SIAM J. CONTROL OPTIM.
Vol. 45, No. 4, pp. 1508–1539

A TWO-PERSON GAME FOR PRICING CONVERTIBLE BONDS*

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Abstract. A firm issues a convertible bond. At each subsequent time, the bondholder must decide whether to continue to hold the bond, thereby collecting coupons, or to convert it to stock. The bondholder wishes to choose a conversion strategy to maximize the bond value. Subject to some restrictions, the bond can be called by the issuing firm, which presumably acts to maximize the equity value of the firm by minimizing the bond value. This creates a two-person game. We show that if the coupon rate is below the interest rate times the call price, then conversion should precede call. On the other hand, if the dividend rate times the call price is below the coupon rate, call should precede conversion. In either case, the game reduces to a problem of optimal stopping.

Key words. convertible bonds, optimal stopping, two-person game, viscosity solutions, free boundary

AMS subject classifications. 90A09, 60H30, 60G44

DOI. 10.1137/050630222

1. Introduction.

Firms raise capital by issuing debt (bonds) and equity (shares of stock). The *convertible bond* is intermediate between these two instruments. A convertible bond is a bond in that it entitles its owner to receive coupons plus the return of principal at maturity. However, prior to maturity the holder may *convert* the bond, surrendering it for a preset number of shares of stock. The price of the bond is thus dependent on the price of the firm's stock. Finally, prior to maturity, the firm may *call* the bond, forcing the bondholder to either surrender it to the firm for a previously agreed price or convert it to stock as above.

After issuing a convertible bond, the firm's objective is to exercise its call option in order to maximize the value of shareholder equity. The bondholder's objective is to exercise the conversion option in order to maximize the value of the bond. Because the firm must pay coupons to the bondholder, it may call the bond if it can subsequently reissue a bond with a lower coupon rate. This happens as the firm's fortunes improve, for then the risk of default has diminished and investors will accept a lower coupon rate on the firm's bonds. In the case of a convertible bond, the firm has a second incentive to call: as the firm's fortunes improve, the investor may convert, becoming a shareholder of a profitable firm and diluting the value of the stock owned by the original shareholders. The firm can prevent this by calling the bond. The bondholder has an incentive to convert the bond to stock before maturity under exactly the same scenario; the bondholder may want to become a shareholder of a profitable firm, for the promise of future dividends may be more valuable than the promise of future coupons.

If stock and convertible bonds are the only assets issued by a firm, then the value of the firm is the aggregate value of these two types of assets. In idealized

*Received by the editors April 28, 2005; accepted for publication (in revised form) May 24, 2006; published electronically October 20, 2006. This material is based upon work supported by the National Science Foundation under grants DMS-0139911 and DMS-0404682. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

http://www.siam.org/journals/sicon/45-4/63022.html

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markets, where the Miller–Modigliani [32], [33] assumptions hold (see Hennessy and Tserlukevich [20] for a model in which they do not), changes in corporate capital structure do not affect firm value. In particular, the value of the firm does not change at the time of conversion, and the only change in the value of the firm at the time of call is a reduction by the call price paid to the bondholder if the bondholder surrenders rather than converting the bond. By acting to maximize the value of equity, the firm is in fact minimizing the value of the convertible bond. By acting to maximize the value of the bond, the bondholder is in fact minimizing the value of equity. This creates a *two-person, zero-sum game*. The game is complicated by the fact that one can expect the dividend payment policy of the firm to depend on the bond price, a feature explicitly modeled in this paper. This feature causes the bond price to be governed by a *nonlinear* second-order partial differential equation, a novel feature of this paper.

This is a companion paper to Sîrbu, Pikovsky, and Shreve [36]. In [36], the bond did not mature and hence time was not a variable, whereas in the present paper, the bond has finite maturity and the bond price depends on the time to maturity.

Brennan and Schwartz [8] and Ingersoll [22] address the convertible bond pricing problem via the arbitrage pricing theory developed by Merton [30], [31] and underlying the option pricing formula of Black and Scholes [7]. In the Brennan–Schwartz [8] model, dividends and coupons are paid at discrete dates. Between these dates, the value of the firm is a geometric Brownian motion and the price of the convertible bond is governed by the *linear* partial differential equation developed by Black and Scholes [7]. This sets up a backward recursion over payment dates, which permits a numerical solution of the bond pricing problem but is not readily amenable to qualitative analysis. In Ingersoll [22], coupons are paid out continuously. For most of the results obtained in [22], dividends are zero, and because of this the bond price is again governed by a *linear* partial differential equation.

The present paper differs from the classical literature in a second respect. In [8], the bond should not be converted except possibly immediately prior to a dividend payment; in [22], the bond should not be converted except possibly at maturity. Therefore, neither of these papers needs to address the *free boundary problem* that arises if early conversion (other than at discrete dates) is optimal.

Ingersoll [22] provides a heuristic argument that the firm should call as soon as the conversion value of the bond (the value the bondholder would receive if he converts the bond to stock) rises to the call price. It is observed that firms tend to call later than this, and several reasons have been advanced to explain this departure from the model; see, e.g., [2], [3], [16], [19], [23]. We show here by a rigorous analysis of the model that, although the Ingersoll conclusion is often valid, it is also possible that the firm should call *before* the conversion value of the bond rises to the call price. In these cases, explanation of observed firm behavior is more difficult than previously believed.

