. Nevertheless, if we are willing to impose further conditions on the coefficients \hat{b} and $\hat{\sigma}$ appearing in Theorem 3.6, then we can often conclude that the solution to (3.11) is unique in law and strong Markov. In particular, if we assume that \hat{b} appearing in Corollary 3.7 is bounded and measurable and that $\hat{\sigma}\hat{\sigma}^{tr}$ is bounded, strictly positive-definite and continuous, then the results of Stroock and Varadhan [34, 35] ensure that the mimicking process satisfying (3.11) in Corollary 3.7 is unique in law and strong Markov with respect to its natural filtration. We state this observation as a corollary.

COROLLARY 3.13. *Let X denote an \mathbb{R}^d -valued process that satisfies equation (3.9), where W, \mathbb{P}, b and σ are as in Theorem 3.6, and suppose that there exists a locally bounded measurable \mathbb{R}^d -valued function \hat{b} and a measurable $d \times d$ matrix-valued function $\hat{\sigma}$ such that (3.10) holds and the function $\hat{a}(t, x) = \hat{\sigma}(t, x)\hat{\sigma}^{tr}(t, x)$ is continuous and strictly positive definite. Then there exists a weak solution to the SDE (3.11) and all weak solutions have the same law. Moreover, if \hat{X} is a weak solution to (3.11), then \hat{X} is strong Markov with respect to the filtration $\hat{\mathcal{F}}_t = \sigma(\hat{X}_s, 0 \leq s \leq t)$ and has the same one-dimensional marginal distributions as the process X .*

The conditions in this corollary can be weakened. For example, more recent results of Krylov [26] imply that the mimicking process in Corollary 3.7 is unique in law and strong Markov when \hat{b} is bounded and measurable and $\hat{\sigma}\hat{\sigma}^{tr}$ is bounded, locally uniformly positive-definite and continuous in the sense of vanishing mean

oscillation. If we restrict attention to the one-dimensional case, then the mimicking process in Corollary 3.7 is unique in law and strong Markov when \widehat{b} is bounded and measurable, and $\widehat{\sigma}$ is bounded, locally uniformly positive and measurable (Exercise 7.3.3 of [36]).

The two-dimensional process $(\widehat{X}, \widehat{A})$ in Corollary 3.9 is degenerate, so the results of Stroock and Varadhan [34–36] do not apply. However, Theorem 5.10 of [7] asserts that the solution to (3.12) is uniquely determined in law when \widehat{b} is bounded and measurable and $\widehat{\sigma}$ is bounded, strictly positive and continuous. It then follows under these conditions that the pair of mimicking processes in Corollary 3.9 possess the strong Markov property.

We observe finally that the path functional $x \mapsto \max_{s \in [0, t]} x(s)$ is Lipschitz continuous for each fixed $t \geq 0$. This implies that pathwise uniqueness holds for the mimicking equation (3.13) in Corollary 3.10 when \widehat{b} and $\widehat{\sigma}$ are bounded and locally Lipschitz continuous. As a result, it is easy to check that the process $(\widehat{X}, \widehat{M})$ in Corollary 3.10 is strong Markov under these conditions.

To summarize, we cannot conclude in general that the mimicking process \widehat{Z} in Theorem 3.6 is unique in law and strong Markov. In many cases of interest, it is possible to identify conditions that may be imposed on the mimicking equation to ensure that the solution is unique and that the mimicking process possesses the strong Markov property. However, these conditions vary from case to case, and depend in an essential way on the structure of the updating function.

4. Concatenated measure. In this section we begin with a measure \mathbb{P} and a partition Π of $[0, \infty)$ and construct a *concatenated measure*. This is the continuous-time analogue of the measure induced on path space by the process Y or the pair (Z, \overline{Z}) in Section 2. We use the notation introduced at the beginning of Section 3. On the space $\Omega^{\mathcal{E}, d} = \mathcal{E} \times C_0^d$, we introduce the σ -field $\mathcal{F}^{\mathcal{E}, d} \triangleq \mathfrak{E} \otimes \sigma(X)$ and the filtration $\mathcal{F}_t^{\mathcal{E}, d} \triangleq \mathfrak{E} \otimes \sigma(X^t)$, $t \geq 0$, where \mathfrak{E} is the Borel σ -field in \mathcal{E} .

DEFINITION 4.1. Let $0 = T_0 \leq T_1 \leq \dots \leq T_n$ be a sequence of finite (for every ω) $\{\mathcal{F}_t^{\mathcal{E}, d}\}_{t \geq 0}$ -stopping times and let $\{\mathcal{G}_i\}_{i=0}^n$ be a collection of σ -fields satisfying $\mathcal{G}_i \subset \mathcal{F}_{T_i}^{\mathcal{E}, d}$ for $i = 0, \dots, n$. Set $T_{n+1} = \infty$, set $\mathcal{H}_0 = \mathcal{F}_0^{\mathcal{E}, d}$ and define $\mathcal{H}_{i+1} \triangleq \mathcal{G}_i \vee \sigma(\Delta(X^{T_{i+1}}, T_i))$, $i = 0, 1, \dots, n$. We say that $\Pi \triangleq (T_i, \mathcal{G}_i)_{i=0}^n$ is an *extended partition* provided:

- (a) $T_{i+1} - T_i \in \mathcal{G}_i \vee \sigma(\Delta(X, T_i))$, $i = 0, 1, \dots, n - 1$,
- (b) $\mathcal{G}_i \subset \mathcal{H}_i$, $i = 0, 1, \dots, n$.

REMARK 4.2. Because $T_{i+1} - T_i$ is $\mathcal{F}_{T_{i+1}}^{\mathcal{E}, d}$ -measurable and $\mathcal{F}_{T_{i+1}}^{\mathcal{E}, d} = \mathfrak{E} \otimes \sigma(X^{T_{i+1}})$, condition (a) in Definition 3.1 is equivalent to the apparently stronger condition:

- (a') $T_{i+1} - T_i \in \mathcal{H}_{i+1}$, $i = 0, 1, \dots, n - 1$.

Because $\mathcal{G}_i \subset \mathcal{F}_{T_i}^{\mathcal{E},d} \subset \mathcal{F}_{T_{i+1}}^{\mathcal{E},d}$ and $\sigma(\Delta(X^{T_{i+1}}, T_i)) \subset \mathcal{F}_{T_{i+1}}^{\mathcal{E},d}$, we have $\mathcal{H}_{i+1} \subset \mathcal{F}_{T_{i+1}}^{\mathcal{E},d}$, or equivalently,

$$(4.1) \quad \mathcal{H}_i \subset \mathcal{F}_{T_i}^{\mathcal{E},d}, \quad i = 0, 1, \dots, n, n + 1.$$

An extended partition is a model for observing and partially forgetting information over time. Partial forgetting occurs in Section 2 when we condition on the value of a process at time n rather than on \mathcal{F}_n . With an extended partition, at time T_i we retain the information in \mathcal{G}_i as we move forward into the interval $[T_i, T_{i+1}]$, but carry no other information from $\mathcal{F}_{T_i}^{\mathcal{E},d}$ forward. We then observe increments in X over the interval $[T_i, T_{i+1}]$, so that the information we have at time T_{i+1} is \mathcal{H}_{i+1} . This information is sufficient to tell us the length of time $T_{i+1} - T_i$ we conduct the observations. We then remember only the information in the sub- σ -field \mathcal{G}_{i+1} of \mathcal{H}_{i+1} as we go forward into the interval $[T_{i+1}, T_{i+2}]$.

4.1. *Existence and uniqueness of concatenated measure.*

THEOREM 4.3 (*n-fold concatenation*). *Let \mathbb{P} be a probability measure on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$, and let $(T_i, \mathcal{G}_i)_{i=0}^n$ be an extended partition. Then there exists a unique measure $\mathbb{P}^{\otimes \Pi}$ satisfying*

$$(4.2) \quad \mathbb{P}^{\otimes \Pi}[A] = \mathbb{P}[A], \quad A \in \mathcal{H}_i, \quad i = 0, 1, \dots, n + 1,$$

$$(4.3) \quad \mathbb{P}^{\otimes \Pi}[B|\mathcal{F}_{T_i}^{\mathcal{E},d}] = \mathbb{P}[B|\mathcal{G}_i], \quad B \in \mathcal{H}_{i+1}, \quad i = 0, 1, \dots, n.$$

We interpret (4.3) to mean that every \mathbb{P} -version of $\mathbb{P}[B|\mathcal{G}_i]$ is a $\mathbb{P}^{\otimes \Pi}$ -version of $\mathbb{P}^{\otimes \Pi}[B|\mathcal{F}_{T_i}^{\mathcal{E},d}]$.

EXAMPLE 4.4 (Simple concatenated measure). Let $\mathcal{E} = \{0\}$, so that $\Omega^{\mathcal{E},1}$ is isomorphic to C_0^1 . Then $\mathcal{F}_0^{\mathcal{E},1}$ is the trivial σ -algebra $\{\emptyset, C_0^1\}$. We consider the extended partition $\Pi = (T_i, \mathcal{G}_i)_{i=0}^1$ with $\mathcal{G}_0 = \mathcal{G}_1 = \{\emptyset, C_0^1\}$ and $T_0 = 0, T_1 = 1$ and, by convention, $T_2 = \infty$. Then $\mathcal{H}_0 = \mathcal{F}_0^{\mathcal{E},1}, \mathcal{H}_1 = \mathcal{F}_1^{\mathcal{E},1} = \sigma(X(t), 0 \leq t \leq 1)$ and $\mathcal{H}_2 = \sigma(X(t) - X(1), t \geq 1)$. We define four elements of C_0^1 by $\omega^0(t) = 0, \omega^1(t) = t, \omega^2(t) = t \wedge 1$ and $\omega^3(t) = (t - 1)^+$ for $t \geq 0$. Let δ^i be the probability measure on C_0^1 assigning probability 1 to ω^i , and set $\mathbb{P} = (\delta^0 + \delta^1)/2$. The sets

$$A_0 = \{x \in C_0^1 : x(t) = 0 \forall t \in [0, 1]\} \quad \text{and} \quad A_1 = \{x \in C_0^1 : x(t) = t \forall t \in [0, 1]\}$$

are in $\mathcal{H}_1 = \mathcal{F}_1^{\mathcal{E},1}$ and $\mathbb{P}(A_0) = \mathbb{P}(A_1) = \frac{1}{2}$. According to (4.2), we must also have $\mathbb{P}^{\otimes \Pi}(A_0) = \mathbb{P}^{\otimes \Pi}(A_1) = \frac{1}{2}$. The sets

$$B_0 = \{x \in C_0^1 : x(t) - x(1) = 0 \forall t \in [1, \infty)\},$$

$$B_1 = \{x \in C_0^1 : x(t) - x(1) = t - 1 \forall t \in [1, \infty)\}$$

are in \mathcal{H}_2 , and (4.3) implies that

$$\mathbb{P}^{\otimes \Pi}[B_0 | \mathcal{F}_1^{\mathcal{E},1}] = \mathbb{P}[B_0 | \mathcal{G}_1] = \mathbb{P}[B_0] = \frac{1}{2}.$$

Integrating this equation over A_1 with respect to $\mathbb{P}^{\otimes \Pi}$, we see that

$$\frac{1}{4} = \mathbb{P}^{\otimes \Pi}(A_1 \cap B_0) = \mathbb{P}^{\otimes \Pi}(\omega^2).$$

Considering all combinations of A_j and B_k , we conclude that $\mathbb{P}^{\otimes \Pi}(\omega^i) = \frac{1}{4}$ for $i = 0, 1, 2, 3$, that is, $\mathbb{P}^{\otimes \Pi} = (\delta_0 + \delta_1 + \delta_2 + \delta_3)/4$.

The remainder of this subsection is devoted to the proof of Theorem 4.3. Let \mathfrak{C}^d denote the Borel σ -field in C^d and let \mathfrak{C}_0^d denote the trace σ -field in C_0^d . We first concatenate a deterministic initial path and a probability measure at a deterministic time. Given a fixed point $\bar{\omega} = (\bar{e}, \bar{x}) \in \Omega^{\mathcal{E},d}$, a time $t \geq 0$ and a probability measure \mathbb{Q} on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$, let $\Psi_{\bar{\omega},t} : \Omega^{\mathcal{E},d} \rightarrow \Omega^{\mathcal{E},d}$ denote the function

$$(4.4) \quad \Psi_{\bar{\omega},t}(e, x) = (\bar{e}, \nabla(\bar{x}, t) + x - \nabla(x, t)),$$

and set $\delta_{\bar{\omega}} \otimes_t \mathbb{Q} = \mathbb{Q} \circ \Psi_{\bar{\omega},t}^{-1}$. The reader can easily check that the measure $\delta_{\bar{\omega}} \otimes_t \mathbb{Q}$ is uniquely determined by the properties

$$(4.5) \quad (\delta_{\bar{\omega}} \otimes_t \mathbb{Q})[E = \bar{e}, X_s = \bar{x}(s) \forall s \leq t] = 1,$$

$$(4.6) \quad (\delta_{\bar{\omega}} \otimes_t \mathbb{Q})[\Delta(X, t) \in A] = \mathbb{Q}[\Delta(X, t) \in A] \quad \forall A \in \mathfrak{C}_0^d.$$

If $\mathbb{Q}[X_t = \bar{x}(t)] = 1$, $\mathcal{E} = \mathbb{R}^d$ and we identify $\Omega^{\mathcal{E},d}$ with C^d in the natural way, then this notation reduces to the construction given in Lemma 6.1.1 of [36].

In the next step, we concatenate an initial probability measure and a probability kernel at a stopping time.

DEFINITION 4.5. Let (Ω', \mathcal{F}') and $(\Omega'', \mathcal{F}'')$ be measurable spaces. We say that a function $Q : \Omega' \times \mathcal{F}'' \rightarrow [0, 1]$ is a probability kernel from (Ω', \mathcal{F}') to $(\Omega'', \mathcal{F}'')$ provided:

- (a) $Q(\omega', A'')$ is an \mathcal{F}' -measurable function of $\omega' \in \Omega'$ for each $A'' \in \mathcal{F}''$,
- (b) $Q(\omega', \cdot)$ is a probability measure on $(\Omega'', \mathcal{F}'')$ for each $\omega' \in \Omega'$.

PROPOSITION 4.6. Let \mathbb{P} be a probability measure on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$, let T be a finite (for every ω) $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -stopping time and let Q be a probability kernel from $(\Omega^{\mathcal{E},d}, \mathcal{F}_T^{\mathcal{E},d})$ to $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$. Then there exists a unique probability measure $\mathbb{P} \otimes_T Q$ on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$ such that:

- (a) $\mathbb{P} \otimes_T Q[A] = \mathbb{P}[A]$, $A \in \mathcal{F}_T^{\mathcal{E},d}$,
- (b) the random variable $\omega \mapsto (\delta_{\omega} \otimes_{T(\omega)} Q(\omega, \cdot))[F]$ is a version of the conditional probability $(\mathbb{P} \otimes_T Q)[F | \mathcal{F}_T^{\mathcal{E},d}]$ for all $F \in \mathcal{F}^{\mathcal{E},d}$.

PROOF. When the initial condition $Q(\bar{\omega}, \{E = \bar{e}, X_{T(\bar{\omega})} = X_{T(\bar{\omega})}(\bar{\omega})\}) = 1$ holds for each $\bar{\omega} \in \Omega^{\mathcal{E},d}$, the result follows in the same way as Theorem 6.1.2 of [36]. To handle the general case, we modify the initial segment of each path to ensure that the proper initial condition holds. Let $\Psi_{\bar{\omega},t}$ be defined as in (4.4) and set $\widehat{Q}(\bar{\omega}, \cdot) = Q(\bar{\omega}, \cdot) \circ \Psi_{\bar{\omega},T(\bar{\omega})}^{-1}$. The map $(\bar{\omega}, \omega) \mapsto \Psi_{\bar{\omega},T(\bar{\omega})}(\omega)$ is $\mathcal{F}_T^{\mathcal{E},d} \otimes \mathcal{F}^{\mathcal{E},d} / \mathcal{F}^{\mathcal{E},d}$ -measurable, so \widehat{Q} is a probability kernel from $(\Omega^{\mathcal{E},d}, \mathcal{F}_T^{\mathcal{E},d})$ to $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$. It follows from the definition of Ψ that $\widehat{Q}(\bar{\omega}, \{E = \bar{e}, X^{T(\bar{\omega})} = X^{T(\bar{\omega})}(\bar{\omega})\}) = 1$ for each $\bar{\omega} \in \Omega^{\mathcal{E},d}$, so we may apply the previous case to conclude that there exists a unique measure $\mathbb{P} \otimes_T \widehat{Q}$ such that (a) and (b) hold when Q is replaced with \widehat{Q} . But the operator $\omega \mapsto \Psi_{\bar{\omega},t}(\omega)$ is idempotent, so $\delta_{\bar{\omega}} \otimes_{T(\bar{\omega})} Q(\bar{\omega}, \cdot) = \delta_{\bar{\omega}} \otimes_{T(\bar{\omega})} \widehat{Q}(\bar{\omega}, \cdot)$ for each $\bar{\omega} \in \Omega^{\mathcal{E},d}$, and $\mathbb{P} \otimes_T Q = \mathbb{P} \otimes_T \widehat{Q}$ is in fact the unique measure which satisfies (a) and (b). \square

We now begin concatenating probability measures.

