Perpetual Convertible Bonds

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What is a convertible bond?

It is a bond issued by a firm with the following provisions:

- Pays coupons.

  In our model, coupons are paid continuously at rate $c > 0$.

- Can be called by the firm.

  In our model, the firm may call at any time the firm value exceeds K by offering to pay K.

  $K > 0$ is the *call price*.

- Can be converted to stock by the bondholder.

  In our model, the bondholder may convert the bond to stock worth a fraction γ of the value of the firm.

  γ is the *conversion factor*.

- Rules of the call.

  If the firm calls, the bondholder may *surrender* the bond in exchange for payment K or may *convert* it to stock worth a fraction γ of the value of the firm.

Two-person zero-sum game. The firm seeks a call strategy which *minimizes* the value of the bond. The bondholder seeks a conversion strategy which *maximizes* its value.

Notation.

- $X(t)$ – Value of the firm at time t.

$$dX(t) = h\big(X(t)\big)\,dt - c\,dt + \sigma X(t)\,dW(t).$$

- $S(t)$ – Total value of the firm's stock. This is to be determined by solving the two-person game.
- $D(t)$ – Value of the convertible bond (firm's debt). This is to be determined by solving the two-person game.
- Before call and conversion:

$$X(t) = D(t) + S(t).$$

- At call without conversion:

$$D(t) = K, \quad S(t) = X(t-) - K.$$

- At conversion:

$$D(t) = \gamma X(t), \quad S(t) = (1-\gamma)X(t).$$

Additional ingredients

- Firm pays dividends.

  In our model, dividends are paid continuously at rate $\delta S(t)$, where $\delta > 0$.
- There is a money market.

  In our model, the money market has constant rate of interest $r > \delta$.

Assumptions

- Our bond is *perpetual*. It never matures.
- Because of payments of dividends and coupons, the firm may see its value drop to zero. *Bankruptcy* occurs at time

$$\tau(0) \triangleq \min\{t \geq 0; X(t) = 0\}.$$

  We have

$$S(t) = D(t) = 0 \text{ for all } t \geq \tau(0).$$

Strategies.

- Call strategy.

  Firm chooses a level $Ca > K$ and calls at the first time $\tau(Ca)$ that $X(t)$ rises to Ca.

- Conversion strategy.

  Bondholder chooses a level $Co > 0$ and converts at the first time $\tau(Co)$ that $X(t)$ rises to Co.

- Bondholder's option.

  At the time of a call, the bondholder has the option of surrendering the bond for K or converting to stock valued at γCa. If the conversion value γCa at the time of a call exceeds the call price K, the bondholder will convert. Thus, we always assume but do not always write explicitly

$$Co \leq Ca \text{ if } \gamma Ca \geq K.$$

Pricing by no-arbitrage condition.

Suppose Ca and Co have been chosen, perhaps not optimally. A third player can build a portfolio which

- Holds $\Delta_1(t)$ shares of stock,
- Holds $\Delta_2(t)$ convertible bonds,
- Finances this by investing in or borrowing from the money market.

Prior to call, conversion and bankruptcy, the value $V(t)$ of this portfolio satifies

$$\begin{aligned} dV(t) &= \Delta_1(t)\big[dS(t) + \delta S(t)\,dt\big] + \Delta_2(t)\big[dD(t) + c\,dt\big] \\ &\quad + r\big[V(t) - \Delta_1(t)S(t) - \Delta_2(t)D(t)\big]\,dt. \end{aligned}$$

This portfolio is an *arbitrage* if

- $V(0) = 0$, and
- For some stopping time τ prior to call, conversion and bankruptcy, we have

$$\mathbb{P}\{V(\tau) \geq 0\} = 1, \quad \mathbb{P}\{V(\tau) > 0\} > 0.$$

Theorem.

Assume that prior to call, conversion and bankruptcy,

$$D(t) = f\big(X(t)\big)$$

for some C^2 function f. If there is no arbitrage, then f must satisfy

$$rf(x) - (rx - c)f'(x) + \delta\big(x - f(x)\big)f'(x) - \frac{1}{2}\sigma^2 x^2 f''(x) = c \qquad (1)$$

for $x \in (0, Ca \wedge Co)$.

Remark (Risk-neutral pricing).