The present paper assumes that a firm's value comprises equity and convertible bonds. To simplify the discussion, we assume that equity is in the form of a single share of stock, and that there is a single convertible bond. We assume that the value of the issuing firm has constant volatility, the bond continuously pays coupons at a fixed rate, and the firm continuously pays dividends at a rate that is a fixed fraction of equity. Default occurs if the coupon payments cause the firm value to fall to zero, in which case the bond has zero recovery. In this model, both the bond price and the stock price are functions of the underlying firm value. Because the stock price is the difference between firm value and bond price, and dividends are paid proportionally to the stock price, the differential equation characterizing the bond price as a function of the firm value is *nonlinear*.

In section 2 we provide a no-arbitrage argument, which states that once the firm and the bondholder choose their call and conversion strategies, the price of the bond is the expected value under the *risk-neutral measure* of the cash flows that accrue from ownership of the bond. The determination of the optimal call and conversion strategies then becomes a Dynkin game between the firm and the bondholder, and the bond is almost a game option in the sense of Kifer [26]. In contrast to [26], here the evolution of the underlying process, the firm value, depends on the solution to the game. Kallsen and Kühn [24] consider a game option setting that includes this possibility.

Recognizing that convertible bond pricing is a game is implicit in previous work. For example, [8] observes that the pricing problem " ... results in a pair of conversion-call strategies which are in equilibrium in the sense that neither party could improve his position by adopting any other strategy." Here we make the game explicit and obtain a good qualitative description of its value. In particular, if the dividend rate is below the interest rate, then the game reduces to one of two possible optimal stopping problems, either the problem of optimal call or the problem of optimal conversion, and we are able to determine in advance from the model parameters which of these two problems is relevant.

Convertible bonds can have several features that must be captured by any model intended for practical application; see [29]. These include periods of call protection, time-dependent conversion factors, and exposure to interest rate and default risk. The model of this paper captures only the default risk, and that via a simple structural model in which default occurs at the time the firm value falls to zero. Loschak [27] allows nonconvertible senior debt and uses a more sophisticated structural model for default. Brennan and Schwartz [9] also allow senior debt. Another interesting issue is the process of conversion when bonds are held by competing investors; see Constantinides [11] and Constantinides and Rosenthal [12].

Practical models have been built around the idea that the cash flow from a convertible bond can be separated into an "equity" part, which should be discounted at the interest rate, and a "bond" part, which should be discounted at the interest rate plus a credit spread. Papers taking this approach are McConnell and Schwartz [28], Cheung and Nelken [10], Ho and Pteffer [21], Tsiveriotis and Fernandes [39], and Yigitbasioglu [40]. Ayache, Forsyth, and Vetzal [4] analyze some of this work and conclude that its failure to account for the effect of default on equity introduces significant pricing errors. This deficiency is corrected in Davis and Lischka [14], Takahashi, Kobayashi, and Nakagawa [38], and Andersen and Buffum [1], who build intensity-based models for default affecting equity value.

We describe our model in section 2 and report our main results in section 3. In particular, the Dynkin game that describes the bond price reduces to one of two optimal stopping problems and a fixed point problem. Section 4 provides a probabilistic justification for the reduction of the game to optimal stopping. Viscosity solution results concerning the Hamilton–Jacobi–Bellman equations governing the optimal stopping problems are provided in section 5. This permits the proof in section 6 of the existence and uniqueness of the solution to the fixed point problem, and this solution is the bond pricing function. Section 7 relates this paper to perpetual convertible bonds. In section 8 we provide some results on the nature of the stopping and continuation regions of the optimal stopping problems of this paper.

2. The model. We denote the value of the firm at each time t by X_t. We assume the value of the firm consists of equity and debt. The debt D_t is due to a single outstanding convertible bond. This assumption of a single bond means that all debt is called and/or converted simultaneously. We denote by S_t the total value of equity, which, following the standard finance model (e.g., Merton [31, bottom of p. 453]) is given by

$$S_t = X_t - D_t. \tag{2.1}$$

Equity owners receive dividends paid continuously over time at a rate δS_t, and the bondholder receives coupons paid continuously over time at a rate c. We assume that $\delta \geq 0$ and $c > 0$ are both constant. If there is no call or conversion prior to *maturity* T, then at maturity the bondholder receives the *par value* L from the firm, provided $X_T \geq L$. Otherwise, the bondholder receives X_T. However, at any time $t \in [0, T]$, the bondholder may *convert* the bond to stock, thereby immediately receiving stock valued at the *conversion factor* $\gamma \in (0, 1)$ times the firm value X_t. The firm value is not affected by this conversion. On the other hand, at any time t when $X_t \geq K$, the firm may *call* the bond, forcing the bondholder to either immediately surrender the bond in exchange for the *call price* K or else immediately convert the bond as described above. We assume $K \geq L > 0$; it is common to have $L = K$. If K were less than L, then L would be irrelevant since the firm could always call at maturity to avoid paying L.