COROLLARY 4.7 (Two-fold concatenation). *Let \mathbb{P}_1 and \mathbb{P}_2 be probability measures on $\Omega^{\mathcal{E},d}$, let T be a finite (for every ω) $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -stopping time, let \mathcal{G} be a sub- σ -field of $\mathcal{F}_T^{\mathcal{E},d}$ and assume that $\mathbb{P}_1|_{\mathcal{G}} \ll \mathbb{P}_2|_{\mathcal{G}}$. Then there exists a unique measure, denoted $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2$, such that:*

- (a) $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2[A] = \mathbb{P}_1[A]$, $A \in \mathcal{F}_T^{\mathcal{E},d}$,
- (b) for every set $B \in \mathcal{G} \vee \sigma(\Delta(X, T))$, every version of $\mathbb{P}_2[B|\mathcal{G}]$ is a version of $(\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2)[B|\mathcal{F}_T^{\mathcal{E},d}]$,
- (c) if \mathbb{P}_1 and \mathbb{P}_2 agree on \mathcal{G} , then $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2$ and \mathbb{P}_2 agree on $\mathcal{G} \vee \sigma(\Delta(X, T))$.

PROOF. Because $\Omega^{\mathcal{E},d}$ is a Polish space, there exists a \mathcal{G} -measurable probability kernel Q from $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$ to $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$ such that for every $F \in \mathcal{F}^{\mathcal{E},d}$, $Q(\cdot, F)$ is a version of $\mathbb{P}_2[F|\mathcal{G}]$ ([36], Theorem 1.1.6). Using Proposition 4.6, we define $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2 \triangleq \mathbb{P}_1 \otimes_T Q$. Property (a) of the corollary is property (a) of Proposition 4.6.

Given $\omega \in \Omega^{\mathcal{E},d}$ and $F \in \mathcal{F}^{\mathcal{E},d}$, set $\widehat{Q}(\omega, F) = (\delta_{\omega} \otimes_{T(\omega)} Q(\omega, \cdot))[F]$. Property (b) of Proposition 4.6 asserts that $\widehat{Q}(\cdot, F)$ is a version of $(\mathbb{P}_1 \otimes_T Q)[F|\mathcal{F}_T^{\mathcal{E},d}]$ for all $F \in \mathcal{F}^{\mathcal{E},d}$. Galmarino’s test ([9], Theorem IV.100) for the filtered space $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d}, \{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0})$ says that $E(\omega) = E(\bar{\omega})$ and $X_u(\omega) = X_u(\bar{\omega})$ for $0 \leq u \leq T(\bar{\omega})$ imply $Y(\omega) = Y(\bar{\omega})$ for every $\mathcal{F}_T^{\mathcal{E},d}$ -measurable random variable Y . In particular, if $A \in \mathcal{F}_T^{\mathcal{E},d}$, then $\omega \in A$ if and only if $(E(\omega), X^{T(\omega)}(\omega)) \in A$. Therefore, $\widehat{Q}(\omega, A \cap F) = \mathbb{I}_A(\omega) \widehat{Q}(\omega, F)$ for all $\omega \in \Omega^{\mathcal{E},d}$, $A \in \mathcal{F}_T^{\mathcal{E},d}$ and $F \in \mathcal{F}^{\mathcal{E},d}$ by (4.5). If $B = A \cap \{\Delta(X, T) \in D\}$ with $A \in \mathcal{G}$ and $D \in \mathcal{C}_0^d$, then (4.6) implies

$$(4.7) \quad \widehat{Q}(\cdot, B) = \mathbb{I}_A Q(\cdot, \{\Delta(X, T) \in D\}) = \mathbb{P}_2[B|\mathcal{G}], \quad \mathbb{P}_2\text{-a.s.}$$

It then follows from Dynkin’s π - λ theorem that (4.7) holds for all $B \in \mathcal{G} \vee \sigma(\Delta(X, T))$. From (a) we have $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2|_{\mathcal{G}} = \mathbb{P}_1|_{\mathcal{G}}$, and we have assumed

$\mathbb{P}_1|_{\mathcal{G}} \ll \mathbb{P}_2|_{\mathcal{G}}$, so the fact that $\widehat{Q}(\cdot, B)$ is a version of both $(\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2)[B|\mathcal{F}_T^{\mathcal{E},d}]$ and $\mathbb{P}_2[B|\mathcal{G}]$ implies that every version of $\mathbb{P}_2[B|\mathcal{G}]$ is also a version of $(\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2)[B|\mathcal{F}_T^{\mathcal{E},d}]$.

For (c), assume that \mathbb{P}_1 and \mathbb{P}_2 agree on \mathcal{G} . Property (a) implies that $\mathbb{P}_1|_{\mathcal{G}} = (\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2)|_{\mathcal{G}}$, and hence, $\mathbb{P}_2|_{\mathcal{G}} = \mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2|_{\mathcal{G}}$. For $B \in \mathcal{G} \vee \sigma(\Delta(X, T))$, we have from (b) that $\mathbb{P}_2[B|\mathcal{G}] = (\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2)[B|\mathcal{G}]$, and we can integrate both sides over $\Omega^{\mathcal{E},d}$ with respect to $\mathbb{P}_2|_{\mathcal{G}} = \mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2|_{\mathcal{G}}$ to obtain (c).

Uniqueness of $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2$ follows from the fact that (b) specifies this measure on $\mathcal{G} \vee \sigma(\Delta(X, T))$ conditioned on $\mathcal{F}_T^{\mathcal{E},d}$, up to $\mathbb{P}_2|_{\mathcal{G}}$ -equivalence. Furthermore, (a) specifies this measure to be \mathbb{P}_1 on $\mathcal{F}_T^{\mathcal{E},d}$, and hence, on \mathcal{G} . But $\mathbb{P}_1 \ll \mathbb{P}_2$, and hence, the integral in the equation $(\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2)[B] = \int_{\Omega^{\mathcal{E},d}} \mathbb{P}_2[B|\mathcal{G}] d\mathbb{P}_1$ for $B \in \mathcal{G} \vee \sigma(\Delta(X, T))$ is well defined. We see then that properties (a) and (b) specify the measure $\mathbb{P}_1 \otimes_{T,\mathcal{G}} \mathbb{P}_2$ on $\mathcal{G} \vee \sigma(\Delta(X, T))$ and on $\mathcal{F}_T^{\mathcal{E},d}$. These two σ -fields generate $\mathcal{F}^{\mathcal{E},d}$, and thus the measure is uniquely determined on $\mathcal{F}^{\mathcal{E},d}$ by properties (a) and (b). \square

PROPOSITION 4.8 (Three-fold concatenation). *Let $\mathbb{P}_1, \mathbb{P}_2$ and \mathbb{P}_3 be probability measures on $\Omega^{\mathcal{E},d}$ and let $0 \leq S \leq T$ be finite (for every ω) $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -stopping times. Let \mathcal{G} be a sub- σ -field of $\mathcal{F}_S^{\mathcal{E},d}$ and let \mathcal{H} be a sub- σ -field of $\mathcal{G} \vee \sigma(\Delta(X^T, S))$, which is a sub- σ -field of $\mathcal{F}_T^{\mathcal{E},d}$. Assume that $T - S$ is $\mathcal{G} \vee \sigma(\Delta(X, S))$ -measurable. If $\mathbb{P}_1|_{\mathcal{G}} \ll \mathbb{P}_2|_{\mathcal{G}}$ and $\mathbb{P}_2|_{\mathcal{H}} \ll \mathbb{P}_3|_{\mathcal{H}}$, then:*

- (a) $\mathbb{P}_1|_{\mathcal{G}} \ll (\mathbb{P}_2 \otimes_{T,\mathcal{H}} \mathbb{P}_3)|_{\mathcal{G}}$,
- (b) $(\mathbb{P}_1 \otimes_{S,\mathcal{G}} \mathbb{P}_2)|_{\mathcal{H}} \ll \mathbb{P}_3|_{\mathcal{H}}$,

so that both $\mathbb{P}_1 \otimes_{S,\mathcal{G}} (\mathbb{P}_2 \otimes_{T,\mathcal{H}} \mathbb{P}_3)$ and $(\mathbb{P}_1 \otimes_{S,\mathcal{G}} \mathbb{P}_2) \otimes_{T,\mathcal{H}} \mathbb{P}_3$ are defined, and

- (c) $\mathbb{P}_1 \otimes_{S,\mathcal{G}} (\mathbb{P}_2 \otimes_{T,\mathcal{H}} \mathbb{P}_3) = (\mathbb{P}_1 \otimes_{S,\mathcal{G}} \mathbb{P}_2) \otimes_{T,\mathcal{H}} \mathbb{P}_3$.

PROOF. We simplify notation by writing $\mathbb{P}_{12} = \mathbb{P}_1 \otimes_{S,\mathcal{G}} \mathbb{P}_2$, $\mathbb{P}_{23} = \mathbb{P}_2 \otimes_{T,\mathcal{H}} \mathbb{P}_3$, $\mathbb{P}_{1,23} = \mathbb{P}_1 \otimes_{S,\mathcal{G}} (\mathbb{P}_2 \otimes_{T,\mathcal{H}} \mathbb{P}_3)$ and $\mathbb{P}_{12,3} = (\mathbb{P}_1 \otimes_{S,\mathcal{G}} \mathbb{P}_2) \otimes_{T,\mathcal{H}} \mathbb{P}_3$. For (a), we note from Corollary 4.7(a) that \mathbb{P}_{23} agrees with \mathbb{P}_2 on $\mathcal{F}_T^{\mathcal{E},d}$, and hence, on \mathcal{G} . Property (a) follows from $\mathbb{P}_1|_{\mathcal{G}} \ll \mathbb{P}_2|_{\mathcal{G}}$. For (b), let $A \in \mathcal{H}$ satisfy $\mathbb{P}_3[A] = 0$. By assumption, we also have $\mathbb{P}_2[A] = 0$, and hence, 0 is a version of $\mathbb{P}_2[A|\mathcal{G}]$. Being in \mathcal{H} , A is also in $\mathcal{G} \vee \sigma(\Delta(X, S))$, and according to Corollary 4.7(b), 0 is a version of $\mathbb{P}_{12}[A|\mathcal{F}_S^{\mathcal{E},d}]$. Therefore, $\mathbb{P}_{12}[A] = 0$.

The collection of sets of the form $A \cap B \cap C$, where $A \in \mathcal{F}_S^{\mathcal{E},d}$, $B \in \sigma(\Delta(X^T, S))$ and $C \in \sigma(\Delta(X, T))$, is closed under finite intersections and generates $\mathcal{F}^{\mathcal{E},d}$. Thus, to prove (c), it suffices to show that the desired equation holds when both sides are evaluated for a set of this form. Let A, B and C be as described, and let G be in \mathcal{G} . Let Z be a version of $\mathbb{E}_3[\mathbb{1}_C|\mathcal{H}]$ and Y a version of $\mathbb{E}_2[\mathbb{1}_B Z|\mathcal{G}]$. Corollary 4.7(b) implies that Z is a version of $\mathbb{P}_{23}[C|\mathcal{F}_T^{\mathcal{E},d}]$. This, combined with Corollary 4.7(a),

implies

$$\mathbb{E}_{23}[\mathbb{I}_G Y] = \mathbb{E}_2[\mathbb{I}_G Y] = \mathbb{E}_2[\mathbb{I}_{G \cap B} Z] = \mathbb{E}_{23}[\mathbb{I}_{G \cap B} Z] = \mathbb{E}_{23}[G \cap B \cap C].$$

We see then that $Y = \mathbb{E}_2[\mathbb{I}_B \mathbb{E}_3[\mathbb{I}_C | \mathcal{H}] | \mathcal{G}]$ is a version of $\mathbb{E}_{23}[\mathbb{I}_{B \cap C} | \mathcal{G}]$, a fact we use along with repeated applications of Corollary 4.7(a), (b) and (c) in the chain of equalities

$$\begin{aligned} \mathbb{P}_{1,23}[A \cap B \cap C] &= \mathbb{E}_{1,23}[\mathbb{I}_A \mathbb{E}_{1,23}[\mathbb{I}_{B \cap C} | \mathcal{F}_S^{\mathcal{E},d}]] \\ &= \mathbb{E}_{1,23}[\mathbb{I}_A \mathbb{E}_{23}[\mathbb{I}_{B \cap C} | \mathcal{G}]] \\ &= \mathbb{E}_{1,23}[\mathbb{I}_A \mathbb{E}_2[\mathbb{I}_B \mathbb{E}_3[\mathbb{I}_C | \mathcal{H}] | \mathcal{G}]] \\ &= \mathbb{E}_1[\mathbb{I}_A \mathbb{E}_2[\mathbb{I}_B \mathbb{E}_3[\mathbb{I}_C | \mathcal{H}] | \mathcal{G}]] \\ &= \mathbb{E}_{12}[\mathbb{I}_A \mathbb{E}_{12}[\mathbb{I}_B \mathbb{E}_3[\mathbb{I}_C | \mathcal{H}] | \mathcal{F}_S^{\mathcal{E},d}]] \\ &= \mathbb{E}_{12}[\mathbb{I}_{A \cap B} \mathbb{E}_3[\mathbb{I}_C | \mathcal{H}]] \\ &= \mathbb{E}_{12,3}[\mathbb{I}_{A \cap B} \mathbb{E}_{12,3}[\mathbb{I}_C | \mathcal{F}_S^{\mathcal{E},d}]] \\ &= \mathbb{P}_{12,3}[A \cap B \cap C]. \end{aligned} \quad \square$$

PROOF OF THEOREM 4.3. Let m satisfy $0 \leq m \leq n - 1$. According to Definition 4.1,

$$\mathcal{G}_{m+1} \subset \mathcal{H}_{m+1} = \mathcal{G}_m \vee \sigma(\Delta(X^{T_{m+1}}, T_m)) \subset \mathcal{G}_m \vee \sigma(\Delta(X, T_m)).$$

If $0 \leq m \leq n - 2$, we further have

$$\mathcal{G}_{m+2} \subset \mathcal{H}_{m+2} = \mathcal{G}_{m+1} \vee \sigma(\Delta(X^{T_{m+2}}, T_{m+1})) \subset \mathcal{G}_m \vee \sigma(\Delta(X, T_m)).$$

Iterating this process, we obtain the relation $\mathcal{G}_j \subset \mathcal{G}_m \vee \sigma(\Delta(X, T_m))$ for $j = m, m + 1, \dots, n$. Consequently,

$$(4.8) \quad \mathcal{G}_j \vee \sigma(\Delta(X^{T_{j+1}}, T_j)) \subset \mathcal{G}_m \vee \sigma(\Delta(X, T_m)), \quad 0 \leq m \leq j \leq n.$$

We now proceed by induction on m . The induction hypothesis corresponding to m , where $m = 0, \dots, n$, is the existence of a measure \mathbb{P}^m such that:

- (i) $\mathbb{P}^m[A] = \mathbb{P}[A]$ for all $A \in \mathcal{H}_i$ and $0 \leq i \leq n + 1$,
- (ii) for $B \in \mathcal{H}_{i+1}$ and $0 \leq i \leq m - 1$, every \mathbb{P} -version of $\mathbb{P}[B | \mathcal{G}_i]$ is a \mathbb{P}^m -version of $\mathbb{P}^m[B | \mathcal{F}_{T_i}^{\mathcal{E},d}]$.

The base case is $\mathbb{P}^0 = \mathbb{P}$, a case for which (i) trivially holds and (ii) is vacuous.

Assume the induction hypothesis for some integer m . Because $\mathcal{G}_m \subset \mathcal{H}_m$ and the measures \mathbb{P}^m and \mathbb{P} agree on \mathcal{H}_m , we may invoke Corollary 4.7 to define $\mathbb{P}^{m+1} \triangleq \mathbb{P}^m \otimes_{T_m, \mathcal{G}_m} \mathbb{P}$. If $A \in \mathcal{H}_j$ for some $j, 0 \leq j \leq m$, then $A \in \mathcal{F}_{T_m}^{\mathcal{E},d}$ and $\mathbb{P}^{m+1}[A] =$

$\mathbb{P}^m[A] = \mathbb{P}[A]$ by Corollary 4.7(a) and part (i) of the induction hypothesis. If $m \leq j \leq n$, then (4.8) implies

$$(4.9) \quad \mathcal{H}_{j+1} \triangleq \mathcal{G}_j \vee \sigma(\Delta(X^{T_{j+1}}, T_j)) \subset \mathcal{G}_m \vee \sigma(\Delta(X, T_m)).$$

But Corollary 4.7(c) implies that \mathbb{P}^{m+1} agrees with \mathbb{P} on $\mathcal{G}_m \vee \sigma(\Delta(X, T_m))$. Hence, \mathbb{P}^{m+1} satisfies (i).

For some i , $0 \leq i \leq m - 1$, let $B \in \mathcal{H}_{i+1}$ and $A \in \mathcal{F}_{T_i}^{\mathcal{E},d}$ be given. Suppose Z is a version of $\mathbb{P}[B|\mathcal{G}_i]$, so that both $A \cap B$ and $\mathbb{I}_A Z$ are $\mathcal{F}_{T_m}^{\mathcal{E},d}$ -measurable [recall (4.1)]. Corollary 4.7(a) (used twice) and part (ii) of the induction hypothesis imply

$$\mathbb{E}^{m+1}[\mathbb{I}_A Z] = \mathbb{E}^m[\mathbb{I}_A Z] = \mathbb{P}^m[A \cap B] = \mathbb{P}^{m+1}[A \cap B],$$

showing that Z is a version of $\mathbb{P}^{m+1}[B|\mathcal{F}_{T_i}^{\mathcal{E},d}]$. Finally, suppose B is in \mathcal{H}_{m+1} , which is a sub- σ -field of $\mathcal{G}_m \vee \sigma(\Delta(X, T_m))$. Corollary 4.7(b) says that every version of $\mathbb{P}[B|\mathcal{G}_m]$ is a version of $\mathbb{P}^{m+1}[B|\mathcal{F}_{T_m}^{\mathcal{E},d}]$. This establishes (ii) with $m + 1$ replacing m .