The operator applied to f in (1) corresponds to the stochastic differential equation

$$\begin{aligned} dX(t) &= rX(t)\,dt - c\,dt - \delta\big(X(t) - f(X(t))\big)\,dt \\ &\quad + \sigma X(t)\,dW(t). \end{aligned} \qquad (2)$$

In other words, a solution to (1) is

$$\begin{aligned} f(x) \triangleq \mathbb{E}\Bigg[& \int_0^{\tau(0)\wedge\tau(Ca)\wedge\tau(Co)} e^{-rt} c\,dt \qquad (3) \\ & + \mathbb{I}_{\{\tau(Co)\le\tau(Ca)\wedge\tau(0)\}} e^{-r\tau(Co)} \gamma X(\tau(Co)) \\ & + \mathbb{I}_{\{\tau(Ca)<\tau(Co)\wedge\tau(0)\}} e^{-r\tau(Ca)} K \,\Bigg|\, X(0) = x \Bigg]. \end{aligned}$$

Indeed, this is the solution we seek.

The problem.

$$\begin{aligned} & f(x; Ca, Co) \\ & \triangleq \mathbb{E}\Bigg[\int_0^{\tau(0)\wedge\tau(Ca)\wedge\tau(Co)} e^{-rt} c\,dt \qquad (3) \\ & \quad + \mathbb{I}_{\{\tau(Co)\le\tau(Ca)\wedge\tau(0)\}} e^{-r\tau(Co)} \gamma X(\tau(Co)) \\ & \quad + \mathbb{I}_{\{\tau(Ca)<\tau(Co)\wedge\tau(0)\}} e^{-r\tau(Ca)} K \,\Bigg|\, X(0) = x \Bigg], \end{aligned}$$

where

$$\begin{aligned} dX(t) &= rX(t)\,dt - c\,dt - \delta\big(X(t) - f(X(t))\big)\,dt \\ &\quad + \sigma X(t)\,dW(t). \end{aligned} \qquad (2)$$

We want to show that the game has a value:

$$\sup_{Ca\ge K} \inf_{Co>0} f(x; Ca, Co) = \inf_{Co>0} \sup_{Ca\ge K} f(x; Ca, Co).$$

But the function $f(x) = f(x; Ca, Co)$ we seek to define by (3) appears in the evolution equation (2). The firm's value depends on the dividends paid by the firm, and these depend on the division of the firm's value between stock and the convertible bond.

Until we solve the convertible bond pricing problem, we cannot specify the evolution of the value of the firm.

Rough idea for resolution of the problem.

Seek a solution on $(0, Ca \wedge Co)$ to the nonlinear ordinary differential equation

$$rf(x) - (rx - c)f'(x) + \delta\big(x - f(x)\big)f'(x) - \frac{1}{2}\sigma^2 x^2 f''(x) = c \tag{1}$$

satisfying the boundary conditions

$$f(0) = 0, \quad f(Ca \wedge Co) = \begin{cases} \gamma Co & \text{if } 0 < Co \leq Ca, \\ K & \text{if } Ca < Co. \end{cases} \tag{4}$$

Instead seek a solution to a linear ordinary differential equation

$$rf(x) - (rx - c)f'(x) + \delta\big(x - g(x)\big)f'(x) - \frac{1}{2}\sigma^2 x^2 f''(x) = c, \tag{5}$$

with boundary conditions (4), where g is a given function. This defines a mapping

$$T: g \to f.$$

Seek a fixed point for T.

The fixed point is a function $f(x) = f(x; Ca, Co)$ depending on Ca and Co. Then show the game has a value:

$$\sup_{Ca \geq K} \inf_{Co > 0} f(x; Ca, Co) = \inf_{Co > 0} \sup_{Ca \geq K} f(x; Ca, Co).$$

Execution of the rough idea.

Let $\mathcal{G}$ be the set of functions g on $[0, \infty)$ satisfying

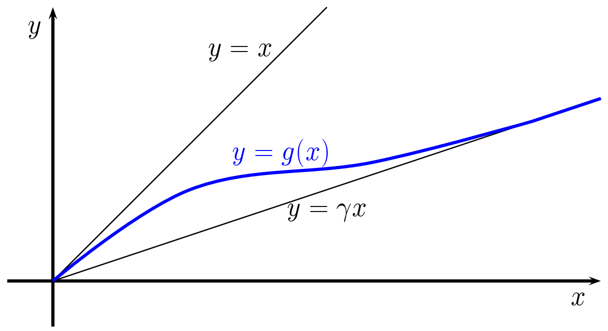
(i) g is continuous on $[0, \infty)$ and C^1 on $(0, \infty)$,

(ii) $g(0) = 0$,

(iii) $g(x) \geq \gamma x$ for all $x \geq 0$,

(iv) $0 \leq g'(x) < 1$ for all $x > 0$,

(v) $g(x) = \gamma x$ for x sufficiently large.