In order to model the firm value process, which is the primitive in our analysis, we note that, prior to maturity, as long as the bond has not been called or converted and the firm value has not fallen to zero, there are three financial instruments in the market: the stock, the convertible bond(s), and a money market account with risk-free rate of interest $r > 0$. The wealth V_t of an investor holding Δ_t shares of stock and Γ_t convertible bonds at each time t, investing or borrowing in the money market account as necessary in order to finance this, evolves according to the stochastic differential equation

$$dV_t = \Delta_t[dS_t + \delta S_t\,dt] + \Gamma_t[dD_t + c\,dt] + r[V(t) - \Delta_t S_t - \Gamma_t D_t]\,dt, \tag{2.2}$$

and the discounted wealth thus satisfies

$$d\big(e^{-rt}V_t\big) = \Delta_t\big[d\big(e^{-rt}S_t\big) + e^{-rt}\delta S_t\,dt\big] + \Gamma_t\big[d\big(e^{-rt}D_t\big) + e^{-rt}c\,dt\big]. \tag{2.3}$$

Such an investor should not be able to produce arbitrage. To ensure this, according to the *first fundamental theorem of asset pricing*, both the discounted stock price plus the cumulative discounted dividend payments,

$$M_t = e^{-rt}S_t + \int_0^t e^{-ru}\delta S_u\,du,$$

and the discounted convertible bond price plus the cumulative discounted coupons,

$$N_t = e^{-rt}D_t + \int_0^t e^{-ru}c\,du,$$

must be local martingales under some *risk-neutral* probability measure $\mathbb{P}$ (see the argument due to [18] and developed in great generality by [15]). Adding the above equations, using the relation $X_t = S_t + D_t$, we obtain

$$dX_t = rX_t\,dt - c\,dt - \delta S_t\,dt + e^{rt}d(M_t + N_t). \tag{2.4}$$

Assuming constant volatility $\sigma > 0$ for X_t means that the local martingale term $e^{rt}d(M_t + N_t)$ is in fact equal to $\sigma X_t\,dW_t$, where W is a Brownian motion. To summarize, in our model the value of the firm evolves according to the equation

$$dX_t = rX_t\,dt - c\,dt - \delta S_t\,dt + \sigma X_t\,dW_t, \tag{2.5}$$

where W_t, $0 \le t \le T$, is a Brownian motion under the risk-neutral probability measure $\mathbb{P}$. This is the starting point of our model, and it is a common starting point for treatments of the convertible bond pricing problem; see, e.g., [8], [22]. Equation (2.5) says that under the risk-neutral measure the mean rate of growth of X_t is the interest rate r adjusted by the payouts being made.

Finally, we observe from (2.5) that $X_t \le X_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$. Therefore, since $0 \le S_t \le X_t$ and $0 \le D_t \le X_t$, we conclude that the local martingales M_t and N_t are martingales under $\mathbb{P}$.

We adopt throughout the *standing assumption*

$$0 \le \delta < r, \tag{2.6}$$

but see Remark 3.5 below.

We generalize slightly the previous discussion by permitting the initial time to be $s \in [0,T]$ rather than requiring it to be 0. We shall price the bond at time s under the assumption that $X_s = x$. Given these initial conditions, we denote by $X_t^{s,x}$ the solution to (2.5) at time $t \in [s,T]$ and set

$$\theta_y^{s,x} \triangleq \min\{t \in [s,T] : X_t^{s,x} = y\}, \quad y \ge 0,$$

where we adopt the convention that $\min \emptyset = \infty$. The firm defaults on the bond at time $\theta_0^{s,x}$ if $\theta_0^{s,x} \le T$, and $\theta_0^{s,x} = \infty$ corresponds to no default.

The firm adopts a *call strategy* ρ and the bondholder adopts a *conversion strategy* τ. Both of these are stopping times for the filtration generated by $W_u - W_s$, $u \in [s,T]$ (augmented by $\mathbb{P}$-null sets), and they must satisfy $\rho, \tau \in [s, T \wedge \theta_0^{s,x}] \cup \{\theta_0^{s,x}\}$. We denote the set of all such stopping times by $\mathcal{S}^{s,x}$. We interpret ρ and τ to be the times of call and conversion, respectively, except on the set $\{\rho = \theta_0^{s,x}\}$, where there is no call. Similarly, on the set $\{\tau = \theta_0^{s,x}\}$, there is no conversion. On the set $\{\rho = \tau < \theta_0^{s,x}\}$, there is simultaneous call and conversion, and the conversion takes priority. This is the standard contractual specification for convertible bonds. There is no requirement that call or conversion must take place, and we capture the absence of call (respectively, conversion) by permitting $\rho = \infty$ (respectively, $\tau = \infty$) if $\theta_0^{s,x} = \infty$. The firm can call at time $\rho < \theta_0^{s,x}$ only if $X_\rho^{s,x} \ge K$. We denote by $\mathcal{S}_K^{s,x}$ the set of stopping times in $\mathcal{S}^{s,x}$ satisfying the additional condition that $X_\rho^{s,x} \ge K$ on $\rho < \theta_0^{s,x}$, and we require that $\rho \in \mathcal{S}_K^{s,x}$. Once the call and conversion strategies $\rho \in \mathcal{S}_K^{s,x}$ and $\tau \in \mathcal{S}^{s,x}$ are chosen, we use the fact that N_t stopped at $\rho \wedge \tau \wedge T$ is a martingale under $\mathbb{P}$ to write the value of the bond D_s at time s as