The induction argument above constructs $\mathbb{P}^{\otimes \Pi} \triangleq \mathbb{P}^{n+1}$ that satisfies (4.2) and (4.3). To see that this measure is unique, we show that (4.2) and (4.3) determine its value on sets of the form $\bigcap_{i=0}^{n+1} B_i$, where $B_0 \in \mathcal{F}_0^{\mathcal{E},d} = \mathcal{H}_0$ and $B_{i+1} \in \sigma(\Delta(X^{T_{i+1}}, T_i)) \subset \mathcal{H}_{i+1}$ for $i = 0, \dots, n$. This collection of sets is closed under finite intersections and generates $\mathcal{F}^{\mathcal{E},d}$. For such a set, repeated application of (4.3), followed by a final application of (4.2), yields

$$\begin{aligned} & \mathbb{P}^{\otimes \Pi} \left[\bigcap_{i=0}^{n+1} B_i \right] \\ &= \mathbb{E}^{n+1}[\mathbb{I}_{B_0} \mathbb{E}^{n+1}[\mathbb{I}_{B_1} \cdots \mathbb{E}^{n+1}[\mathbb{I}_{B_n} \mathbb{E}^{n+1}[\mathbb{I}_{B_{n+1}} |\mathcal{F}_{T_n}^{\mathcal{E},d}] | \mathcal{F}_{T_{n-1}}^{\mathcal{E},d}] \cdots | \mathcal{F}_0^{\mathcal{E},d}]] \\ &= \mathbb{E}^{n+1}[\mathbb{I}_{B_0} \mathbb{E}^{n+1}[\mathbb{I}_{B_1} \cdots \mathbb{E}^{n+1}[\mathbb{I}_{B_n} \mathbb{E}[\mathbb{I}_{B_{n+1}} | \mathcal{G}_n] | \mathcal{F}_{T_{n-1}}^{\mathcal{E},d}] \cdots | \mathcal{F}_0^{\mathcal{E},d}]] \\ (4.10) \quad &= \mathbb{E}^{n+1}[\mathbb{I}_{B_0} \mathbb{E}^{n+1}[\mathbb{I}_{B_1} \cdots \mathbb{E}[\mathbb{I}_{B_n} \mathbb{E}[\mathbb{I}_{B_{n+1}} | \mathcal{G}_n] | \mathcal{G}_{n-1}] \cdots | \mathcal{F}_0^{\mathcal{E},d}]] \\ & \vdots \\ &= \mathbb{E}^{n+1}[\mathbb{I}_{B_0} \mathbb{E}[\mathbb{I}_{B_1} \cdots \mathbb{E}[\mathbb{I}_{B_n} \mathbb{E}[\mathbb{I}_{B_{n+1}} | \mathcal{G}_n] | \mathcal{G}_{n-1}] \cdots | \mathcal{G}_0]] \\ &= \mathbb{E}[\mathbb{I}_{B_0} \mathbb{E}[\mathbb{I}_{B_1} \cdots \mathbb{E}[\mathbb{I}_{B_n} \mathbb{E}[\mathbb{I}_{B_{n+1}} | \mathcal{G}_n] | \mathcal{G}_{n-1}] \cdots | \mathcal{G}_0]]. \end{aligned}$$

The proof of Theorem 4.3 is complete. \square

REMARK 4.9. We see from the proof of Theorem 4.3 that

$$\mathbb{P}^{\otimes \Pi} = \mathbb{P} \otimes_{T_0, \mathcal{G}_0} \mathbb{P} \otimes_{T_1, \mathcal{G}_1} \cdots \otimes_{T_n, \mathcal{G}_n} \mathbb{P},$$

where the associative property of Proposition 4.8(c) makes the grouping of the $\otimes_{T_i, \mathcal{G}_i}$ operators irrelevant. Equation (4.10) provides insight into the nature of

$\mathbb{P}^{\otimes \Pi}$. If \mathcal{G}_i is equal to $\mathcal{F}_{T_i}^{\mathcal{E},d}$ for each i , then the last iterated conditional expectation in (4.10) collapses to $\mathbb{P}[\bigcap_{i=0}^{n+1} B_i]$, and $\mathbb{P}^{\otimes \Pi}$ agrees with \mathbb{P} . At the other extreme, if \mathcal{G}_i is the trivial σ -field $\{\emptyset, \Omega^{\mathcal{E},d}\}$ for each i , then this iterated conditional expectation becomes $\prod_{i=0}^{n+1} \mathbb{P}[B_i]$, and increments of the path fragments over $[T_i, T_{i+1}]$ are independent of one another under $\mathbb{P}^{\otimes \Pi}$ but have the same unconditional distribution as under \mathbb{P} .

4.2. *Properties preserved by concatenation.*

PROPOSITION 4.10. *Let \mathbb{P} be a probability measure and let $(T_i, \mathcal{G}_i)_{i=0}^n$ be an extended partition on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$. Let A be an $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -adapted continuous real-valued process on $\Omega^{\mathcal{E},d}$, and assume that $\Delta(A, T_i)$ is $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable for $i = 0, \dots, n$.*

- (a) *The total variation of A on $[0, \infty)$ is \mathbb{P} -almost surely finite if and only if it is $\mathbb{P}^{\otimes \Pi}$ -almost surely finite.*
- (b) *The process A is \mathbb{P} -almost surely absolutely continuous if and only if it is $\mathbb{P}^{\otimes \Pi}$ -almost surely absolutely continuous.*

PROOF. We set $\mathbb{P}^0 = \mathbb{P}$ and $\mathbb{P}^{i+1} = \mathbb{P}^i \otimes_{T_i, \mathcal{G}_i} \mathbb{P}$, $i = 1, \dots, n$. Then $\mathbb{P}^{\otimes \Pi} = \mathbb{P}^{n+1}$. For (a), we proceed by induction on $i = 0, 1, \dots, n$, assuming that:

- (a_i) the total variation of A on $[0, \infty)$ is \mathbb{P} -almost surely finite if and only if it is \mathbb{P}^i -almost surely finite.

On $\mathcal{F}_{T_i}^{\mathcal{E},d}$, the probability measures \mathbb{P}^i and \mathbb{P}^{i+1} agree [Corollary 4.7(a)], and hence, A restricted to $[0, T_i]$ is \mathbb{P}^i -a.s. of finite variation if and only if A restricted to $[0, T_i]$ is \mathbb{P}^{i+1} -a.s. of finite variation. The variation of A on subintervals in $[T_i, \infty)$ is a function of $\Delta(A, T_i)$, which is $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable, and on this σ -field, the measures \mathbb{P}^i and \mathbb{P}^{i+1} agree [Corollary 4.7(c)]. Therefore, A restricted to $[T_i, \infty)$ is \mathbb{P}^i -a.s. of finite variation if and only if A restricted to $[T_i, \infty)$ is \mathbb{P}^{i+1} -a.s. of finite variation. We conclude that A has finite total variation on $[0, \infty)$ \mathbb{P}^i -almost surely if and only if it has finite total variation \mathbb{P}^{i+1} -almost surely. Combining this with the induction hypothesis (a_i), we obtain the induction hypothesis with $i + 1$ replacing i .

The continuous process A is absolutely continuous on $[0, \infty)$ if and only if it is absolutely continuous on $[0, T_i]$ and absolutely continuous on $[T_i, \infty)$. Therefore, we can imitate the proof of (a) to obtain (b). \square

PROPOSITION 4.11. *Let \mathbb{P} be a probability measure and let $(T_i, \mathcal{G}_i)_{i=0}^n$ be an extended partition on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$. Let A be an $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -adapted continuous \mathbb{R}^d -valued process on $\Omega^{\mathcal{E},d}$ with $A_0 = 0$, and assume that $\Delta(A, T_i)$ is*

$\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable for $i = 0, \dots, n$. Assume there exists a measurable \mathbb{R}^d -valued process α such that the set

$$(4.11) \quad J(\omega) \triangleq \left\{ t \in [0, \infty) : \frac{\partial}{\partial t} A_t(\omega) \text{ exists but is not equal to } \alpha_t(\omega) \right\}$$

has Lebesgue measure zero for \mathbb{P} -almost every and $\mathbb{P}^{\otimes \Pi}$ -almost every $\omega \in \Omega^{\mathcal{E},d}$. Then

$$(4.12) \quad \mathbb{P} \left[A_t = \int_0^t \alpha_u \, du \, \forall t \in [0, \infty) \right] = 1$$

if and only if

$$(4.13) \quad \mathbb{P}^{\otimes \Pi} \left[A_t = \int_0^t \alpha_u \, du \, \forall t \in [0, \infty) \right] = 1.$$

When the equalities (4.12) and (4.13) hold, we also have

$$(4.14) \quad \mathbb{E} \int_0^S f(\alpha_u) \, du = \mathbb{E}^{\otimes \Pi} \int_0^S f(\alpha_u) \, du$$

for every nonnegative, Borel-measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -stopping time S satisfying $(S - T_i)^+ \in \mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ for $i = 0, \dots, n$.

PROOF. Assume (4.12). Then each component of A is \mathbb{P} -a.s. absolutely continuous. Proposition 4.10 implies that the components of A are $\mathbb{P}^{\otimes \Pi}$ -a.s. absolutely continuous as well. Therefore, for $\mathbb{P}^{\otimes \Pi}$ -almost every ω , the set

$$C(\omega) \triangleq \left\{ t \in [0, \infty) : \frac{\partial}{\partial t} A_t(\omega) \text{ exists} \right\}$$

has full Lebesgue measure, and by the assumption about $J(\omega)$, the set

$$D(\omega) \triangleq \left\{ t \in [0, \infty) : \frac{\partial}{\partial t} A_t(\omega) \text{ exists and is equal to } \alpha_t(\omega) \right\}$$

also has full Lebesgue measure for $\mathbb{P}^{\otimes \Pi}$ -almost every ω . This implies (4.13). This argument is reversible; (4.13) implies (4.12).

We now assume (4.12) and (4.13). The $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurability of $\Delta(A, T_i)$ together with the $\mathcal{F}_{T_{i+1}}^{\mathcal{E},d}$ -measurability of $\Delta(A^{T_{i+1}}, T_i)$ implies the \mathcal{H}_{i+1} -measurability of $\Delta(A^{T_{i+1}}, T_i)$. Because A is adapted and continuous, $A_{T_i(\omega)+t}(\omega) \mathbb{1}_{\{0 \leq t < T_{i+1}(\omega) - T_i(\omega)\}}$ is a jointly $\mathcal{H}_{i+1} \otimes \mathcal{B}[0, \infty)$ -measurable function of (ω, t) , where $\mathcal{B}[0, \infty)$ is the Borel σ -field on $[0, \infty)$ (recall Remark 4.2). The same is then true for the right-hand derivative $\frac{\partial^+}{\partial t} A_{T_i(\omega)+t} \mathbb{1}_{\{0 \leq t < T_{i+1}(\omega) - T_i(\omega)\}}$, where we set this right-hand derivative equal to an arbitrary value whenever the limit of the relevant difference quotient does not exist. By assumption, $(S - T_i)^+$ is also $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable. Therefore, $(T_{i+1} - T_i) \wedge (S - T_i)^+ = T_{i+1} \wedge S - T_i \wedge S$ is \mathcal{H}_{i+1} -measurable. But on each \mathcal{H}_{i+1} , the measures \mathbb{P} and

$\mathbb{P}^{\otimes \Pi}$ agree, which implies that for every nonnegative Borel-measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\mathbb{E} \int_0^{T_{i+1} \wedge S - T_i \wedge S} f(\alpha_{T_i+u}) du = \mathbb{E}^{\otimes \Pi} \int_0^{T_{i+1} \wedge S - T_i \wedge S} f(\alpha_{T_i+u}) du.$$

Summing over $i = 0, 1, \dots, n$, we obtain (4.14). \square

EXAMPLE 4.12 (Example 4.4 continued). Consider the extended partition and probability measures \mathbb{P} and $\mathbb{P}^{\otimes \Pi}$ of Example 4.4. We take $A = X$ so that $A_0 = 0$ and $\Delta(A, T_i)$ is $\mathcal{G}_1 \vee \sigma(\Delta(X, T_i))$ -measurable for $i = 0, 1$. We define the adapted processes

$$\begin{aligned} \alpha_t(\omega) &= \left\{ \limsup_{\varepsilon \downarrow 0} 0 \vee \frac{\omega(\varepsilon)}{\varepsilon} \wedge 1 \right\} \mathbb{I}_{(0, \infty]}(t), \\ \beta_t(\omega) &= \left\{ \limsup_{\varepsilon \downarrow 0} 0 \vee \frac{\omega(\varepsilon)}{\varepsilon} \wedge 1 \right\} \mathbb{I}_{(0, 1]}(t) \\ &\quad + \left\{ \limsup_{\varepsilon \downarrow 0} 0 \vee \frac{\omega(1 + \varepsilon) - \omega(1)}{\varepsilon} \wedge 1 \right\} \mathbb{I}_{(1, \infty]}(t) \end{aligned}$$

and the sets $E = \{A_t = \int_0^t \alpha_u du \ \forall t \in [0, \infty)\}$ and $F = \{A_t = \int_0^t \beta_u du \ \forall t \in [0, \infty)\}$. Then we have $\mathbb{P}[E] = \mathbb{P}[F] = \mathbb{P}^{\otimes \Pi}[F] = 1$, but $\mathbb{P}^{\otimes \Pi}[E] = 1/2$.

If we let $K(\omega)$ denote the set obtained by replacing α with β in (4.11), then we see that $K(\omega)$ is a Lebesgue-null set \mathbb{P} -almost surely and $\mathbb{P}^{\otimes \Pi}$ -almost surely. On the other hand, $J(\omega)$ defined by (4.11) is a Lebesgue-null set \mathbb{P} -almost surely, but has strictly positive Lebesgue measure with strictly positive $\mathbb{P}^{\otimes \Pi}$ -probability. In particular, we see that (4.12) and (4.13) may not be equivalent in this situation.

COROLLARY 4.13. Let \mathbb{P} be a probability measure on $(\Omega^{\mathcal{E}, d}, \mathcal{F}^{\mathcal{E}, d})$ and for each positive integer m , let $\Pi^m \triangleq (T_i^m, \mathcal{G}_i^m)_{i=0}^{N(m)}$ be an extended partition. Let A be an $\{\mathcal{F}_t^{\mathcal{E}, d}\}_{t \geq 0}$ -adapted continuous \mathbb{R}^d -valued process on $\Omega^{\mathcal{E}, d}$ with $A_0 = 0$, and assume that T_i^m and $\Delta(A, T_i^m)$ are $\mathcal{G}_i^m \vee \sigma(\Delta(X, T_i^m))$ -measurable for $i = 1, \dots, N(m)$ and $m = 1, 2, \dots$. Let α be a measurable \mathbb{R}^d -valued process such that $A_t = \int_0^t \alpha_u du$ for every $t \geq 0$, \mathbb{P} -almost surely, and assume that the set $J(\omega)$ defined by (4.11) has Lebesgue measure zero for every $\omega \in \Omega^{\mathcal{E}, d}$. Finally, assume

$$(4.15) \quad \mathbb{E} \int_0^t \|\alpha_u\| du < \infty, \quad t \geq 0.$$

Then the following hold.

- (a) For every $t \in [0, \infty)$, α restricted to $[0, t]$ is uniformly integrable with respect to the collection of product measures $\{\mathbb{P}^{\otimes \Pi^m} \times \lambda_{[0, t]}\}_{m=1}^\infty$, where $\lambda_{[0, t]}$ denotes Lebesgue measure on $[0, t]$.

(b) *The collection of measures $\{\mathbb{P}^{\otimes \Pi^m} \circ A^{-1}\}_{m=1}^\infty$ on C_0^d is tight.*

PROOF. For (a), fix $t \in [0, \infty)$. Given $\varepsilon > 0$, (4.15) guarantees that there exists $M_\varepsilon > 0$ so large that $\mathbb{E} \int_0^t \|\alpha_u\| \mathbb{I}_{\{\|\alpha_u\| \geq M_\varepsilon\}} du \leq \varepsilon$. Applying Proposition 4.11 with $f(x) = \|x\| \mathbb{I}_{\{\|x\| \geq M_\varepsilon\}}$ and $S = t$, we obtain $\mathbb{E}^{\otimes \Pi^m} \int_0^t \|\alpha_u\| \mathbb{I}_{\{\|\alpha_u\| \geq M_\varepsilon\}} du \leq \varepsilon$ for all m .

For (b) it suffices to verify that for every $\varepsilon > 0$, there exists a set $\Omega_\varepsilon \in \mathcal{F}^{\mathcal{E},d}$ such that $\mathbb{P}^{\otimes \Pi^m}(\Omega_\varepsilon) \geq 1 - \varepsilon$ for every m and

$$(4.16) \quad \limsup_{\delta \downarrow 0} \sup_{\omega \in \Omega_\varepsilon} \sup_{0 \leq s \leq v \leq t \wedge (s+\delta)} \|A_v(\omega) - A_s(\omega)\| = 0, \quad t \geq 0.$$

Fix $\varepsilon > 0$ and let $\{t_n\}_{n=1}^\infty$ be an increasing sequence of positive numbers with $\lim_{n \rightarrow \infty} t_n = \infty$. For fixed n , we construct Ω_n such that $\mathbb{P}^{\otimes \Pi^m}(\Omega_n) \geq 1 - 2^{-n}\varepsilon$ for every m and

$$(4.17) \quad \limsup_{\delta \downarrow 0} \sup_{\omega \in \Omega_n} \sup_{0 \leq s \leq v \leq t_n \wedge (s+\delta)} \|A_v(\omega) - A_s(\omega)\| = 0.$$

Then $\Omega_\varepsilon = \bigcap_{n=1}^\infty \Omega_n$ satisfies (4.16) and $\mathbb{P}^{\otimes \Pi^m}(\Omega_\varepsilon) \geq 1 - \varepsilon$ for every m .