$$\begin{aligned} X^x(0) &= x, \\ dX^x(t) &= rX^x(t)\,dt - c\,dt - \delta\big(X^x(t) - g(X^x(t))\big)\,dt \\ &\quad + \sigma X^x(t)\,dW(t), \\ Tg(x) &\triangleq \sup_{0 \leq \tau \leq \tau(0)} \mathbb{E}\left[\int_0^\tau e^{-ru} c\,du + \mathbb{I}_{\{\tau < \infty\}} e^{-r\tau}\gamma X^x(\tau)\right]. \end{aligned}$$

Properties of Tg.

$$Tg(x) \triangleq \sup_{0\le\tau\le\tau(0)} \mathbb{E}\left[\int_0^\tau e^{-ru}c\,du + I\!\!I_{\{\tau<\infty\}}e^{-r\tau}\gamma X^x(\tau)\right].$$

When $g \in \mathcal{G}$, the function Tg satisfies

(i) Tg is continuous on $[0,\infty)$ and C^1 on $(0,\infty)$,

(ii) $Tg(0) = 0$,

(iii) $Tg(x) \ge \gamma x$ for all $x \ge 0$,

(iv) $0 \le \frac{d}{dx}Tg(x) < 1$ for all $x > 0$,

(v) $Tg(x) = \gamma x$ for all x sufficiently large.

In other words, T maps $\mathcal{G}$ into $\mathcal{G}$.

Properties (ii) and (iii) are obvious. The other proofs are based on the observation that Tg is a viscosity solution of the variational inequality

$$\min\{\mathcal{L}_g\psi(x) - c, \psi(x) - \gamma x\} = 0,$$

where $\mathcal{L}_g$ is the *linear* second-order differential operator

$$\begin{aligned}\mathcal{L}_g\psi(x) \;&\triangleq\; r\psi(x) - (rx-c)\psi'(x) + \delta(x-g(x))\psi'(x)\\ &\qquad -\frac{1}{2}\sigma^2x^2\psi''(x).\end{aligned}$$

Theorem (Contraction).

Let f and g in $\mathcal{G}$ be given, and define $\varphi = Tf$, $\psi = Tg$. Then

$$\|\varphi-\psi\|_\infty \le \frac{\delta}{r}\|f-g\|_\infty.$$

Proof: Because $\varphi(0) = \psi(0) = 0$ and $\varphi(x) = \psi(x) = \gamma x$ for sufficiently large x, there exists x_* such that

$$\varphi(x_*) - \psi(x_*) = \max_{x\ge0}\big[\varphi(x) - \psi(x)\big].$$

Either $\varphi(x_*) - \psi(x_*) = 0$ or $\varphi(x_*) - \psi(x_*) > 0$. In the latter case,

$$\varphi(x_*) > \psi(x_*) \ge \gamma x_*.$$

The inequality $\min\{\mathcal{L}_f\varphi(x) - c, \varphi(x) - \gamma x\} = 0$ implies

$$\mathcal{L}_f\varphi(x_*) - c = 0. \tag{6}$$

The inequality $\min\{\mathcal{L}_g\psi(x) - c, \psi(x) - \gamma x\} = 0$ implies

$$\mathcal{L}_g\psi(x_*) - c \ge 0. \tag{7}$$

Subtracting (7) from (6), we obtain

$$\begin{aligned}0 \;&\ge\; \mathcal{L}_f\varphi(x_*) - \mathcal{L}_g\psi(x_*)\\ &=\; r\big(\varphi(x_*) - \psi(x_*)\big) - \delta\big(f(x_*) - g(x_*)\big)\varphi'(x_*)\\ &\qquad -\frac{1}{2}\sigma^2x_*^2\big(\varphi''(x_*) - \psi''(x_*)\big),\end{aligned}$$

where we have used $\varphi'(x_*) = \psi'(x_*)$.

From the previous page we have

$$r\big(\varphi(x_*) - \psi(x_*)\big) - \delta\big(f(x_*) - g(x_*)\big)\varphi'(x_*) \le \frac{1}{2}\sigma^2 x_*^2\big(\varphi''(x_*) - \psi''(x_*)\big)$$

Becasue $\varphi - \psi$ attains a maximum at x_*, we must have $\varphi''(x_*) - \psi''(x_*) \le 0$. Therefore,

$$\begin{aligned} r\big(\varphi(x_*) - \psi(x_*)\big) &\le \delta\big(f(x_*) - g(x_*)\big)\varphi'(x_*) \\ &\le \delta\|f - g\|_\infty \cdot 1. \end{aligned}$$

In other words,

$$\max_{x \ge 0}\big[\varphi(x) - \psi(x)\big] = \varphi(x_*) - \psi(x_*) \le \frac{\delta}{r}\|f - g\|_\infty.$$

Reversing the roles of f and g, we obtain

$$\|\varphi - \psi\|_\infty \le \frac{\delta}{r}\|f - g\|_\infty.$$

$\square$

$\mathcal{G}$ is not closed, but we can close it, extend T to the closure $\overline{\mathcal{G}}$, and show that the extended T is still a contraction. The **Banach Fixed Point Theorem** implies the existence of a unique fixed f_∞. We can show that $f_\infty \in \mathcal{G}$, i.e., the fixed point is in the interior of $\overline{\mathcal{G}}$.