$$\begin{aligned} J(s,x;\rho,\tau) &\triangleq e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\tau\wedge T} e^{-ru}c\,du + e^{-r(\rho\wedge\tau\wedge T)} D_{\rho\wedge\tau\wedge T}\right] \\ &= e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\tau\wedge T} e^{-ru}c\,du + e^{-r(\rho\wedge\tau\wedge T)}\big(\mathbb{I}_{\{\tau\le\rho\wedge T\}}\gamma X_\tau^{s,x}\right. \\ &\qquad \left. + \mathbb{I}_{\{\rho<\tau\}}K + \mathbb{I}_{\{\rho\wedge\tau=\infty\}}(X_T^{s,x}\wedge L)\big)\right]. \end{aligned} \tag{2.7}$$

3. The method and principal results. We must deal with the fact that the process S_t in section 2 is endogenous. In fact, the bond price, the firm value, and the equity value S_t are related by (2.1). Just as in [8], [22], [31], and even [7], for the case of options rather than convertible bonds, we make the ansatz that there is a function $g(t,x)$ such that prior to call and conversion, $D_t = g(t,X_t)$ and hence $S_t = X_t - g(t,X_t)$. This is a reasonable step because the only source of uncertainty in the model is the uncertainty in the firm value (equivalently, the uncertainty in the Brownian motion driving the firm value), and thus all asset prices should depend on only this and the time variable.

We eventually see (Lemma 4.1 below) that if $\gamma X_t \geq K$, then it is optimal to convert, and hence $D_t = \gamma X_t$. Hence, the function $g(t,x)$ should satisfy

$$g(t,x) = \gamma x \text{ for } 0 \leq t \leq T \text{ and } x \geq \frac{K}{\gamma}. \tag{3.1}$$

Also, we expect both the value of the bond and the value of the equity to increase with increasing firm value, which is equivalent to

$$0 \leq g(t,y) - g(t,x) \leq y - x \text{ for } 0 \leq t \leq T \text{ and } 0 \leq x \leq y. \tag{3.2}$$

The bond is never worth less than its conversion value and never worth more than the firm value. Since the firm can always call when $\gamma x \leq K$, in which case the call does not result in conversion, the bond is not worth more than the call price. In other words,

$$\gamma x \leq g(t,x) \leq x \wedge K \text{ for } 0 \leq t \leq T \text{ and } 0 \leq x \leq \frac{K}{\gamma}. \tag{3.3}$$

We shall show that within the collection of functions

$$\mathcal{G} = \{g\colon [0,T] \times [0,\infty) \to [0,\infty) : g \text{ is continuous and (3.1)–(3.3) hold}\},$$

there exists a unique function g^* such that $g^*(t,X_t)$ gives a bond price consistent with our modeling assumptions.

To get started, we simply choose an arbitrary $g \in \mathcal{G}$ and define

$$S_t = X_t - g(t,X_t). \tag{3.4}$$

We substitute this value of S_t into (2.5), thereby obtaining a stochastic differential equation for X. The Lipschitz continuity (3.2) guarantees that corresponding to every initial condition $(s,x) \in [0,T] \times [0,\infty)$ this equation has a strong solution, and we thus obtain $X^{s,x}$. We proceed as in section 2 and conclude with the function J of (2.7), which we now denote J_g.

For each fixed $g \in \mathcal{G}$, we can construct a Dynkin game, where now the evolution of the underlying process is specified by (2.5) and (3.4). Kallsen and Kühn [24] show that the value of this game will be the no-arbitrage price of the bond, provided the function g has been chosen "correctly" (see the next paragraph). This game has lower and upper values

$$\underline{v}_g(s,x) \triangleq \sup_{\tau \in \mathcal{S}^{s,x}} \inf_{\rho \in \mathcal{S}_K^{s,x}} J_g(s,x;\rho,\tau), \quad \overline{v}_g(s,x) \triangleq \inf_{\rho \in \mathcal{S}_K^{s,x}} \sup_{\tau \in \mathcal{S}^{s,x}} J_g(s,x;\rho,\tau),$$

respectively. Clearly, $\underline{v}_g \leq \overline{v}_g$. In fact,

$$\underline{v}_g(s,x) = \overline{v}_g(s,x) \text{ for } 0 \leq s \leq T \text{ and } x \geq 0. \tag{3.5}$$

This is a consequence of the theory of Dynkin games, but rather than appeal to that theory, we obtain (3.5) as a by-product of our characterization of the solution of the game; see Lemma 4.1 and Propositions 4.5 and 4.6 below.