We fix n and construct Ω_n by working through the proof of the Borel–Cantelli lemma. For each positive integer k , part (a) implies the existence of $\delta_k > 0$ for which

$$\mathbb{E}^{\otimes \Pi^m} \left[\sup_{0 \leq s \leq v \leq t_n \wedge (s+\delta_k)} \|A_v - A_s\| \right] \leq \mathbb{E}^{\otimes \Pi^m} \left[\sup_{0 \leq s \leq v \leq t_n \wedge (s+\delta_k)} \int_s^v \|\alpha_u\| du \right] \leq 2^{-2k}$$

for all m . We define $F_k = \{\sup_{0 \leq s \leq v \leq t_n \wedge (s+\delta_k)} \int_s^v \|\alpha_u\| du \geq 2^{-k}\}$, and note from Chebyshev’s inequality that $\mathbb{P}^{\otimes \Pi^m}(F_k) \leq 2^{-k}$ for every m and k . Choose j such that $2^{-(j-1)} \leq 2^{-n}\varepsilon$ and set $\Omega_n = \bigcap_{k \geq j} F_k^c$. We have $\mathbb{P}^{\otimes \Pi^m}(\Omega_n^c) \leq \sum_{k=j}^\infty \mathbb{P}^{\otimes \Pi^m}(F_k) \leq 2^{-n}\varepsilon$ for every m , as desired. Also, $\omega \in \Omega_n$ implies that $\sup_{0 \leq s \leq v \leq t_n \wedge (s+\delta_k)} \int_s^v \|\alpha_u\| du \leq 2^{-k}$ for all $k \geq j$, and hence, (4.17) holds. \square

DEFINITION 4.14. Let Y be an adapted continuous \mathbb{R}^d -valued process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, let B be an adapted continuous \mathbb{R}^d -valued process whose components are of finite variation and for which $B_0 = 0$ and let C be an adapted continuous, $d \times d$ -matrix-valued process whose components are of finite variation and for which $C_0 = 0$. We further assume that outside a \mathbb{P} -null set that does not depend on s and t , the increment $C_t - C_s$ is positive semidefinite whenever $0 \leq s < t < \infty$. We say that Y is a semimartingale with characteristic pair (B, C) if the components of $Y - B$ and $(Y - B)(Y - B)^{tr} - C$ are local martingales.

PROPOSITION 4.15. *Let \mathbb{P} be a probability measure and let $(T_i, \mathcal{G}_i)_{i=0}^n$ be an extended partition on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$. Let Y be a continuous $\mathbb{R}^{d'}$ -valued process (d' may be different from d), and suppose that Y is a semimartingale with*

characteristic pair (B, C) under \mathbb{P} . If $\Delta(Y, T_i)$, $\Delta(B, T_i)$ and $\Delta(C, T_i)$ are all $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable for $i = 1, \dots, n$, then under $\mathbb{P}^{\otimes \Pi}$ the process Y is still a semimartingale with characteristic pair (B, C) .

The proof of Proposition 4.15 depends on some preliminary results.

LEMMA 4.16. *Let \mathbb{P}_1 and \mathbb{P}_2 be probability measures on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$, and let T be a finite (for every ω) $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -stopping time. Let M be a continuous local martingale relative to $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ under \mathbb{P}_1 and \mathbb{P}_2 . Let \mathcal{G} be a sub- σ -field of $\mathcal{F}_T^{\mathcal{E},d}$ such that $\mathbb{P}_1|_{\mathcal{G}} \ll \mathbb{P}_2|_{\mathcal{G}}$ and assume that $\widehat{M} \triangleq \Delta(M, T)$ is $\mathcal{G} \vee \sigma(\Delta(X, T))$ -measurable. Then $(M_t, \mathcal{F}_t^{\mathcal{E},d})_{t \geq 0}$ is a continuous local martingale under $\mathbb{P}_{12} \triangleq \mathbb{P}_1 \otimes_{T, \mathcal{G}} \mathbb{P}_2$.*

PROOF. It is sufficient to show that M^T and $M - M^T$ are both \mathbb{P}_{12} -local martingales relative to $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$. As M^T is a \mathbb{P}_1 -local martingale, M^T is $\mathcal{F}_T^{\mathcal{E},d}$ -measurable and \mathbb{P}_1 and \mathbb{P}_2 agree on $\mathcal{F}_T^{\mathcal{E},d}$, we may immediately conclude that M^T is a \mathbb{P}_{12} -local martingale.

For each integer $n > 0$, define the stopping time $S^n \triangleq \inf\{t \geq T : |M_t - M_T| \geq n\}$. Then $M^{S^n} - (M^{S^n})^T = M^{S^n} - M^T$ is bounded, and $\Delta(M^{S^n}, T)$ is $\mathcal{G} \vee \sigma(\Delta(X, T))$ -measurable. As a result, we may assume without loss of generality that $M - M^T$ is a uniformly integrable \mathbb{P}_2 -martingale.

We now show that \widehat{M} is a \mathbb{P}_{12} -martingale with respect to the filtration $\widehat{\mathcal{F}}_t = \mathcal{F}_{T+t}$. The process \widehat{M} is clearly $\{\widehat{\mathcal{F}}_t\}_{t \geq 0}$ -adapted, and it follows from the optional sampling theorem that \widehat{M} is a \mathbb{P}_2 -martingale with respect to the filtration $\{\widehat{\mathcal{F}}_t\}_{t \geq 0}$. For $0 \leq s \leq t$, $A \in \mathcal{F}_T^{\mathcal{E},d}$ and $B \in \sigma(\Delta_r(X, T) : 0 \leq r \leq s)$, we have from Corollary 4.7(b) that

$$\begin{aligned} \mathbb{E}_{12}[\mathbb{I}_{A \cap B}(\widehat{M}_t - \widehat{M}_s)] &= \mathbb{E}_{12}[\mathbb{I}_A \mathbb{E}_{12}[\mathbb{I}_B(\widehat{M}_t - \widehat{M}_s) | \mathcal{F}_T^{\mathcal{E},d}]] \\ &= \mathbb{E}_{12}[\mathbb{I}_A \mathbb{E}_2[\mathbb{I}_B(\widehat{M}_t - \widehat{M}_s) | \mathcal{G}]] \\ &= \mathbb{E}_{12}[\mathbb{I}_A \mathbb{E}_2[\mathbb{I}_B \mathbb{E}_2[\widehat{M}_t - \widehat{M}_s | \widehat{\mathcal{F}}_s] | \mathcal{G}]] = 0, \end{aligned}$$

where we have used the fact that \widehat{M} is \mathbb{P}_2 -martingale in the last step. Writing $X_t^{T+s} = X_t^T + \Delta_{0 \vee (t-T) \wedge s}(X, T)$, we see that sets of the form $A \cap B$ generate $\widehat{\mathcal{F}}_s$. It then follows from Dynkin's π - λ theorem that \widehat{M} is \mathbb{P}_{12} -martingale relative to $\{\widehat{\mathcal{F}}_t\}_{t \geq 0}$.

To conclude the proof, we observe that $(r - T)^+$ is a bounded $\{\widehat{\mathcal{F}}_t\}_{t \geq 0}$ -stopping time and $\mathcal{F}_r \subset \widehat{\mathcal{F}}_{(r-T) \vee 0}$ for each $r \geq 0$. Fixing $0 \leq s < t$ and $A \in \mathcal{F}_s$, we have

$$\begin{aligned} \mathbb{E}_{12}[\mathbb{I}_A(M_t - M_s^T)] &= \mathbb{E}_{12}[\mathbb{I}_A \widehat{M}_{(t-T) \vee 0}] = \mathbb{E}_{12}[\mathbb{I}_A \widehat{M}_{(s-T) \vee 0}] \\ &= \mathbb{E}_{12}[\mathbb{I}_A(M_s - M_s^T)], \end{aligned}$$

so $M - M^T$ is \mathbb{P}_{12} -martingale relative to $\{\mathcal{F}_t\}_{t \geq 0}$. \square

LEMMA 4.17. *Let \mathbb{P} be a probability measure on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$, and let M be a uniformly integrable \mathbb{P} -martingale relative to $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$. Let S, T and U be stopping times with $T \leq U$ almost surely, and let Z be an $\mathcal{F}_T^{\mathcal{E},d}$ -measurable bounded random variable. Then $\mathbb{E}[(M_U - M_T)Z | \mathcal{F}_S^{\mathcal{E},d}] = (M_{U \wedge S} - M_{T \wedge S})Z$.*

PROOF. Because

$$\mathbb{I}_{\{S \leq T\}} \mathbb{E}[(M_U - M_T)Z | \mathcal{F}_S^{\mathcal{E},d}] = \mathbb{I}_{\{S \leq T\}} \mathbb{E}[Z \mathbb{E}[(M_U - M_T) | \mathcal{F}_T^{\mathcal{E},d}] | \mathcal{F}_S^{\mathcal{E},d}] = 0,$$

we have

$$\begin{aligned} & \mathbb{E}[(M_U - M_T)Z | \mathcal{F}_S^{\mathcal{E},d}] \\ &= \mathbb{I}_{\{T < S \leq U\}} \mathbb{E}[(M_U - M_T)Z | \mathcal{F}_S^{\mathcal{E},d}] + \mathbb{I}_{\{U < S\}} \mathbb{E}[(M_U - M_T)Z | \mathcal{F}_S^{\mathcal{E},d}] \\ &= \mathbb{I}_{\{T < S \leq U\}} (M_S - M_T)Z + \mathbb{I}_{\{U < S\}} (M_U - M_T)Z \\ &= (M_{U \wedge S} - M_{T \wedge S})Z. \quad \square \end{aligned}$$

LEMMA 4.18. *Let \mathbb{P}_1 and \mathbb{P}_2 be probability measures on $(\Omega^{\mathcal{E},d}, \mathcal{F}^{\mathcal{E},d})$ and let T be a finite (for every ω) $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -stopping time. Let M^1, M^2 and C be continuous $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ -adapted real-valued processes such that M^1, M^2 and $M^3 \triangleq M^1 M^2 - C$ are local martingales relative to $\{\mathcal{F}_t^{\mathcal{E},d}\}_{t \geq 0}$ under \mathbb{P}_1 and \mathbb{P}_2 . Let \mathcal{G} be a sub- σ -field of $\mathcal{F}_T^{\mathcal{E},d}$ such that $\mathbb{P}_1 |_{\mathcal{G}} \ll \mathbb{P}_2 |_{\mathcal{G}}$ and assume that $\widehat{M}^1 \triangleq \Delta(M^1, T), \widehat{M}^2 \triangleq \Delta(M^2, T)$ and $\widehat{C} \triangleq \Delta(C, T)$ are $\mathcal{G} \vee \sigma(\Delta(X, T))$ -measurable. Then $(M^3, \mathcal{F}_t^{\mathcal{E},d})_{t \geq 0}$ is a local martingale under $\mathbb{P}_{12} \triangleq \mathbb{P}_1 \otimes_{\mathcal{G}, T} \mathbb{P}_2$.*

PROOF. We cannot apply Lemma 4.16 directly because we did not assume that \widehat{M}_3 is $\mathcal{G} \vee \sigma(\Delta(X, T))$ -measurable. Instead, we define the process

$$\begin{aligned} Y_t &\triangleq M_{t \wedge T}^1 M_{t \wedge T}^2 + (M_t^1 - M_{t \wedge T}^1)(M_t^2 - M_{t \wedge T}^2) - C_t \\ &= M_t^3 - (M_t^1 - M_{t \wedge T}^1)M_{t \wedge T}^2 - (M_t^2 - M_{t \wedge T}^2)M_{t \wedge T}^1, \quad t \geq 0, \end{aligned}$$

for which $\Delta(Y, T) = \Delta(M^1, T)\Delta(M^2, T) - \Delta(C, T)$ is $\mathcal{G} \vee \sigma(\Delta(X, T))$ -measurable. Define $T_n \triangleq \inf\{t \geq 0 : |M_t^1| \vee |M_t^2| \vee |M_t^3| \vee |C_t| \geq n\}$, and set $M^{i,n} \triangleq (M^i)^{T_n}$ for $i = 1, 2, 3$, $C^n \triangleq C^{T_n}$ and $Y^n \triangleq Y^{T_n}$. For fixed n , the processes $M^{i,n}, i = 1, 2, 3$, and Y^n are bounded. For $0 \leq s \leq t$, we apply Lemma 4.17 with $M = M^{1,n}, Z = M_{t \wedge T}^{2,n}, S = s, T = t \wedge T$ and $U = t$, and use the fact that $M_s^{1,n} - M_{s \wedge T}^{1,n} = 0$ if $T \geq s$ to obtain

$$\begin{aligned} (4.18) \quad \mathbb{E}_k[(M_t^{1,n} - M_{t \wedge T}^{1,n})M_{t \wedge T}^{2,n} | \mathcal{F}_s^{\mathcal{E},d}] &= (M_s^{1,n} - M_{s \wedge T}^{1,n})M_{t \wedge T}^{2,n} \\ &= (M_s^{1,n} - M_{s \wedge T}^{1,n})M_{s \wedge T}^{2,n}, \quad k = 1, 2. \end{aligned}$$

The same equality holds if we reverse the roles of $M^{1,n}$ and $M^{2,n}$. Finally, because $M^{3,n}$ is a martingale,

$$\begin{aligned} \mathbb{E}_k[Y_t^n | \mathcal{F}_s^{\mathcal{E},d}] &= \mathbb{E}_k[M_t^{3,n} | \mathcal{F}_s^{\mathcal{E},d}] - \mathbb{E}_k[(M_t^{1,n} - M_{t \wedge T}^{1,n})M_{t \wedge T}^{2,n} | \mathcal{F}_s^{\mathcal{E},d}] \\ &\quad - \mathbb{E}_k[(M_t^{2,n} - M_{t \wedge T}^{2,n})M_{t \wedge T}^{1,n} | \mathcal{F}_s^{\mathcal{E},d}] \\ &= M_s^{3,n} - (M_s^{1,n} - M_{s \wedge T}^{1,n})M_{s \wedge T}^{2,n} - (M_s^{2,n} - M_{s \wedge T}^{2,n})M_{s \wedge T}^{1,n} \\ &= Y_s^n, \quad k = 1, 2, \end{aligned}$$

so Y is a local martingale under both \mathbb{P}_1 and \mathbb{P}_2 . Lemma 4.16 implies that M^1, M^2 and Y are \mathbb{P}_{12} -local martingales. Therefore, (4.18) holds under \mathbb{P}_{12} as well, from which we conclude that

$$\begin{aligned} \mathbb{E}_{12}[M_t^{3,n} | \mathcal{F}_s^{\mathcal{E},d}] &= \mathbb{E}_{12}[Y_t^n | \mathcal{F}_s^{\mathcal{E},d}] + \mathbb{E}_{12}[(M_t^{1,n} - M_{t \wedge T}^{1,n})M_{t \wedge T}^{2,n} | \mathcal{F}_s^{\mathcal{E},d}] \\ &\quad + \mathbb{E}_{12}[(M_t^{2,n} - M_{t \wedge T}^{2,n})M_{t \wedge T}^{1,n} | \mathcal{F}_s^{\mathcal{E},d}] \\ &= Y_s^n + (M_s^{1,n} - M_{s \wedge T}^{1,n})M_{s \wedge T}^{2,n} + (M_s^{2,n} - M_{s \wedge T}^{2,n})M_{s \wedge T}^{1,n} \\ &= M_s^{3,n}, \quad 0 \leq s \leq t. \quad \square \end{aligned}$$

PROOF OF PROPOSITION 4.15. According to Remark 4.9, $\mathbb{P}^{\otimes \Pi} = \mathbb{P}^{n+1}$, where \mathbb{P}^i is defined recursively by $\mathbb{P}^0 = \mathbb{P}$ and $\mathbb{P}^{i+1} = \mathbb{P}^i \otimes_{T_i, \mathcal{G}_i} \mathbb{P}$, $i = 0, \dots, n$. If M is a continuous local martingale under \mathbb{P} and $\Delta(M, T_i)$ is $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable for $i = 0, \dots, n$, then repeated application of Lemma 4.16 shows that M is a \mathbb{P}^i -local martingale for $i = 1, \dots, n, n + 1$, and in particular, M is a continuous local martingale under $\mathbb{P}^{\otimes \Pi}$. Similarly, if M^1 and M^2 are continuous local martingales under \mathbb{P} , C is a finite variation process such that $M^3 \triangleq M^1 M^2 - C$ is a local martingale under \mathbb{P} , and $\Delta(M^1, T_i), \Delta(M^2, T_i)$ and $\Delta(C, T_i)$ are $\mathcal{G}_i \vee \sigma(\Delta(X, T_i))$ -measurable for $0 = 1, \dots, n$, then repeated application of Lemma 4.18 shows that $M^1 M^2 - C$ is a \mathbb{P}^i -local martingale for $i = 1, \dots, n, n + 1$. In particular, $M^1 M^2 - C$ is a continuous local martingale under $\mathbb{P}^{\otimes \Pi}$. These observations combined with Proposition 4.10(a) prove the desired result. \square

5. Conditional expectations. The results of this section are implicit in Krylov [25] and Gyöngy [18]. We use the notation introduced in Sections 3 and 4. In addition, we denote the Borel σ -field on $[0, t]$ by $\mathcal{B}[0, t]$ and the Borel σ -field on $[0, \infty)$ by $\mathcal{B}[0, \infty)$.

PROPOSITION 5.1. *Let Z be an \mathcal{E} -valued process and let Γ be an \mathbb{R}^d -valued process (resp., a $d \times d$ matrix-valued process) taking values in a closed convex set K , and satisfying $\mathbb{E}[\int_0^t \|\Gamma_u\| du] < \infty$ for all $t \geq 0$. Then there exists an \mathbb{R}^d -valued measurable function (resp., a $d \times d$ matrix-valued measurable function) $\widehat{\Gamma}$,*

defined on $[0, \infty) \times \mathcal{E}$, taking values in K , and there exists a Lebesgue-null set $N \subset [0, \infty)$, so that

$$(5.1) \quad \widehat{\Gamma}(t, Z_t) = \mathbb{E}[\Gamma_t | Z_t], \quad \mathbb{P}\text{-a.s.}, t \in N^c.$$

The proof of Proposition 5.1 depends on the following lemma.