Reintroducing the call price (large call prices).

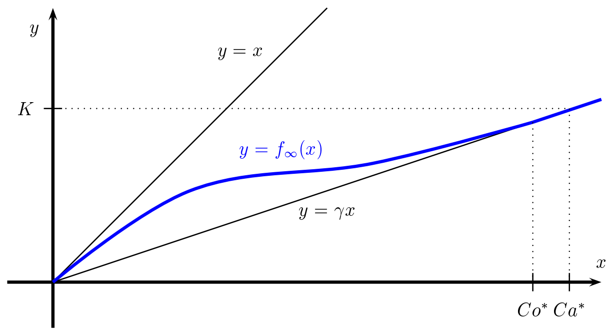


Figure 1: Large call prices K

For large call prices K, the optimal conversion level Co^* and call level Ca^* are shown in Figure 1. When the firm value is x, the value of the two-person game is $f_\infty(x)$.

Small call prices.

Let $a \in (0,\infty)$ be given. Define

$$T_a g(x) \triangleq \sup_{0\le\tau\le\tau(0)\wedge\tau(a)} \mathbb{E}\left[\int_0^\tau e^{-ru} c\,du + \mathbb{I}_{\{\tau<\infty\}} e^{-r\tau}\gamma X^x(\tau)\right].$$

T_a has a unique fixed point f_a in an appropriate set of functions. Existence uses the **Schauder Fixed Point Theorem** and uniqueness uses a viscosity solution comparison argument.

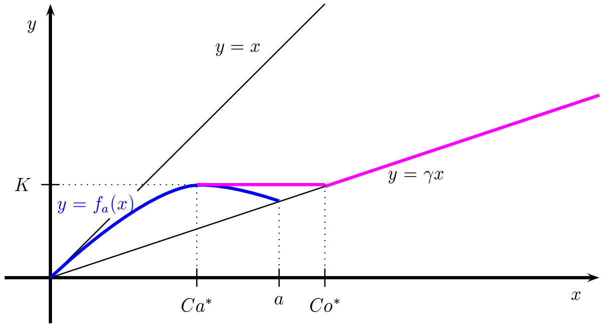


Figure 2: Small call prices K

When the firm value is $x \le Ca^*$, the value of the game is $f_a(x)$. When $x \ge Ca^*$, the value of the game is $\max\{K,\gamma x\}$.

Intermediate call prices.

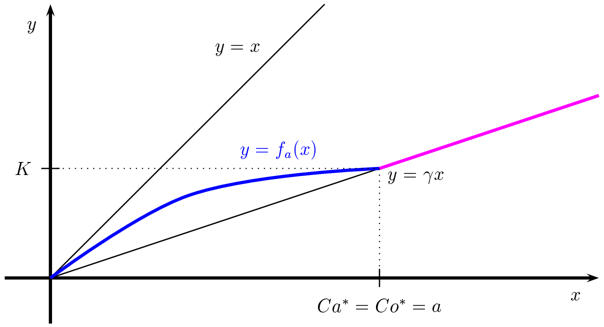


Figure 3: Intermediate call prices K

Optimal call and conversion levels agree. When the firm value is $x \le Ca^* = Co^*$, the value of the game is $f_a(x)$. When $x \ge Ca^* = Co^*$, the value of the game is γx.

Finite maturity convertible bonds. (Mihai Sîrbu's Ph.D. dissertation, in progress.)

Recall $0 < \delta < r$.

Large call price.

When $K \geq c/r$, the game reduces to the optimal stopping problem of *optimal conversion*. Call occurs at firm value K/γ, which is greater than or equal to the optimal conversion level.

Small call price.

When $K \leq c/\delta$, the game reduces to the optimal stopping problem of *optimal call*. Conversion occurs at firm value K/γ, which is greater than or equal to the optimal call level.

Intermediate call price.

When $\frac{c}{r} \leq K \leq \frac{c}{\delta}$, *call and conversion occur simultaneously* at firm value K/γ.