The function $\underline{v}_g = \overline{v}_g$ provides the price of the convertible bond if we choose g to be the pricing function of the convertible bond. That is to say, we want to find a function $g^* \in \mathcal{G}$ such that $\underline{v}_{g^*} = \overline{v}_{g^*} = g^*$. Let us define the operator $\mathcal{T}$ on $\mathcal{G}$ by $\mathcal{T}g \triangleq \underline{v}_g = \overline{v}_g$. We shall prove the following.

THEOREM 3.1. *$\mathcal{T}$ maps $\mathcal{G}$ into $\mathcal{G}$ and has a unique fixed point g^*.*

When $x = 0$, the only stopping time in $\mathcal{S}^{T,x}$ is $\tau \equiv T$, whereas when $0 < x \leq \frac{K}{\gamma}$, the set $\mathcal{S}^{T,x}$ also contains the stopping time $\tau \equiv \infty$. When $x = 0$, the only stopping in $\mathcal{S}_K^{T,x}$ is $\rho = T$, whereas when $0 < x < K$, the only stopping time in $\mathcal{S}_K^{T,x}$ is $\rho \equiv \infty$. When $K \leq x \leq \frac{K}{\gamma}$, the set $\mathcal{S}_K^{T,x}$ consists of $\rho \equiv T$ and $\rho \equiv \infty$. We further have

$$J_g(T, x; \rho, \tau) = \mathbb{I}_{\{\tau \leq \rho \wedge T\}} \gamma x + \mathbb{I}_{\{\rho < \tau\}} K + \mathbb{I}_{\{\rho \wedge \tau = \infty\}} (x \wedge L).$$

It is now straightforward to compute $\underline{v}_g(T, x)$ and $\overline{v}_g(T, x)$ for $0 \leq x \leq \frac{K}{\gamma}$, and irrespective of the choice of g, this results in the terminal convertible bond pricing function $g^*(T, \cdot)$ given by

$$g^*(T, x) = (x \wedge L) \vee (\gamma x) \text{ for } 0 \leq x \leq \frac{K}{\gamma}. \tag{3.6}$$

Because g^* also satisfies (3.1), we need only describe this function on $[0, T) \times \big[0, \frac{K}{\gamma}\big)$. From (3.3) we have the boundary conditions

$$g^*(t, 0) = 0, \quad g^*\Big(t, \frac{K}{\gamma}\Big) = K \text{ for } 0 \leq t \leq T. \tag{3.7}$$

THEOREM 3.2 (Case I). *If $c \leq rK$, the time of optimal call is the first time the conversion value γX_t rises to the call price K. The bond pricing function g^* is determined by solving the problem of optimal conversion in $[0, T] \times \big[0, \frac{K}{\gamma}\big]$. In particular, g^* is the unique continuous viscosity solution of the variational inequality*

$$\min\left\{-v_t + rv - (rx - c)v_x + \delta(x - v)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c, v - \gamma x\right\} = 0 \tag{3.8}$$

on $[0, T] \times [0, \frac{K}{\gamma}]$ satisfying (3.6) *and* (3.7).

(Case II). *If $\delta K \leq c$, the time of optimal conversion is at the first time the conversion value γX_t rises to the call price K, or at maturity if the conversion value exceeds the par value. The bond pricing function g^* is determined by solving the problem of optimal call in $[0, T] \times \big[0, \frac{K}{\gamma}\big]$. In particular, g^* is the unique continuous viscosity solution of the variational inequality*

$$\max\left\{-v_t + rv - (rx - c)v_x + \delta(x - v)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c, v - K\right\} = 0 \tag{3.9}$$

on $[0, T] \times [0, \frac{K}{\gamma}]$ satisfying (3.6) *and* (3.7).

Remark 3.3. By a "continuous viscosity solution on $[0, T] \times [0, \frac{K}{\gamma}]$" we mean a continuous function on $[0, T] \times [0, \frac{K}{\gamma}]$ that is a solution in the interior of the domain, namely, $(0, T) \times (0, \frac{K}{\gamma})$.

Remark 3.4. Because of standing assumption (2.6), Cases I and II overlap. In other words, we can have $\delta K \le c \le rK$, and optimal call and conversion both occur the first time γX_t rises to K. In this case, g^* is the unique continuous viscosity solution on $[0,T] \times \left[0, \frac{K}{\gamma}\right]$ of the partial differential equation

$$-v_t + rv - (rv - c)v_x + \delta(x - v)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} = c$$

satisfying (3.6) and (3.7).

Remark 3.5. The proofs in this paper do not actually require standing assumption (2.6), but rather that either $c \le rK$ or $\delta K \le c$. Under either of these conditions, Theorems 3.1 and 3.2 hold. However, Theorem 3.6 below requires (2.6) for the pricing of the perpetual convertible bond; see [36].