LEMMA 5.2. *Let Z be an \mathcal{E} -valued process and let Γ be a real-valued process satisfying $\mathbb{E} \int_0^t |\Gamma_u| du < \infty$ for all $t \geq 0$. Let $\widehat{\Gamma}$ be a real-valued measurable function on $[0, \infty) \times \mathcal{E}$. There exists a Lebesgue-null set $N \subset [0, \infty)$ so that (5.1) holds if and only if for every bounded $\mathcal{B}[0, \infty) \otimes \mathfrak{E}$ -measurable real-valued function f ,*

$$(5.2) \quad \mathbb{E} \int_0^t \widehat{\Gamma}(u, Z_u) f(u, Z_u) du = \mathbb{E} \int_0^t \Gamma_u f(u, Z_u) du, \quad t \geq 0.$$

PROOF. If (5.1) holds, then (5.2) follows from Fubini’s theorem.

To prove the converse, we assume (5.2). Taking $f(u, Z_u) = \text{sgn}(\widehat{\Gamma}(u, Z_u))$ and using the integrability of Γ , we see that $\mathbb{E} \int_0^t |\widehat{\Gamma}(u, Z_u)| du < \infty$ for all $t \geq 0$.

The σ -field \mathfrak{E} is generated by a collection of open balls intersected with \mathcal{E} , each ball having a rational radius and centered at a point in a countable dense subset of the separable metric space containing \mathcal{E} . Let \mathcal{O} denote the collection of finite intersections of this countable collection of sets. Then \mathcal{O} is itself countable and $\mathfrak{E} = \sigma(\mathcal{O})$. We enumerate the sets in \mathcal{O} as O_1, O_2, \dots . Define $g_n(t) \triangleq \mathbb{E}[(\widehat{\Gamma}(t, Z_t) - \Gamma_t)\mathbb{I}_{\{Z_t \in O_n\}}]$. For $B \in \mathcal{B}[0, t]$, (5.2) implies

$$\int_B g_n(u) du = \mathbb{E} \int_0^t (\widehat{\Gamma}(u, Z_u) - \Gamma_u)\mathbb{I}_{\{(u, Z_u) \in B \times O_n\}} du = 0.$$

Since both $t \geq 0$ and $B \in \mathcal{B}[0, t]$ are arbitrary, we conclude that $g_n = 0$ for Lebesgue-almost every $t \geq 0$. Thus, $N \triangleq \{t \geq 0 | g_n(t) \neq 0 \text{ for some } n\}$ is a Lebesgue-null set.

The collection of sets $A \in \mathfrak{E}$ for which

$$(5.3) \quad \mathbb{E}[(\widehat{\Gamma}(t, Z_t) - \Gamma_t)\mathbb{I}_{\{Z_t \in A\}}] = 0, \quad t \in N^c,$$

is a λ system containing \mathcal{O} , and the Dynkin π - λ theorem implies that (5.3) holds for every $A \in \mathfrak{E}$. This gives us (5.1). \square

PROOF OF PROPOSITION 5.1. Except for the assertion that $\widehat{\Gamma}$ takes values in the set K , it suffices to prove the proposition for the case that Γ is real-valued. We can then apply the one-dimensional result to each component of the Γ in the proposition.

In the one-dimensional case, we define the σ -finite measure

$$\mu(A) \triangleq \mathbb{E} \int_0^\infty \mathbb{I}_A(u, Z_u) du, \quad A \in \mathcal{B}[0, \infty) \otimes \mathfrak{E},$$

and the σ -finite signed measure

$$\nu(A) \triangleq \mathbb{E} \int_0^\infty \Gamma_u \mathbb{I}_A(u, Z_u) du, \quad A \in \mathcal{B}[0, \infty) \otimes \mathfrak{E}.$$

Obviously, $\nu \ll \mu$, so we can define $\widehat{\Gamma}(t, z) = \frac{d\nu}{d\mu}(t, z)$ for $(t, z) \in [0, \infty) \otimes \mathfrak{E}$. Let f be a bounded $\mathcal{B}[0, \infty) \otimes \mathfrak{E}$ -measurable real-valued function. For $t \geq 0$,

$$\begin{aligned} \mathbb{E} \int_0^t \widehat{\Gamma}(u, Z_u) f(u, Z_u) du &= \int_{[0,t] \times \mathfrak{E}} \widehat{\Gamma}(u, z) f(u, z) \mu(du, dz) \\ &= \int_{[0,t] \times \mathfrak{E}} f(u, z) \nu(du, dz) \\ &= \mathbb{E} \int_0^t \Gamma_u f(u, Z_u) du. \end{aligned}$$

Equation (5.1) follows from Lemma 5.2.

Let us now consider the case of a multi-dimensional Γ taking values in a closed convex set K . We have already shown the existence of $\widehat{\Gamma}$ such that (5.1) holds, and it remains to show that $\widehat{\Gamma}$ takes values in K . Define $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ (resp., $\varphi: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$) by $\varphi(\gamma) = \min_{\kappa \in K} \|\gamma - \kappa\|$, which is the distance from γ to K . One can verify from the triangle inequality that for each constant c , the set $\{\gamma | \varphi(\gamma) \leq c\}$ is convex, and hence, φ is a continuous convex real-valued function. Such a function has the property that $\varphi(\gamma) = \max\{\ell(\gamma) | \ell \text{ is linear and } \ell \leq \varphi\}$. This permits us to establish the Jensen inequality

$$\begin{aligned} \mathbb{E}[\varphi(\Gamma_t) | Z_t] &\geq \max\{\mathbb{E}[\ell(\Gamma_t) | Z_t] | \ell \text{ is linear and } \ell \leq \varphi\} \\ &= \max\{\ell(\mathbb{E}[\Gamma_t | Z_t]) | \ell \text{ is linear and } \ell \leq \varphi\} \\ &= \varphi(\mathbb{E}[\Gamma_t | Z_t]) \\ &= \varphi(\widehat{\Gamma}(t, Z_t)), \quad t \in N^c. \end{aligned}$$

But Γ takes values in K , so the left-hand side of this inequality is zero. Thus the right-hand side is zero, implying $\widehat{\Gamma}(t, Z_t) \in K$ almost surely for each $t \in N^c$. We can modify $\widehat{\Gamma}(t, z)$ so that it takes values in K for every t , and (5.1) still holds. \square

DEFINITION 5.3. Let $\{\Gamma^i\}_i$ be a collection of processes on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let T be a $[0, \infty)$ -valued random variable. We say the collection $\{\Gamma^i\}_i$ is *strongly independent* of T if there is a σ -field $\mathcal{G} \subset \mathcal{F}$ such that each Γ_i is $\mathcal{B}[0, \infty) \times \mathcal{G}$ -measurable and \mathcal{G} is independent of $\sigma(T)$.

PROPOSITION 5.4. *Within the setting of Proposition 5.1, let T be a $[0, \infty)$ -valued random variable whose distribution $\mu \triangleq \mathbb{P} \circ T^{-1}$ is absolutely continuous with respect to Lebesgue measure. Assume also that the pair of processes (Γ, Z) is strongly independent of T and $\mathbb{E}|\Gamma_T| < \infty$. Then*

$$(5.4) \quad \widehat{\Gamma}(T, Z_T) = \mathbb{E}[\Gamma_T | T, Z_T], \quad \mathbb{P}\text{-a.s.}$$

PROOF. We first observe that

$$(5.5) \quad \int_0^\infty \mathbb{E}[\Xi_t] \mu(dt) = \mathbb{E}[\Xi_T]$$

for any process Ξ that is strongly independent of T and satisfies $\mathbb{E}|\Xi_T| < \infty$. To see this, consider the case $\Xi_t = \sum_{i=1}^n \mathbb{I}_{A_i} \mathbb{I}_{B_i}(t)$, where $A_i \in \mathcal{G}$, the σ -field in Definition 5.3 and $B_i \in \mathcal{B}[0, \infty)$. Then use the monotone class theorem.

Now let $f : [0, \infty) \times \mathcal{E} \rightarrow \mathbb{R}$ be a bounded, $\mathcal{B}[0, \infty) \otimes \mathcal{E}$ -measurable real-valued function. Proposition 5.1 implies $\mathbb{E}[\widehat{\Gamma}(t, Z_t) f(t, Z_t)] = \mathbb{E}[\Gamma_t f(t, Z_t)]$ for all $t \in N^c$. Integrating both sides of this equation with respect to $\mu(dt)$ and using (5.5), we obtain $\mathbb{E}[\widehat{\Gamma}(T, Z_T) f(T, Z_T)] = \mathbb{E}[\Gamma_T f(T, Z_T)]$. Equation (5.4) follows. \square

6. Approximation. We collect in this section three approximation results needed to prove Theorem 3.6. We denote by \mathbb{N} the set of natural numbers and define $\overline{\mathbb{N}} \triangleq \mathbb{N} \cup \{\infty\}$. We recall that $\lambda_{[0,t]}$ denotes Lebesgue measure on $[0, t]$.

6.1. *Convergence of the integral of a process.*

PROPOSITION 6.1. *Let $\{Z^m\}_{m \in \overline{\mathbb{N}}}$ be a collection of continuous \mathcal{E} -valued processes, possibly defined on different probability spaces under different probability measures \mathbb{Q}^m . Let $f : [0, \infty) \times \mathcal{E} \rightarrow \mathbb{R}^d$ be a measurable function. Assume:*

- (i) *for each $t \in [0, \infty)$, the distribution of Z_t^m under \mathbb{Q}^m is independent of $m \in \mathbb{N}$,*
- (ii) *the distribution on $C^\mathcal{E}$ of Z^m under \mathbb{Q}^m converges weakly to the distribution of Z^∞ under \mathbb{Q}^∞ , that is, $\mathbb{Q}^m \circ (Z^m)^{-1} \Rightarrow \mathbb{Q}^\infty \circ (Z^\infty)^{-1}$ and*
- (iii) *$\mathbb{E}^{\mathbb{Q}^1} \int_0^t \|f(u, Z_u^1)\| du < \infty$ for every $t \in [0, \infty)$.*

Then:

- (iv) *for every $m \in \overline{\mathbb{N}}$ the integral process $F_t^m \triangleq \int_0^t f(s, Z_s^m) ds, t \in [0, \infty)$, is defined \mathbb{Q}^m -almost surely,*
- (v) *$\mathbb{Q}^m[F^m \in C^d] = 1$ for every $m \in \overline{\mathbb{N}}$,*
- (vi) *$\{f(\cdot, Z^m), \lambda_{[0,t]} \times \mathbb{Q}^m\}_{m \in \overline{\mathbb{N}}}$ is uniformly integrable for every $t \in [0, \infty)$,*
- (vii) *$(Z^m, F^m) \Rightarrow (Z^\infty, F^\infty)$.*

PROOF. It suffices to prove parts (iv)–(vi) of the lemma for the case $d = 1$, since these results can be applied component-wise to the d -dimensional f .

Define the measure μ on $[0, \infty) \times \mathcal{E}$ by $\mu(A) \triangleq \mathbb{E}^{\mathbb{Q}^m} \int_0^\infty \mathbb{I}_A(s, Z_s^m) ds$. Assumption (i) and the convergence in (ii) imply that the distribution of Z_t^m is independent of $m \in \overline{\mathbb{N}}$, so it does not matter which $m \in \overline{\mathbb{N}}$ we use in the definition of μ .

Therefore, for each $m \in \overline{\mathbb{N}}$ and $M > 0$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^m} \int_0^t |f(s, Z_s^m)| \mathbb{I}_{\{|f(s, Z_s^m)| \geq M\}} ds &= \int_{[0,t] \times \mathcal{E}} |f(s, e)| \mathbb{I}_{\{|f(s, e)| \geq M\}} \mu(ds, de) \\ &= \mathbb{E}^{\mathbb{Q}^1} \int_0^t |f(s, Z_s^1)| \mathbb{I}_{\{|f(s, Z_s^1)| \geq M\}} ds. \end{aligned}$$

Setting $M = 0$, we obtain (iv) and (v) from (iii). Condition (iii) implies that the last term can be made arbitrarily small by choosing M large, and (vi) also follows.

To prove (v), we use (vi) and Lusin’s theorem to choose for each $k \in \mathbb{N}$ a bounded continuous function $f^k : [0, k] \times \mathcal{E} \rightarrow \mathbb{R}^d$ such that

$$\lim_{k \rightarrow \infty} \int_{[0,k] \times \mathcal{E}} \|f(t, e) - f^k(t, e)\| \mu(dt, de) = 0.$$

The mapping $z \mapsto \int_0^{\wedge k} f^k(s, z(s)) ds$ is continuous from $C^{\mathcal{E}}$ to C^d , which implies that

$$(6.1) \quad (Z^m, F^{m,k}) \Rightarrow (Z^\infty, F^{\infty,k}) \quad \text{as } m \rightarrow \infty,$$

where $F_t^{m,k} \triangleq \int_0^{t \wedge k} f^k(s, Z_s^m) ds$. But for each fixed T and $k \geq T$,

$$\begin{aligned} (6.2) \quad & \sup_{m \in \overline{\mathbb{N}}} \mathbb{Q}^m \left[\sup_{0 \leq t \leq T} \|F_t^m - F_t^{m,k}\| > \varepsilon \right] \\ & \leq \sup_{m \in \overline{\mathbb{N}}} \frac{1}{\varepsilon} \mathbb{E}^{\mathbb{Q}^m} \sup_{0 \leq t \leq T} \|F_t^m - F_t^{m,k}\| \\ & \leq \frac{1}{\varepsilon} \int_{[0,k] \times \mathcal{E}} \|f(s, e) - f^k(s, e)\| \mu(ds, de), \end{aligned}$$

which has limit zero as $k \rightarrow \infty$. In particular, the convergence $F^{m,k} \Rightarrow F^m$ as $k \rightarrow \infty$ is uniform in $m \in \overline{\mathbb{N}}$.

Let $\Psi : C^{\mathcal{E}} \times C^1 \rightarrow \mathbb{R}$ be a uniformly continuous bounded function. To prove weak convergence of measures on a metric space, it suffices to consider such functions (see [29], Chapter II, Theorem 6.1). We have

$$\begin{aligned} & |\mathbb{E}^m[\Psi(Z^m, F^m)] - \mathbb{E}^\infty[\Psi(Z^\infty, F^\infty)]| \\ & = |\mathbb{E}^m[\Psi(Z^m, F^m) - \Psi(Z^m, F^{m,k})]| \\ & \quad + |\mathbb{E}^m[\Psi(Z^m, F^{m,k})] - \mathbb{E}^\infty[\Psi(Z^\infty, F^{\infty,k})]| \\ & \quad + |\mathbb{E}^\infty[\Psi(Z^\infty, F^{\infty,k}) - \Psi(Z^\infty, F^\infty)]|. \end{aligned}$$

Given $\varepsilon > 0$, (6.2) guarantees that we can choose k so large that the first and third terms on the right-hand side are less than ε , independently of m . For this value of k , we can then use (6.1) to choose M so that for all $m \geq M$, the second term is also less than ε . \square

6.2. *Approximation by step functions.* We show in Proposition 6.3 below that an arbitrary integrable process can be approximated in $L_1(\mathbb{P} \times \lambda_{[0,t]})$ by step functions obtained by sampling the process at random partition points.

LEMMA 6.2. *Let $f : [0, \infty) \rightarrow \mathbb{R}^d$ be a measurable function with $\int_0^t \|f(s)\| ds < \infty$ for every $t \in [0, \infty)$. Define the sets*

$$(6.3) \quad I_i^n \triangleq \left\{ (t, u) \in [0, \infty) \times [0, 1] : \frac{u+i-1}{n} \leq t < \frac{u+i}{n} \right\}, \quad i = 1, 2, \dots,$$

and define the sequence of functions $f_n(t, u) = \sum_{i=1}^{\infty} f\left(\frac{u+i-1}{n}\right) \mathbb{I}_{I_i^n}(t, u)$. Then

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^t \|f(s) - f_n(s, u)\| ds du = 0, \quad t \in [0, \infty).$$

PROOF. Fix $t > 0$ and $\varepsilon > 0$. Choose a continuous, \mathbb{R}^d -valued function g defined on $[0, t + 1]$ for which $\int_0^{t+1} \|f(s) - g(s)\| ds \leq \varepsilon$. Set $m = \lceil t \rceil \in [t, t + 1) \cap \mathbb{N}$ and set $g_n(s, u) \triangleq \sum_{i=1}^{mn} g\left(\frac{u+i-1}{n}\right) \mathbb{I}_{I_i^n}(s, u)$. We have

$$\begin{aligned} & \int_0^1 \int_0^t \|f_n(s, u) - g_n(s, u)\| ds du \\ & \leq \sum_{i=1}^{mn} \int_0^1 \int_{(u+i-1)/n}^{(u+i)/n} \left\| f\left(\frac{u+i-1}{n}\right) - g\left(\frac{u+i-1}{n}\right) \right\| ds du \\ & = \sum_{i=1}^{mn} \int_0^1 \left\| f\left(\frac{u+i-1}{n}\right) - g\left(\frac{u+i-1}{n}\right) \right\| \frac{du}{n} \\ & = \sum_{i=1}^{mn} \int_{(i-1)/n}^{i/n} \|f(v) - g(v)\| dv \leq \varepsilon. \end{aligned}$$

Because g is uniformly continuous on $[0, t + 1]$, we may choose N so that $\|g(s_2) - g(s_1)\| \leq \varepsilon/t$ whenever $|s_2 - s_1| \leq 1/N$. By enlarging N if necessary, we can also ensure that $\int_0^{1/N} \|g(s)\| ds \leq \varepsilon$. Therefore, for $n \geq N$, we have

$$\begin{aligned} & \int_0^1 \int_0^t \|f(s) - f_n(s, u)\| ds du \\ & \leq \int_0^t \|f(s) - g(s)\| ds + \int_0^1 \int_{u/n}^t \|g(s) - g_n(s, u)\| ds du \\ & \quad + \int_0^{1/n} \|g(s)\| ds + \int_0^1 \int_0^t \|g_n(s, u) - f_n(s, u)\| ds du \leq 4\varepsilon. \quad \square \end{aligned}$$

PROPOSITION 6.3. *Let $(\Omega', \mathcal{F}', \mathbb{Q}')$ be a probability space that supports an \mathbb{R}^d -valued process a satisfying*

$$(6.4) \quad \mathbb{E}^{\mathbb{Q}'} \int_0^t \|a_s\| ds < \infty, \quad t \geq 0.$$

Set $\Omega \triangleq [0, 1] \times \Omega'$, with generic point $\omega = (u, \omega')$, and define $U(u, \omega') = u$. Set $\mathcal{F} = \mathcal{B}[0, 1] \otimes \mathcal{F}'$, $\mathbb{Q} = \lambda_{[0,1]} \times \mathbb{Q}'$ and extend a to Ω via the abuse of notation $a(u, \omega') \triangleq a(\omega')$. Finally, define the random times $T_0^n \triangleq 0$, $T_i^n \triangleq (U + i - 1)/n$ for $i = 1, 2, \dots, n^2$ and $T_{n^2+1}^n \triangleq \infty$. Then the sampled process

$$a_t^n(\omega) \triangleq \sum_{i=1}^{n^2} a_{T_i^n(\omega)}(\omega) \mathbb{I}_{[T_i^n(\omega), T_{i+1}^n(\omega))}(t) = \sum_{i=1}^{n^2} a_{(u+i-1)/n}(\omega') \mathbb{I}_{I_i^n}(t, u),$$

where I_i^n is defined by (6.3), satisfies

$$(6.5) \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \int_0^t \|a_s - a_s^n\| ds = 0, \quad t \geq 0.$$

PROOF. Define $A_t^n(\omega') \triangleq \int_0^1 \int_0^t \|a_s(\omega') - a_s^n(u, \omega')\| ds du$ for $t \geq 0$ and $\omega' \in \Omega'$. Assumption (6.4) implies that $\int_0^t \|a_s(\omega')\| ds < \infty$ for all $t \geq 0$ for \mathbb{Q}' -almost every ω' . For fixed ω' satisfying this condition, Lemma 6.2 then shows that $\lim_{n \rightarrow \infty} A_t^n(\omega') = 0$ for every $t \geq 0$. Equation (6.5) is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}'} A_t^n = 0, \quad t \geq 0,$$

and to obtain this result it now suffices to show that for each fixed $t \geq 0$, the collection of random variables $\{A_t^n\}_{n=1}^\infty$ is uniformly integrable under \mathbb{Q}' .

We first show that $\{a^n\}_{n=1}^\infty$ is uniformly integrable with respect to $\lambda_{[0,t]} \times \mathbb{Q}$ for every $t \geq 0$. Toward this end, fix $t \geq 0$ and set $m \triangleq \lceil t \rceil \in [t, t + 1) \cap \mathbb{N}$, so that $t \leq T_{mn+1}^n$. Then

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} [\|a_{T_i^n}^n\| \mathbb{I}_{\{\|a_{T_i^n}^n\| \geq M\}}] \\ &= \int_0^1 \mathbb{E}^{\mathbb{Q}'} [\|a_{(u+i-1)/n}^n\| \mathbb{I}_{\{\|a_{(u+i-1)/n}^n\| \geq M\}}] \frac{du}{n} \\ &= \int_{(i-1)/n}^{i/n} \mathbb{E}^{\mathbb{Q}'} [\|a_s\| \mathbb{I}_{\{\|a_s\| \geq M\}}] ds, \quad i = 1, \dots, mn, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} \int_0^t \|a_s^n\| \mathbb{I}_{\{\|a_s^n\| \geq M\}} ds &\leq \mathbb{E}^{\mathbb{Q}} \int_0^{T_{mn+1}^n} \|a_s^n\| \mathbb{I}_{\{\|a_s^n\| \geq M\}} ds \\ &= \frac{1}{n} \sum_{i=1}^{mn} \mathbb{E}^{\mathbb{Q}} [\|a_{T_i^n}^n\| \mathbb{I}_{\{\|a_{T_i^n}^n\| \geq M\}}] \\ &= \mathbb{E}^{\mathbb{Q}'} \int_0^m \|a_s\| \mathbb{I}_{\{\|a_s\| \geq M\}} ds. \end{aligned}$$

The uniform integrability of $\{a^n\}_{n=1}^\infty$ under $\lambda_{[0,t]} \times \mathbb{Q}$ follows from (6.4). This implies the uniform integrability of $\{\|a - a^n\|\}_{n=1}^\infty$. Jensen’s inequality implies

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}'}[(A_t^n - M)^+] &= \mathbb{E}^{\mathbb{Q}'}\left[\left(\int_0^1 \int_0^t \|a_s(\cdot) - a_s^n(u, \cdot)\| ds du - M\right)^+\right] \\ &\leq \mathbb{E}^{\mathbb{Q}'}\left[\int_0^1 \left(\int_0^t \|a_s(\cdot) - a_s^n(u, \cdot)\| ds - M\right)^+ du\right] \\ &= \mathbb{E}^{\mathbb{Q}'}\left[\left(\int_0^t \|a_s - a_s^n\| ds - M\right)^+\right] \\ &\leq \mathbb{E}^{\mathbb{Q}'}\left[\int_0^t \left(\|a_s - a_s^n\| - \frac{M}{t}\right)^+ ds\right], \end{aligned}$$

and the uniform integrability of $\{\|a - a^n\|\}_{n=1}^\infty$ under $\lambda_{[0,t]} \times \mathbb{Q}$ implies that for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}'}[(A_t^n - M_\varepsilon)^+] \leq \varepsilon.$$

Consequently,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}'}[A_t^n \mathbb{I}_{\{A_t^n \geq 2M_\varepsilon\}}] &= \sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}'}[(A_t^n - 2M_\varepsilon)^+ + 2M_\varepsilon \mathbb{I}_{\{A_t^n \geq 2M_\varepsilon\}}] \\ &\leq \sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}'}[(A_t^n - 2M_\varepsilon)^+ + 2(A_t^n - M_\varepsilon)^+] \\ &\leq 3 \sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}'}[(A_t^n - M_\varepsilon)^+] \\ &\leq 3\varepsilon. \end{aligned}$$

This proves the uniform integrability of $\{A_t^n\}_{n=1}^\infty$ under \mathbb{Q}' . \square

6.3. *Sequence of discrete-time martingales with zero limit.* For our final approximation result, we construct a sequence of continuous-time, finite-variation processes that are martingales when sampled at certain discrete times. We provide conditions under which this sequence must converge to zero.

DEFINITION 6.4. A random partition Π is a set of random times $0 = T_0 \leq T_1 \leq \dots \leq T_n$. We set $|\Pi|(\omega) \triangleq \sup_{1 \leq i \leq n} |T_i(\omega) - T_{i-1}(\omega)|$. Let $\{\Pi^m\}_{m=1}^\infty$ be a sequence of random partitions, possibly defined on different spaces $\{\Omega^m\}_{m=1}^\infty$, where the random times in the partitions Π^m are denoted $T_0^m \leq T_1^m \leq \dots \leq T_{N(m)}^m$. We say that $\{\Pi^m\}_{m=1}^\infty$ converges uniformly to the identity if

$$(6.6) \quad \lim_{m \rightarrow \infty} \sup_{\omega \in \Omega^m} |\Pi^m(\omega)| = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \inf_{\omega \in \Omega^m} T_{N(m)}^m(\omega) = \infty.$$

PROPOSITION 6.5. *Let $(\Omega^m, \mathcal{F}^m, \mathbb{P}^m)_{m=1}^\infty$ be a sequence of probability spaces. Assume that on each space there is defined an \mathbb{R}^d -valued process X^m and a random partition $\Pi^m = \{T_0^m, T_1^m, \dots, T_{N(m)}^m\}$, and these partitions converge uniformly to the identity. Assume further that the set of processes and measures $(X^m, \lambda_{[0,t]} \times \mathbb{P}^m)_{m=1}^\infty$ is uniformly integrable for every $t \geq 0$. For $k = 0, 1, \dots, N(m)$, define $Y_k^m \triangleq \int_0^{T_k^m} X_u^m du$ and $\mathcal{F}_k^m \triangleq \sigma(Y_j^m, T_j^m \mid 0 \leq j \leq k)$, and assume that $(Y_k^m, \mathcal{F}_k^m)_{0 \leq k \leq N(m)}$ is a martingale for each m . Then*

$$\lim_{m \rightarrow \infty} \mathbb{E}^m \sup_{0 \leq s \leq t} \left\| \int_0^s X_u^m du \right\| = 0, \quad t \geq 0.$$

PROOF. By considering components of $\int_0^s X_u^m du$, we may reduce the proof to the case $d = 1$. Fix $t \geq 0$. Fix m large enough that $\sup_{\omega \in \Omega^m} |\Pi^m|(\omega) \leq 1$ and $\inf_{\omega \in \Omega^m} T_{N(m)}^m(\omega) > t$. Define $\rho \triangleq \min\{k : T_k^m \geq t\}$, so that T_ρ^m is the first random time after t and $T_\rho^m \leq T_{N(m)}^m \wedge (t + 1)$. The discrete-time martingale Y^m stopped at T_ρ^m is still a martingale. For $0 \leq s \leq t$, set $\tau(s) \triangleq \max\{k : T_k^m \leq s\}$, so that $T_{\tau(s)}^m$ is the last random time before s . Then $\tau(s) \leq \rho$ and $0 \leq s - T_{\tau(s)}^m \leq |\Pi^m|$. For $M > 0$,

$$\begin{aligned} \left| \int_0^s X_u^m du \right| &\leq |Y_{\tau(s)}^m| + \int_{T_{\tau(s)}^m}^s |X_u^m| du \\ &\leq |Y_{\tau(s)}^m| + \int_{T_{\tau(s)}^m}^s [(|X_u^m| - M)^+ + M] du \\ &\leq \max_{1 \leq k \leq \rho} |Y_k^m| + \int_0^t (|X_u^m| - M)^+ du + M |\Pi^m|. \end{aligned}$$

Maximizing over $s \in [0, t]$ and taking expectations, we obtain

$$\begin{aligned} \mathbb{E}^m \sup_{s \in [0,t]} \left| \int_0^s X_u^m du \right| &\leq \mathbb{E}^m \max_{1 \leq k \leq \rho} |Y_k^m| \\ (6.7) \qquad &+ \mathbb{E}^m \int_0^t (|X_u^m| - M)^+ du \\ &+ M \mathbb{E}^m |\Pi^m|. \end{aligned}$$

We bound the first term on the right-hand side of (6.7). The discrete-time Burkholder–Davis–Gundy inequality (e.g., [16], inequality II.1.1) implies the existence of a universal constant C for which

$$(6.8) \qquad \mathbb{E}^m \max_{1 \leq k \leq \rho} |Y_k^m| \leq C \mathbb{E}^m \left[\left(\sum_{1 \leq k \leq \rho} (Y_k^m - Y_{k-1}^m)^2 \right)^{1/2} \right].$$

The right-hand side of (6.8) can be bounded using Hölder’s inequality. In particular,

$$\begin{aligned}
 & \mathbb{E}^m \left[\left(\sum_{1 \leq k \leq \rho} (Y_k^m - Y_{k-1}^m)^2 \right)^{1/2} \right] \\
 & \leq \mathbb{E}^m \left[\max_{1 \leq k \leq \rho} |Y_k^m - Y_{k-1}^m|^{1/2} \cdot \left(\sum_{1 \leq k \leq \rho} |Y_k^m - Y_{k-1}^m| \right)^{1/2} \right] \\
 (6.9) \quad & \leq \mathbb{E}^m \left[\max_{1 \leq k \leq \rho} |Y_k^m - Y_{k-1}^m|^{1/2} \cdot \left(\int_0^{t+1} |X_u^m| du \right)^{1/2} \right] \\
 & \leq \sqrt{\mathbb{E}^m \max_{1 \leq k \leq \rho} |Y_k^m - Y_{k-1}^m|} \cdot \sqrt{\mathbb{E}^m \int_0^{t+1} |X_u^m| du} \\
 & \leq \sqrt{\mathbb{E}^m \int_0^{t+1} (|X_u^m| - M)^+ du} + M \mathbb{E}^m |\Pi^m| \cdot \sqrt{\mathbb{E}^m \int_0^{t+1} |X_u^m| du}.
 \end{aligned}$$

Combining (6.7)–(6.9), we obtain

$$\begin{aligned}
 & \mathbb{E}^m \sup_{s \in [0, t]} \left| \int_0^s X_u^m du \right| \\
 & \leq C \sqrt{\mathbb{E}^m \int_0^{t+1} (|X_u^m| - M)^+ du} + M \mathbb{E}^m |\Pi^m| \cdot \sqrt{\mathbb{E}^m \int_0^{t+1} |X_u^m| du} \\
 & \quad + \mathbb{E}^m \int_0^t (|X_u^m| - M)^+ du + M \mathbb{E}^m |\Pi|,
 \end{aligned}$$

where C does not depend on X and $M \geq 0$ is arbitrary. The uniform integrability of $(X^m, \lambda_{[0, t+1]} \times \mathbb{P}^m)_{m=1}^\infty$ implies that $\sup_m \mathbb{E}^m \int_0^{t+1} |X_u^m| du$ is a finite constant C' . Given $\varepsilon > 0$, uniform integrability further permits us to choose M so large that $\sup_m \mathbb{E}^m \int_0^{t+1} (|X_u^m| - M)^+ du \leq \varepsilon$. For such an M ,

$$\mathbb{E}^m \sup_{s \in [0, t]} \left| \int_0^s X_u^m du \right| \leq C \sqrt{C'} \sqrt{\varepsilon + M \mathbb{E}^m |\Pi^m|} + \varepsilon + M \mathbb{E}^m |\Pi^m|.$$

Letting $m \rightarrow \infty$ and using the first part of (6.6), we conclude that

$$\limsup_{m \rightarrow \infty} \mathbb{E}^m \sup_{s \in [0, t]} \left| \int_0^s X_u^m du \right| \leq C \sqrt{C'} \varepsilon + \varepsilon. \quad \square$$

7. Proof of Theorem 3.6. We prove a theorem that is little more than a re-statement of Theorem 3.6 without reference to the driving Brownian motions W and \widehat{W} in that theorem. We develop this connection immediately after the statement of Theorem 7.1 below. Recall Definition 4.14.

THEOREM 7.1. *Let \mathcal{E} be a Polish space. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space that supports an \mathcal{E} -valued random variable Z_0 and an adapted continuous \mathbb{R}^d -valued semimartingale Y with $Y_0 = 0$ and with characteristic pair (B, C) , where*

$$(7.1) \quad B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds,$$

and the adapted \mathbb{R}^d -valued process b and the adapted $\mathbb{R}^d \times \mathbb{R}^d$ -valued positive semidefinite process c satisfy

$$(7.2) \quad \mathbb{E} \left[\int_0^t (\|b_s\| + \|c_s\|) ds \right] < \infty, \quad t \geq 0.$$

Let \widehat{b} and \widehat{c} be measurable functions defined on $[0, \infty) \times \mathcal{E}$ with \widehat{b} taking values in \mathbb{R}^d and \widehat{c} taking values in the space of $d \times d$ positive semidefinite matrices, and let $N \subset [0, \infty)$ be a Lebesgue-null set such that

$$(7.3) \quad \widehat{b}(t, Z_t) = \mathbb{E}[b_t | Z_t], \quad \widehat{c}(t, Z_t) = \mathbb{E}[c_t | Z_t], \quad \mathbb{P}\text{-a.s.}, t \in N^c.$$

Define $\Omega^{\mathcal{E},d} \triangleq \mathcal{E} \times C_0^d$, let $\Phi : \Omega^{\mathcal{E},d} \rightarrow C^{\mathcal{E}}$ be a continuous updating function and let Z be the continuous, \mathcal{E} -valued process given by $Z = \Phi(Z_0, Y)$. Let $\widehat{Y} : \Omega^{\mathcal{E},d} \rightarrow C_0^d$ be given by $\widehat{Y}(e, x) = x$ and $\widehat{Z} : \Omega^{\mathcal{E},d} \rightarrow C^{\mathcal{E}}$ be given by $\widehat{Z} = \Phi(e, x)$. Then there exists a measure $\widehat{\mathbb{P}}$ on $\Omega^{\mathcal{E},d}$ such that:

- (i) \widehat{Y} is a semimartingale with characteristic pair $(\widehat{B}, \widehat{C})$ under $\widehat{\mathbb{P}}$, where $\widehat{B}_t \triangleq \int_0^t \widehat{b}(s, \widehat{Z}_s) ds$ and $\widehat{C}_t \triangleq \int_0^t \widehat{c}(s, \widehat{Z}_s) ds$, and
- (ii) for each $t \geq 0$, the distribution of \widehat{Z}_t under $\widehat{\mathbb{P}}$ agrees with the distribution of Z_t under \mathbb{P} .