THEOREM 3.6. *As the time to maturity approaches* ∞*, the price of the finite-maturity convertible bond approaches the price of the perpetual convertible bond of* [36]*, which is characterized in Theorem* 7.2 *below, and this convergence is uniform in the firm value.*

4. Construction and properties of v_g.

4.1. Reduction to $[0,T] \times \left[0, \frac{K}{\gamma}\right]$.

LEMMA 4.1. *Assume that g defined on* $[0,T] \times [0,\infty)$ *satisfies* (3.2)*, so that* (3.4) *and* (2.5) *uniquely determine a process* $X^{s,x}$ *for* $(s,x) \in [0,T] \times [0,\infty)$*. Then*

$$\underline{v}_g(s,x) = \overline{v}_g(s,x) = \gamma x \text{ for } 0 \le s \le T \text{ and } x \ge \frac{K}{\gamma}. \tag{4.1}$$

Proof. With $\tau \equiv s$, (2.7) implies $J_g(s,x;\rho,s) = \gamma x$ for $\rho \in \mathcal{S}_K^{s,x}$, and thus

$$\underline{v}_g(s,x) \ge \inf_{\rho \in \mathcal{S}_K^{s,x}} J_g(s,x;\rho,s) = \gamma x. \tag{4.2}$$

For $x \ge \frac{K}{\gamma}$, we may set $\rho \equiv s$ and then have for every $\tau \in \mathcal{S}^{s,x}$ that $J_g(s,x;s,\tau) = \gamma x \mathbb{I}_{\{\tau = s\}} + K \mathbb{I}_{\{s < \tau\}} \le \gamma x$, and hence

$$\overline{v}_g(s,x) \le \sup_{\tau \in \mathcal{S}_K^{s,x}} J_g(s,x;s,\tau) \le \gamma x. \tag{4.3}$$

But directly from their definitions, we know that $\underline{v}_g \le \overline{v}_g$. □

We fix a function $g \in \mathcal{G}$ for the remainder of section 4.

4.2. Modification of payoffs. The contractual features of convertible bonds require that we define the value of the bond by (2.7) once the call and conversion strategies ρ and τ are specified. This formulation is not readily amenable to analysis, since the stopping times are allowed to take the value ∞, the different players have different sets of stopping times at their disposal, and the payoff in the event of call is not always less than or equal to the payoff in the event of conversion. In this section we create an auxiliary problem that has all these desirable features, and we subsequently show in Propositions 4.5 and 4.6 that both problems have the same value.

We restrict our attention to stopping times in $\mathcal{S}_T^{s,x} \triangleq \{\theta \in \mathcal{S}^{s,x} : \theta \le T\}$. In particular, we do not allow stopping times to take the value ∞ and we do not require the call strategy ρ to satisfy $X_\rho^{s,x} \ge K$ on $\{\rho < \theta_0^{s,x}\}$. We also change the payoffs appearing in (2.7). We define

$$\psi(t,x) \triangleq \begin{cases} \gamma x & \text{for } 0 \le t < T,\ x \ge 0, \\ (x \wedge L) \vee (\gamma x) & \text{for } t = T,\ x \ge 0, \end{cases}$$

$$\varphi(t,x) \triangleq \begin{cases} (x \wedge K) \vee (\gamma x) & \text{for } 0 \le t < T,\ x \ge 0, \\ (x \wedge L) \vee (\gamma x) & \text{for } t = T,\ x \ge 0. \end{cases}$$

Then $\psi < \varphi$ on $[0,T) \times \big(0, \frac{K}{\gamma}\big)$ and $\psi = \varphi$ on the parabolic boundary

$$\partial_p D_0 \triangleq \left([0,T) \times \left\{0, \frac{K}{\gamma}\right\}\right) \cup \left(\{T\} \times \left[0, \frac{K}{\gamma}\right]\right). \tag{4.4}$$

For $\rho, \tau \in \mathcal{S}_T^{s,x}$, we set

$$\widetilde{J}_g(s,x;\rho,\tau) \triangleq e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\tau} e^{-ru} c\,du + e^{-r(\rho\wedge\tau)}\big(\mathbb{I}_{\{\tau<\rho\}}\psi(\tau, X_\tau^{s,x}) + \mathbb{I}_{\{\rho\le\tau\}}\varphi(\rho, X_\rho^{s,x})\big)\right].$$

The interpretation of $\widetilde{J}_g$ is that if the firm value is insufficient to pay the call price at the time of the call, then the bondholder receives the firm value. Also, call takes priority over conversion, but the bondholder receives the conversion value if that is greater than the call price at the time of the call. We show in Propositions 4.5 and 4.6 that changing the payoffs in this way does not change the value of the convertible bond pricing problem. We begin with the following straightforward modification of Lemma 4.1.