PROOF OF THEOREM 3.6. Let us assume Theorem 7.1. Then, under the hypotheses of Theorem 3.6, we may define $c_s \triangleq \sigma_s \sigma_s^{tr}$ and invoke Proposition 5.1 to ensure the existence of functions \widehat{b} and \widehat{c} and a Lebesgue-null set N such that (7.3) holds. We then conclude that there exist \widehat{Y} and \widehat{Z} satisfying properties (i) and (ii) in Theorem 7.1. To show that \widehat{Y} has the representation (3.8), we set $\widehat{\sigma}$ equal to the symmetric square root of \widehat{c} and invoke the Itô integral representation (e.g., [22], Chapter 3, Theorem 4.2) for the d -dimensional local martingale $\widehat{Y} - \widehat{B}$. \square

PROOF OF THEOREM 7.1. The proof, which involves a discretization, as suggested by the example in Section 2, and then passage to the limit, proceeds in several steps.

Step 1: Construction of canonical space and processes. The random object of interest, (Z_0, Y, B, C) , takes values in $\Omega^{\mathcal{E},d} \times C_0^d \times C_0^{d^2}$. In order to show that the discretization has a limit, we need to randomize the discretization times, and thus introduce an extra dimension, defining $\Omega^* \triangleq [0, 1] \times \Omega^{\mathcal{E},d} \times C_0^d \times C_0^{d^2}$. Note that

Ω^* can also be written as $\Omega^{\mathcal{E}^*, d^*}$, where $\mathcal{E}^* = [0, 1] \times \mathcal{E}$ and $d^* = d + d + d^2$. We denote $\mathcal{F}^{\mathcal{E}^*, d^*}$ simply as \mathcal{F}^* and denote $\mathcal{F}_t^{\mathcal{E}^*, d^*}$ simply as \mathcal{F}_t^* . On \mathcal{F}^* we define the measure \mathbb{Q} to be the product of uniform measure on $[0, 1]$ and the measure induced by (Z_0, Y, B, C) under \mathbb{P} on $\Omega^{\mathcal{E}, d} \times C_0^d \times C_0^{d^2}$. The generic element of Ω^* will be denoted $\omega = (\mu, \varepsilon, \eta, \beta, \gamma)$, and we define the projections

$$U^*(\omega) = \mu, \quad Z_0^*(\omega) = \varepsilon, \quad Y^*(\omega) = \eta, \quad B^*(\omega) = \beta, \quad C^*(\omega) = \gamma.$$

On the filtered probability space $(\Omega^*, \mathcal{F}^*, \{\mathcal{F}_t^*\}_{t \geq 0}, \mathbb{Q})$, Y^* is a semimartingale with characteristic pair (B^*, C^*) .

We choose an \mathbb{R}^d -valued predictable process b^* whose i th component at each time $t > 0$, denoted $(b_i^*)_t$, agrees with

$$\liminf_{k \rightarrow \infty} k((B_i^*)_t - (B_i^*)_{(t-1/k)^+}),$$

whenever the latter is finite. Likewise, we choose an \mathbb{R}^{d^2} -valued predictable process c^* whose (i, j) th component at each time $t > 0$, denoted $(c_{i,j}^*)_t$, agrees with

$$\liminf_{k \rightarrow \infty} k((C_{i,j}^*)_t - (C_{i,j}^*)_{(t-1/k)^+}),$$

whenever the latter is finite. By assumption, the components of B^* and C^* are \mathbb{Q} -almost surely absolutely continuous, and so their left derivatives are defined for Lebesgue-almost every $t \geq 0$, \mathbb{Q} -almost surely. By construction, b^* and c^* are these left derivatives whenever they are defined. It follows that

$$(7.4) \quad \mathbb{Q} \left[\int_0^t (\|b_s^*\| + \|c_s^*\|) ds < \infty, B_t^* = \int_0^t b_s^* ds, C_t^* = \int_0^t c_s^* ds \forall t \right] = 1.$$

For $i, j = 1, \dots, d$, the sets

$$(7.5) \quad \left\{ t \in [0, \infty) : \frac{\partial}{\partial t} (B_i^*)_t(\omega) \text{ exists but is not equal to } (b_i^*)_t(\omega) \right\},$$

$$\left\{ t \in [0, \infty) : \frac{\partial}{\partial t} (C_{i,j}^*)_t(\omega) \text{ exists but is not equal to } (c_{i,j}^*)_t(\omega) \right\}$$

are empty for every $\omega \in \Omega^*$.

We set $Z^* \triangleq \Phi(Z_0^*, Y^*)$ and observe that the random time U^* is strongly independent of $(Y^*, Z^*, B^*, C^*, b^*, c^*)$ (recall Definition 5.3). Furthermore, the distribution of (Y^*, Z^*, B^*, C^*) under \mathbb{Q} is the same as the distribution of (Y, Z, B, C) under \mathbb{P} , so (7.1) and (7.4) imply that

$$(7.6) \quad \mathbb{E} \int_0^t f(Y_s, Z_s, b_s, c_s) ds = \mathbb{E}^{\mathbb{Q}} \int_0^t f(Y_s^*, Z_s^*, b_s^*, c_s^*) ds$$

for any $t \geq 0$ and f such that one side of (7.6) is well defined. In particular, (7.2) and (7.6) imply that

$$(7.7) \quad \mathbb{E}^{\mathbb{Q}} \left[\int_0^t (\|b_s^*\| + \|c_s^*\|) ds \right] < \infty, \quad t \geq 0,$$

and (7.3), (7.6) and Lemma 5.2 ensure the existence of a Lebesgue-null set $N^* \subset [0, \infty)$ such that $\widehat{b}(t, Z_t^*) = \mathbb{E}^{\mathbb{Q}}[b_t^* | Z_t^*]$ and $\widehat{c}(t, Z_t^*) = \mathbb{E}^{\mathbb{Q}}[c_t^* | Z_t^*]$ for all $t \notin N^*$. From (7.7) and the conditional version of Jensen's inequality, we also have

$$\mathbb{E}^{\mathbb{Q}} \left[\int_0^t (\|\widehat{b}(s, Z_s^*)\| + \|\widehat{c}(s, Z_s^*)\|) ds \right] < \infty, \quad t \geq 0,$$

or equivalently,

$$\mathbb{E} \left[\int_0^t (\|\widehat{b}(s, Z_s)\| + \|\widehat{c}(s, Z_s)\|) ds \right] < \infty, \quad t \geq 0.$$

Step 2: Construction of extended partitions. For each positive integer m , set $N(m) = m^2$, $T_0^m \triangleq 0$ and for $i = 1, \dots, N(m)$, set $T_i^m \triangleq (U^* + i - 1)/m$. Note that each T_i^m is $\sigma(U^*)$ measurable, and consequently is an $\{\mathcal{F}_t^*\}_{t \geq 0}$ -stopping time. Let Π^m denote this set of stopping times. The sequence of random partitions $\{\Pi^m\}_{m=1}^\infty$ converges uniformly to the identity (Definition 6.4).

For the next step, we adopt the notation $X = (Y^*, B^*, C^*)$. We set $\mathcal{G}_0^m = \mathcal{H}_0^m = \mathcal{F}_0^* = \sigma(U, Z_0^*)$, and for $i = 1, \dots, N(m)$, we set $\mathcal{G}_i^m = \sigma(U^*, Z_{T_i^m}^*)$ and $\mathcal{H}_i^m = \mathcal{G}_{i-1}^m \vee \sigma(\Delta(X^{T_i^m}, T_{i-1}^m))$. Finally, we set $T_{N(m)+1}^m = \infty$ and $\mathcal{H}_{N(m)+1}^m = \mathcal{G}_{N(m)}^m \vee \sigma(\Delta(X, T_{N(m)}^m))$. It is clear that part (a) of Definition 4.1 is satisfied. To show that $(T_i^m, \mathcal{G}_i^m)_{i=1}^{N(m)}$ is an extended partition, it suffices to verify condition (b) of Definition 4.1, that is, that $\mathcal{G}_i^m \subset \mathcal{H}_i^m$ for $i = 1, \dots, N(m)$. In particular, it suffices to show that $Z_{T_i^m}^*$ is measurable with respect to $\sigma(U^*) \vee \sigma(Z_{T_{i-1}^m}^*) \vee \sigma(\Delta(X^{T_i^m}, T_{i-1}^m))$. Let $\tau \geq 0$ be a possibly random time and define $S_i^m = (\tau - T_{i-1}^m)^+$. On the set $T_{i-1}^m \leq \tau \leq T_i^m$, we may use property (3.4) of the updating function Φ to write

$$\begin{aligned} Z_\tau^* &= \Theta_{S_i^m}(Z^*, T_{i-1}^m) \\ &= \Theta_{S_i^m}(\Phi(Z_0^*, Y^*), T_{i-1}^m) \\ (7.8) \quad &= \Phi_{S_i^m}(\Phi_{T_{i-1}^m}(Z_0^*, Y^*), \Delta(Y^*, T_{i-1}^m)) \\ &= \Phi_{S_i^m}(Z_{T_{i-1}^m}^*, \Delta(Y^*, T_{i-1}^m)). \end{aligned}$$

If we take $\tau = T_i^m$, this leads to

$$Z_{T_i^m}^* = \Phi_{T_i^m - T_{i-1}^m}(Z_{T_{i-1}^m}^*, \Delta(Y^*, T_{i-1}^m)) = \Phi_{T_i^m - T_{i-1}^m}^{T_i^m - T_{i-1}^m}(Z_{T_{i-1}^m}^*, \Delta(Y^*, T_{i-1}^m)),$$

and by property (3.3), the last expression depends on the path of $\Delta(Y^*, T_{i-1}^m)$ only up to time $T_i^m - T_{i-1}^m$, which agrees with the path of $\Delta((Y^*)^{T_i^m}, T_{i-1}^m)$ up to time $T_i^m - T_{i-1}^m$. We have thus written $Z_{T_i^m}^*$ in terms of $T_i^m - T_{i-1}^m$, which is nonrandom unless $i = 1$, in which case it is U^*/m , in terms of $Z_{T_{i-1}^m}^*$, and in terms of $\Delta((Y^*)^{T_i^m}, T_{i-1}^m)$.

Step 3: Concatenated measures. We denote by Π^m the extended partition $(T_i^m, \mathcal{G}_i^m)_{i=1}^{N(m)}$. These extended partitions are on the space $\Omega^* = [0, 1] \times \Omega^{\mathcal{E},d} \times C_0^d \times C_0^{d^2}$, which is the same as $\Omega^{\mathcal{E}^*,d^*}$ defined in step 1. Theorem 4.3 implies the existence of concatenated measures $\mathbb{Q}^m \triangleq \mathbb{Q}^{\otimes \Pi^m}$ that satisfy

$$(7.9) \quad \mathbb{Q}^m[A] = \mathbb{Q}[A], \quad A \in \mathcal{H}_i^m, \quad i = 0, 1, \dots, N(m) + 1,$$

$$(7.10) \quad \mathbb{Q}^m[B|\mathcal{F}_{T_i^m}^*] = \mathbb{Q}[B|\mathcal{G}_i^m], \quad B \in \mathcal{H}_{i+1}^m, \quad i = 0, 1, \dots, N(m).$$

Applying Proposition 4.15 with $X = (Y^*, B^*, C^*)$, we see that Y^* is a semimartingale with characteristic pair (B^*, C^*) under each \mathbb{Q}^m .

Step 4: Tightness and convergence. Corollary 4.13(b) shows that the collection of measures induced on $C_0^d \times C_0^{d^2}$ by (B^*, C^*) under $\{\mathbb{Q}^m\}_{m=1}^\infty$ is tight. Theorem VI.4.18 of [21] (Rebolledo’s criterion; see [33]) then implies that the collection of measures induced on C_0^d by Y^* under $\{\mathbb{Q}^m\}_{m=1}^\infty$ is tight. Since Z_0^* has the same distribution under every \mathbb{Q}^m , the set of measures induced on $\Omega^{\mathcal{E},d}$ by (Z_0^*, Y^*) is likewise tight. Passing to a convergent subsequence if necessary, we obtain a limiting measure $\widehat{\mathbb{P}}$ on $\Omega^{\mathcal{E},d}$. To simplify notation, we assume that the passage to a subsequence is not necessary to obtain convergence. We denote the coordinate mappings on $\Omega^{\mathcal{E},d}$ by \widehat{Z}_0 and \widehat{Y} , and we define $\widehat{Z} = \Phi(\widehat{Z}_0, \widehat{Y})$. The continuous mapping theorem implies that the distributions of (Y^*, Z^*) on $C_0^d \times C^{\mathcal{E}}$ under the sequence of measures $\{\mathbb{Q}^m\}_{m=1}^\infty$ converge to the distribution of $(\widehat{Y}, \widehat{Z})$ under $\widehat{\mathbb{P}}$, that is, $\mathbb{Q}^m \circ (Y^*, Z^*)^{-1} \implies \widehat{\mathbb{P}} \circ (\widehat{Y}, \widehat{Z})^{-1}$.

Step 5: Agreement of one-dimensional distributions. Returning to (7.8), we take $\tau = t$, a fixed nonnegative number, so that $S_i^m = (t - T_{i-1}^m)^+$. On the \mathcal{H}_i^m -measurable set $\{T_{i-1}^m \leq t < T_i^m\}$, we have

$$Z_t^* = \Phi_{S_i^m}(Z_{T_{i-1}^m}^*, \Delta(Y^*, T_{i-1}^m)),$$

and the term $\Phi_{S_i^m}(Z_{T_{i-1}^m}^*, \Delta(Y^*, T_{i-1}^m))$ restricted to $\{T_{i-1}^m \leq t < T_i^m\}$ depends only on S_i^m , $Z_{T_{i-1}^m}^*$ and $\Delta((Y^*)^{T_i^m}, T_{i-1}^m)$, all of which are \mathcal{H}_i^m -measurable. Because \mathbb{Q}^m and \mathbb{Q} agree on each \mathcal{H}_i^m , we conclude that for every Borel subset A of \mathcal{E} and for every $t \geq 0$,

$$\begin{aligned} \mathbb{Q}^m[Z_t^* \in A] &= \sum_{i=1}^{N(m)+1} \mathbb{Q}^m[Z_t^* \in A \text{ and } T_{i-1}^m \leq t < T_i^m] \\ (7.11) \quad &= \sum_{i=1}^{N(m)+1} \mathbb{Q}[Z_t^* \in A \text{ and } T_{i-1}^m \leq t < T_i^m] \\ &= \mathbb{Q}[Z_t^* \in A] = \mathbb{P}[Z_t \in A]. \end{aligned}$$

But the distributions of Z^* under the sequence of measures $\{\mathbb{Q}^m\}_{m=1}^\infty$ converge to the distribution of \widehat{Z} under $\widehat{\mathbb{P}}$, and part (ii) of Theorem 7.1 is proved.

Step 6: Semimartingale characteristics of the limit. To complete the proof, we must show that under the measure $\widehat{\mathbb{P}}$ on $\Omega^{\mathcal{E},d}$, \widehat{Y} is a semimartingale with characteristic pair $(\widehat{B}, \widehat{C})$, defined in part (i) of Theorem 7.1. We do this by showing that the distribution of the (Y^*, Z^*, B^*, C^*) under \mathbb{Q}^m converges to the distribution of $(\widehat{Y}, \widehat{Z}, \widehat{B}, \widehat{C})$ under $\widehat{\mathbb{P}}$, that is,

$$(7.12) \quad \mathbb{Q}^m \circ (Y^*, Z^*, B^*, C^*)^{-1} \implies \widehat{\mathbb{P}} \circ (\widehat{Y}, \widehat{Z}, \widehat{B}, \widehat{C})^{-1}.$$

The filtration on $\Omega^{\mathcal{E},d}$, defined at the beginning of Section 4, is generated by \widehat{Y} . Once (7.12) is established, Theorem IX.2.4 of [21] will give the desired result.

On Ω^* we define the processes

$$\begin{aligned} \bar{b}_t &\triangleq \widehat{b}(t, Z_t^*), & \bar{B}_t &\triangleq \int_0^t \bar{b}_s ds, \\ \bar{c}_t &\triangleq \widehat{c}(t, Z_t^*), & \bar{C}_t &\triangleq \int_0^t \bar{c}_s ds, \quad t \geq 0. \end{aligned}$$

According to Proposition 6.1,

$$(7.13) \quad \mathbb{Q}^m \circ (Y^*, Z^*, \bar{B}, \bar{C})^{-1} \implies \widehat{\mathbb{P}} \circ (\widehat{Y}, \widehat{Z}, \widehat{B}, \widehat{C})^{-1},$$

$$(7.14) \quad \begin{aligned} \{\bar{b}, \lambda_{[0,t]} \times \mathbb{Q}^m\}_{m \in \mathbb{N}} &\text{ is uniformly integrable for every } t \in [0, \infty), \\ \{\bar{c}, \lambda_{[0,t]} \times \mathbb{Q}^m\}_{m \in \mathbb{N}} &\text{ is uniformly integrable for every } t \in [0, \infty). \end{aligned}$$

We show that $\mathbb{Q}^m \circ (Y^*, Z^*, B^*, C^*)^{-1}$ and $\mathbb{Q}^m \circ (Y^*, Z^*, \bar{B}, \bar{C})^{-1}$ have the same limit as $m \rightarrow \infty$. We do this by showing that for every $\varepsilon > 0$ and $t \in [0, \infty)$,

$$(7.15) \quad \lim_{m \rightarrow \infty} \mathbb{Q}^m \left[\sup_{0 \leq s \leq t} \|B_s^* - \bar{B}_s\| \geq \varepsilon \right] = 0,$$

$$(7.16) \quad \lim_{m \rightarrow \infty} \mathbb{Q}^m \left[\sup_{0 \leq s \leq t} \|C_s^* - \bar{C}_s\| \geq \varepsilon \right] = 0.$$

Once this has been done, (7.13) will imply (7.12), and we will be done.

Step 7: Proof of (7.15) and (7.16). In fact, we prove only (7.15), because the proof of (7.16) is the same. Without loss of generality, we assume that B^* and \widehat{B} are one dimensional.