LEMMA 4.2. *For $0 \le s \le T$ and $x \ge \frac{K}{\gamma}$, we have*

$$\inf_{\rho\in\mathcal{S}_T^{s,x}} \sup_{\tau\in\mathcal{S}_T^{s,x}} \widetilde{J}_g(s,x;\rho,\tau) = \sup_{\tau\in\mathcal{S}_T^{s,x}} \inf_{\rho\in\mathcal{S}_T^{s,x}} \widetilde{J}_g(s,x;\rho,\tau) = \gamma x.$$

4.3. Technical preparations. Itô's formula implies that if h is a continuous function on $[0,T] \times \big[0, \frac{K}{\gamma}\big]$, h is $C^{1,2}$ on the interior of its domain, and the derivatives of h have limits at the boundary of its domain, then for $(s,x) \in [0,T] \times \big[0, \frac{K}{\gamma}\big]$,

$$\begin{aligned} &d\left(\int_s^t e^{-ru} c\,du + e^{-rt} h(t, X_t^{s,x})\right) \\ &\quad = e^{-rt}\big[-\mathcal{L}_g h(t, X_t^{s,x}) + c\big]\,dt + e^{-rt}\sigma X_t^{s,x} h_x(t, X_t^{s,x})\,dW_t \end{aligned} \tag{4.5}$$

for $t \in [s, \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x} \wedge T]$, where

$$\begin{aligned} \mathcal{L}_g h(t,x) \triangleq{}& -h_t(t,x) + rh(t,x) - (rx - c)h_x(t,x) \\ &+ \delta\big(x - g(t,x)\big)h_x(t,x) - \frac{1}{2}\sigma^2 x^2 h_{xx}(t,x). \end{aligned} \tag{4.6}$$

LEMMA 4.3. *Let $\widetilde{c} > 0$ be given, and for $0 \le s \le T$ and $0 \le x \le \frac{K}{\gamma}$, define*

$$\begin{aligned} &k(s,x) \\ &\triangleq e^{rs}\mathbb{E}\left[\int_s^{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T} e^{-ru}\widetilde{c}\,du + e^{-r(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T)}\psi\left(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T, X^{s,x}_{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T}\right)\right] \\ &= e^{rs}\mathbb{E}\left[\int_s^{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T} e^{-ru}\widetilde{c}\,du + e^{-r(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T)}\varphi\left(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T, X^{s,x}_{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T}\right)\right], \end{aligned} \tag{4.7}$$

where we have used the fact that ψ and φ agree on the parabolic boundary $\partial_p D_0$. Then k is continuous and satisfies $k = \varphi = \psi$ on $\partial_p D_0$.

Remark 4.4. We would expect the function k to satisfy the partial differential equation $\mathcal{L}_g k = \widetilde{c}$, but since g is only continuous and not Hölder continuous with respect to time, we do not know that this equation has a classical solution. Hence, we give a probabilistic proof of Lemma 4.3.

Proof of Lemma 4.3. It is apparent that $k = \psi$ on $\partial_p D_0$. It remains to prove the continuity.

We extend g to a jointly continuous function, globally Lipschitz in its second variable, defined on $[0,T] \times \mathbb{R}$, so that for every $(s,x) \in [0,T] \times \left[0, \frac{K}{\gamma}\right]$, $X_t^{s,x}$ can be defined by (2.5) and (3.4) for all $t \in [s,T]$. We define $X_t^{s,x} = x$ for $t \in [0,s)$. All the processes $X^{s,x}$ are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in $C[0,T]$.

For $(s,x) \in [0,T] \times \left[0, \frac{K}{\gamma}\right]$, denote by $\mathbb{P}^{s,x}$ the distribution of $X^{s,x}$ on $C[0,T]$. According to [37, p. 152], $\mathbb{P}^{s,x}$ is continuous in (s,x). For $(s,x) \in [0,T] \times \mathbb{R}$, we define the measure $\mathbb{Q}^{s,x} \triangleq \delta_s \times \mathbb{P}^{s,x}$ on $[0,T] \times C[0,T]$, where δ_s denotes the unit point mass at s. Then $\mathbb{Q}^{s,x}$ is also continuous in (s,x) [6, Thm. 2.8, p. 23], which means that

$$\int_{C[0,T]} f(s_n, y)\, d\mathbb{P}^{s_n,x_n}(y) \to \int_{C[0,T]} f(s,y)\, d\mathbb{P}^{s,x}(y) \tag{4.8}$$

whenever $s_n \to s$, $x_n \to x$ and f defined on $[0,T] \times C[0,T]$ is a bounded function that is continuous except on a $\mathbb{Q}^{s,x}$-null set.

We define $\tau\colon [0,T] \times C[0,T] \to [0,T]$ by

$$\tau(s,y) \triangleq T \wedge \min\left\{ t \in [s,T] : y(t) \notin \left(0, \frac{K}{\gamma}\right) \right\},$$

so that $\theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x} \wedge T = \tau\big(s, X^{s,x}\big)$. For $\mathbb{P}$-almost every ω, if $\tau(s, X^{s,x}(\omega)) < T$, then there is a sequence $\epsilon_n \downarrow 0$, depending on s, x, and ω, such that $X^{s,x}_{\tau(s,X^{s,x}(\omega))+\epsilon_n} \notin \left[0, \frac{K}{\gamma}\right]$ for every n. Indeed, if $X^{s,x}(\omega)$ exits $\left(0, \frac{K}{\gamma}\right)$ at 0, this follows from the fact that $c > 0$ and all other terms on the right-hand side of (2.5) are zero at the time of exit. On the other hand, if $X^{s,x}(\omega)$ exits $\left(0, \frac{K}{\gamma}\right)$ at $\frac{K}{\gamma}$, this follows from the nondegeneracy of the diffusion term in (2.5). Because of the fact that, almost surely, if $X^{s,x}$ exits $(0, \frac{K}{\gamma})$ strictly prior to time T, then it also exits $[0, \frac{K}{\gamma}]$ immediately thereafter, a small perturbation of the path of $X^{s,x}$ results in a small perturbation of the time of exit. More precisely, for every $(s,x) \in [0,T] \times \mathbb{R}$, τ is continuous except on a $\mathbb{Q}^{s,x}$-null set. We conclude by rewriting (4.7) as