For $i = 1, \dots, N(m)$, define the \mathcal{H}_{i+1}^m -measurable random variable

$$\xi_i^m \triangleq \liminf_{k \rightarrow \infty} k(B_{T_i^m + 1/k}^* - B_{T_i^m}^*),$$

which is the right derivative of B^* at T_i^m whenever this derivative is defined. Recall from step 1 that $b_{T_i^m}^*$ is the left derivative of B^* at T_i^m whenever this derivative is

defined and is finite. By construction, B^* is independent of T_i^m under \mathbb{Q} , and its derivative is defined and is finite Lebesgue-almost everywhere, \mathbb{Q} -almost surely. But T_i^m is uniformly distributed on $[\frac{i-1}{n}, \frac{i}{n}]$. It follows that

$$(7.17) \quad \mathbb{Q}[\xi_i^m = b_{T_i^m}^*] = 1, \quad i = 1, \dots, N(m).$$

We define three sequences of step functions:

$$b_t^m \triangleq \sum_{i=1}^{N(m)} \xi_i^m \mathbb{I}_{[T_i^m, T_{i+1}^m)}(t), \quad \bar{b}_t^m \triangleq \sum_{i=1}^{N(m)} \bar{b}_{T_i^m} \mathbb{I}_{[T_i^m, T_{i+1}^m)},$$

$$b_t^{\Pi^m} \triangleq \sum_{i=1}^{N(m)} b_{T_i^m}^* \mathbb{I}_{[T_i^m, T_{i+1}^m)}.$$

We further define

$$B_t^m \triangleq \int_0^t b_s^m ds, \quad \bar{B}_t^m \triangleq \int_0^t \bar{b}_s^m ds.$$

Because of (7.17), b^m and b^{Π^m} are \mathbb{Q} -indistinguishable.

Each B^m is piecewise linear, and so for every $\omega \in \Omega^*$, $\frac{\partial}{\partial t} B_t^m(\omega) = b_t^m(\omega)$ except at finitely many values of t . In addition, $\Delta(B^m, T_i^m)$ is $\sigma(\xi_j^m : j \geq i)$ -measurable. For $j \geq i$, ξ_j^m is \mathcal{H}_{j+1}^m -measurable, and we have shown in the proof of Theorem 4.3 [see (4.9)] that $\mathcal{H}_{j+1}^m \subset \mathcal{G}_i^m \vee \sigma(\Delta(X, T_i^m))$ for $j = i, i + 1, \dots, N(m)$, so we may conclude that $\Delta(B^m, T_i^m)$ is $\mathcal{G}_i^m \vee \sigma(\Delta(X, T_i^m))$ -measurable for $i = 1, \dots, N(m)$. This measurability condition is trivially satisfied when $i = 0$ as well. We conclude that the pair of processes (B^m, b^m) satisfies the hypotheses of Proposition 4.11, including (4.12), with \mathbb{Q} replacing \mathbb{P} and \mathbb{Q}^m replacing $\mathbb{P}^{\otimes \Pi}$.

Because B^* is a component of X and the set (7.5) is empty, (B^*, b^*) also satisfies the hypothesis of Proposition 4.11, and hence, so does $(B^* - B^m, b^* - b^m)$. We thus obtain from (4.14) that

$$\mathbb{E}^{\mathbb{Q}^m} \int_0^t |b_s^* - b_s^m| ds = \mathbb{E}^{\mathbb{Q}} \int_0^t |b_s^* - b_s^m| ds, \quad t \geq 0.$$

For fixed $t \geq 0$, we use this equality, the \mathbb{Q} -indistinguishability of b^m and b^{Π^m} and Proposition 6.3 to write

$$(7.18) \quad \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^m} \sup_{0 \leq s \leq t} |B_s^* - B_s^m| \leq \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^m} \int_0^t |b_s^* - b_s^m| ds$$

$$= \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \int_0^t |b_s^* - b_s^m| ds$$

$$= \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \int_0^t |b_s^* - b_s^{\Pi^m}| ds$$

$$= 0.$$

We consider the difference between \bar{B} and \bar{B}^m . For $i = 1, \dots, N(m) + 1$,

$$\begin{aligned}
 & \int_{\mathcal{H}_{T_{i-1}^m \wedge t}^m}^{T_i^m \wedge t} |\bar{b}_s - \bar{b}_s^m| ds \\
 (7.19) \quad &= \int_{\mathcal{H}_{T_{i-1}^m \wedge t}^m}^{T_i^m \wedge t} |\widehat{b}(s, Z_s^*) - \widehat{b}(T_{i-1}^m, Z_{T_{i-1}^m}^*)| ds \\
 &= \int_0^{S_i^m} |\widehat{b}(T_{i-1}^m + s, \Theta_s((Z^*)^{T_i^m}, T_{i-1}^m)) - \widehat{b}(\Theta_0((Z^*)^{T_i^m}, T_{i-1}^m))| ds,
 \end{aligned}$$

where $S_i^m = \frac{1}{n} \wedge (t - T_{i-1}^m)^+$ if $i \geq 2$ and $S_1^m = T_1^m \wedge t$. The final expression in (7.19) is \mathcal{H}_t^m -measurable, and so the first expression is as well. But \mathbb{Q}^m and \mathbb{Q} agree on \mathcal{H}_t^m , which together with Proposition 6.3 implies

$$\begin{aligned}
 \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^m} \sup_{0 \leq s \leq t} |\bar{B}_s - \bar{B}_s^m| &\leq \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^m} \int_0^t |\bar{b}_s - \bar{b}_s^m| ds \\
 (7.20) \quad &= \limsup_{m \rightarrow \infty} \sum_{i=1}^{N(m)+1} \mathbb{E}^{\mathbb{Q}^m} \int_{T_{i-1}^m \wedge t}^{T_i^m \wedge t} |\bar{b}_s - \bar{b}_s^m| ds \\
 &= \limsup_{m \rightarrow \infty} \sum_{i=1}^{N(m)+1} \mathbb{E}^{\mathbb{Q}} \int_{T_{i-1}^m \wedge t}^{T_i^m \wedge t} |\bar{b}_s - \bar{b}_s^m| ds \\
 &= \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \int_0^t |\bar{b}_s - \bar{b}_s^m| ds \\
 &= 0.
 \end{aligned}$$

It remains to estimate the difference between B^m and \bar{B}^m . From (7.14) and (7.20) we see that $\{\bar{b}^m, \lambda_{[0,t]} \times \mathbb{Q}^m\}_{m \in \mathbb{N}}$ is uniformly integrable for every $t \in [0, \infty)$. We show that $\{b^m, \lambda_{[0,t]} \times \mathbb{Q}^m\}_{m \in \mathbb{N}}$ is also uniformly integrable by using the \mathcal{H}_{i+1}^m measurability of T_i^m, T_{i+1}^m and ξ_i^m to write

$$\begin{aligned}
 \mathbb{E}^{\mathbb{Q}^m} \int_0^t |b_s^m| \mathbb{I}_{\{|b_s^m| \geq M\}} ds &= \sum_{i=0}^{N(m)} \mathbb{E}^{\mathbb{Q}^m} [(T_{i+1}^m \wedge t - T_i^m \wedge t) |\xi_i^m| \mathbb{I}_{\{|\xi_i^m| \geq M\}}] \\
 (7.21) \quad &= \sum_{i=0}^{N(m)} \mathbb{E}^{\mathbb{Q}} [(T_{i+1}^m \wedge t - T_i^m \wedge t) |\xi_i^m| \mathbb{I}_{\{|\xi_i^m| \geq M\}}] \\
 &= \mathbb{E}^{\mathbb{Q}} \int_0^t |b_s^m| \mathbb{I}_{\{|b_s^m| \geq M\}} ds.
 \end{aligned}$$

Under $\lambda_{[0,t]} \times \mathbb{Q}$, b^* restricted to $[0, t]$ is integrable [see (7.4)]. One consequence of (7.18) is that b^m restricted to $[0, t]$ converges to b^* restricted to $[0, t]$ in $L^1(\lambda_{[0,t]} \times \mathbb{Q})$. This, combined with (7.21), yields the uniform integrability of $\{b^m, \lambda_{[0,t]} \times$

$\mathbb{Q}^m\}_{m \in \mathbb{N}}$. We conclude that $\{b^m - \bar{b}^m, \lambda_{[0,t]} \times \mathbb{Q}^m\}_{m \in \mathbb{N}}$ is uniformly integrable for every $t \geq 0$.

Define

$$\Psi_k^m \triangleq B_{T_k^m} - \bar{B}_{T_k^m} = \int_0^{T_k^m} (b_s^m - \bar{b}_s^m) ds.$$

Let $k = 0, 1, \dots, N(m) - 1$ be given. Because T_{k+1}^m and T_k^m are $\mathcal{F}_{T_k^m}^*$ -measurable, $\xi_k^m - \bar{b}_{T_k^m} = \xi_k^m - \widehat{b}(T_k^m, Z_{T_k^m}^*)$ is \mathcal{H}_{k+1}^m -measurable, and (7.10) and (7.17) hold, we may write

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}^m} [\Psi_{k+1}^m - \Psi_k^m | \mathcal{F}_{T_k^m}^*] \\ &= (T_{k+1}^m - T_k^m) \mathbb{E}^{\mathbb{Q}^m} [\xi_k^m - \widehat{b}(T_k^m, Z_{T_k^m}^*) | \mathcal{F}_{T_k^m}^*] \\ &= (T_{k+1}^m - T_k^m) \mathbb{E}^{\mathbb{Q}} [\xi_k^m - \widehat{b}(T_k^m, Z_{T_k^m}^*) | \mathcal{G}_k^m] \\ &= (T_{k+1}^m - T_k^m) (\mathbb{E}^{\mathbb{Q}} [b_{T_k^m}^* | \mathcal{G}_k^m] - \widehat{b}(T_k^m, Z_{T_k^m}^*)). \end{aligned}$$

Proposition 5.4 implies that

$$\mathbb{E}^{\mathbb{Q}} [b_{T_k^m}^* | \mathcal{G}_k^m] = \mathbb{E}^{\mathbb{Q}} [b_{T_k^m}^* | T_k^m, Z_{T_k^m}^*] = \widehat{b}(T_k^m, Z_{T_k^m}^*).$$

We conclude that $(\Psi_k^m, \mathcal{F}_{T_k^m}^* | 0 \leq k \leq N(m))$ is a discrete-time martingale under \mathbb{Q}^m , which implies that $(\Psi_k^m, \mathcal{F}_k^m | 0 \leq k \leq N(m))$ is also a martingale, where $\mathcal{F}_k^m \triangleq \sigma(\Psi_j^m, T_j^m | 0 \leq j \leq k) \subset \mathcal{F}_{T_k^m}^*$. Proposition 6.5 now implies that

$$(7.22) \quad \lim_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^m} \sup_{0 \leq s \leq t} |B_s^m - \bar{B}_s^m| = 0, \quad t \geq 0.$$

Using the triangle inequality, we combine (7.18), (7.22) and (7.20) to conclude

$$(7.23) \quad \limsup_{m \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^m} \sup_{0 \leq s \leq t} |B_s^* - \bar{B}_s^m| = 0.$$

Equation (7.15) follows. \square

Acknowledgments. We thank Peter Carr for pointing out Gyöngy [18] and an anonymous referee for a number of helpful comments.

REFERENCES

[1] ALBIN, J. M. P. (2008). A continuous non-Brownian motion martingale with Brownian motion marginal distributions. *Statist. Probab. Lett.* **78** 682–686. [MR2409532](#)

[2] ANTONOV, A. and MISIRPASHAEV, T. (2009). Markovian projection onto a displaced diffusion: Generic formulas with applications. *Int. J. Theor. Appl. Finance* **12** 507–522. [MR2541829](#)

[3] BENTATA, A. and CONT, R. (2011). Mimicking the marginal distributions of a semimartingale. Available at [arXiv:0910.3992v3](#).

- [4] BRIGO, D. and MERCURIO, F. (2002). Displaced and mixture diffusions for analytically-tractable smile models. In *Mathematical Finance—Bachelier Congress, 2000 (Paris)* (H. Geman, D. Madan, S. R. Pliska and T. Vorst, eds.) 151–174. Springer, Berlin. [MR1960563](#)
- [5] BRIGO, D. and MERCURIO, F. (2002). Lognormal-mixture dynamics and calibration to market volatility smiles. *Int. J. Theor. Appl. Finance* **5** 427–446. [MR1914680](#)
- [6] BRUNICK, G. (2008). A weak existence result with application to the financial engineer’s calibration problem. Ph.D. dissertation, Dept. Math. Sciences, Carnegie Mellon Univ., Pittsburgh, PA.
- [7] BRUNICK, G. (2013). Uniqueness in law for a class of degenerate diffusions with continuous covariance. *Probab. Theory Related Fields* **155** 265–302. [MR3010399](#)
- [8] COX, A. M. G., HOBSON, D. and OBLÓJ, J. (2011). Time-homogeneous diffusions with a given marginal at a random time. *ESAIM Probab. Stat.* **15** S11–S24. [MR2817342](#)
- [9] DELLACHERIE, C. and MEYER, P. A. (1978). *Probability and Potential, Part I*. North-Holland, New York. Translation of *Probabilités et Potential*, Hermann, Paris.
- [10] DERMAN, E. and KANI, I. (1994). Riding on a smile. *Risk* **7** 32–39.
- [11] DERMAN, E. and KANI, I. (1998). Stochastic implied trees: Arbitrage pricing with stochastic term and strike structure of volatility. *Int. J. Theor. Appl. Finance* **1** 61–110.
- [12] DUPIRE, B. (1994). Pricing with a smile. *Risk* **7** 18–20.
- [13] DUPIRE, B. (1996). A unified theory of volatility. In *Derivative Pricing: The Classic Collection*, 2004 (P. Carr, ed.) 185–198. Risk Books, New York.
- [14] EKSTRÖM, E., HOBSON, D., JANSON, S. and TYSK, J. (2013). Can time-homogeneous diffusions produce any distribution? *Probab. Theory Related Fields* **155** 493–520. [MR3034785](#)
- [15] FORDE, M. (2011). A diffusion-type process with a given joint law for the terminal level and supremum at an independent exponential time. *Stochastic Process. Appl.* **121** 2802–2817. [MR2844541](#)
- [16] GARSIA, A. M. (1973). *Martingale Inequalities: Seminar Notes on Recent Progress*. Benjamin, Amsterdam. [MR0448538](#)
- [17] GATHERAL, J. (2006). *The Volatility Surface: A Practitioner’s Guide*. Wiley, Hoboken, NJ.
- [18] GYÖNGY, I. (1986). Mimicking the one-dimensional marginal distributions of processes having an Itô differential. *Probab. Theory Related Fields* **71** 501–516. [MR0833267](#)
- [19] HAMZA, K. and KLEBANER, F. C. (2007). A family of non-Gaussian martingales with Gaussian marginals. *J. Appl. Math. Stoch. Anal.* Art. ID 92723, 19. [MR2335977](#)
- [20] HESTON, S. (1993). A closed-form solution for options with stochastic volatility and applications to bond and currency options. *Rev. Futures Markets* **9** 54–76.
- [21] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer, Berlin. [MR0959133](#)
- [22] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. [MR1121940](#)
- [23] KELLERER, H. G. (1972). Markov-Komposition und eine Anwendung auf Martingale. *Math. Ann.* **198** 99–122. [MR0356250](#)
- [24] KLEBANER, F. (2002). Option price when the stock is a semimartingale. *Electron. Commun. Probab.* **7** 79–83 (electronic). [MR1887176](#)
- [25] KRYLOV, N. V. (1985). Once more about the connection between elliptic operators and Itô’s stochastic equations. In *Statistics and Control of Stochastic Processes (Moscow, 1984)* 214–229. Optimization Software, New York. [MR0808203](#)
- [26] KRYLOV, N. V. (2007). Parabolic and elliptic equations with VMO coefficients. *Comm. Partial Differential Equations* **32** 453–475. [MR2304157](#)
- [27] MADAN, D. B. and YOR, M. (2002). Making Markov martingales meet marginals: With explicit constructions. *Bernoulli* **8** 509–536. [MR1914701](#)

- [28] OLESZKIEWICZ, K. (2008). On fake Brownian motions. *Statist. Probab. Lett.* **78** 1251–1254. [MR2444313](#)
- [29] PARTHASARATHY, K. R. (2005). *Probability Measures on Metric Spaces*. Amer. Math. Soc., Providence, RI. [MR2169627](#)
- [30] PITERBARG, V. V. (2005). Stochastic volatility model with time-dependent skew. *Appl. Math. Finance* **12** 147–185.
- [31] PITERBARG, V. V. (2006). Smiling hybrids. *Risk* **19** 66–71.
- [32] PITERBARG, V. V. (2007). Markovian projection for volatility calibration. *Risk* **20** 84–89.
- [33] REBOLLEDO, R. (1979). La méthode des martingales appliquée à l'étude de la convergence en loi de processus. *Bull. Soc. Math. France Mém.* **62** v+125 pp. [MR0568153](#)
- [34] STROOCK, D. W. and VARADHAN, S. R. S. (1969). Diffusion processes with continuous coefficients. I. *Comm. Pure Appl. Math.* **22** 345–400. [MR0253426](#)
- [35] STROOCK, D. W. and VARADHAN, S. R. S. (1969). Diffusion processes with continuous coefficients. II. *Comm. Pure Appl. Math.* **22** 479–530. [MR0254923](#)
- [36] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **233**. Springer, Berlin. [MR0532498](#)

DEPARTMENT OF STATISTICS
AND APPLIED PROBABILITY
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
SANTA BARBARA, CALIFORNIA 93106
USA
E-MAIL: brunick@pstat.ucsb.edu

DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213
USA
E-MAIL: shreve@andrew.cmu.edu