$$\begin{aligned} &k(s_n, x_n) \\ &= e^{rs_n} \int_{C[0,T]} \left[\int_{s_n}^{\tau(s_n,y)} e^{-ru} \widetilde{c}\, du + e^{-r\tau(s_n,y)} \psi\big(\tau(s_n,y), y(\tau(s_n,y))\big) \right] d\mathbb{P}^{s_n,x_n}(y) \end{aligned}$$

and observing that because the argument of ψ is in $\partial_p D_0$, where ψ is bounded and continuous, (4.8) implies $k(s_n,x_n) \to k(s,x)$ as $s_n \to s$, $x_n \to x$. □

4.4. Characterization of game value.

PROPOSITION 4.5 (Case I). *Assume $c \le rK$. In this case, we define*

$$v_g(s,x) \triangleq \sup_{\tau \in \mathcal{S}_T^{s,x}, \tau \le \theta^{s,x}_{\frac{K}{\gamma}}} e^{rs} \mathbb{E}\left[\int_s^\tau e^{-ru} c\, du + e^{-r\tau} \psi(\tau, X_\tau^{s,x}) \right] \tag{4.9}$$

for $(s,x) \in [0,T] \times \big[0, \frac{K}{\gamma}\big]$, *and we define* $v_g(s,x) \triangleq \gamma x$ *for* $(s,x) \in [0,T] \times (\frac{K}{\gamma}, \infty)$. *Then* $v_g = \underline{v}_g = \overline{v}_g$ *on* $[0,T] \times \big[0,\infty\big)$. *Furthermore,*

$$v_g(s,x) = \inf_{\rho \in \mathcal{S}_T^{s,x}} \sup_{\tau \in \mathcal{S}_T^{s,x}} \widetilde{J}_g(s,x;\rho,\tau) = \sup_{\tau \in \mathcal{S}_T^{s,x}} \inf_{\rho \in \mathcal{S}_T^{s,x}} \widetilde{J}_g(s,x;\rho,\tau) \tag{4.10}$$

for $0 \le s \le T$ *and* $x \ge 0$.

Proof. The claims about v_g for $x > \frac{K}{\gamma}$ follow immediately from Lemmas 4.1 and 4.2. We thus restrict our attention to $0 \le x \le \frac{K}{\gamma}$.

Step 1: *Construction of an upper bound on* v_g. Define $h_1(t,x) \triangleq K$ and $h_2(t,x) \triangleq x$ for $0 \le t \le T$ and $0 \le x \le \frac{K}{\gamma}$. Both h_1 and h_2 dominate ψ on $[0,T] \times \big[0, \frac{K}{\gamma}\big]$. Because $c \le rK$, we have $-\mathcal{L}_g h_1 + c \le 0$. Therefore, for any stopping time $\tau \in \mathcal{S}_T^{s,x}$ satisfying $\tau \le \theta^{s,x}_{\frac{K}{\gamma}}$, (4.5) implies

$$\begin{aligned} h_1(s,x) &\ge e^{rs}\mathbb{E}\left[\int_s^\tau e^{-ru} c\, du + e^{-r\tau} h_1(\tau, X_\tau^{s,x})\right] \\ &\ge e^{rs}\mathbb{E}\left[\int_s^\tau e^{-ru} c\, du + e^{-r\tau} \psi(\tau, X_\tau^{s,x})\right]. \end{aligned} \tag{4.11}$$

It follows that $v_g \le h_1$ on $[0,T] \times \big[0, \frac{K}{\gamma}\big]$. On the other hand,

$$\mathcal{L}_g h_2 = c + \delta(x - g(x)) \ge c \tag{4.12}$$

because of (3.3), and the above argument applied with h_2 in place of h_1 yields $v_g \le h_2$ on $[0,T] \times \big[0, \frac{K}{\gamma}\big]$. We conclude that

$$v_g(s,x) \le x \wedge K \text{ for } 0 \le s \le T,\ 0 \le x \le \frac{K}{\gamma}. \tag{4.13}$$

By definition, $v_g(T,\cdot) = \psi(T,\cdot) = \varphi(T,\cdot)$. Because of this and (4.13),

$$v_g(s,x) \le \varphi(s,x) \text{ for } 0 \le s \le T,\ 0 \le x \le \frac{K}{\gamma}. \tag{4.14}$$

Step 2: *Optimal stopping time.* The theory of optimal stopping we use here requires that we replace ψ on the right-hand side of (4.9) with a continuous function. Let $\widetilde{c} \in (0,c)$ be given, and let k be the continuous function defined by (4.7). For $0 \le s < T$ and $0 < x < \frac{K}{\gamma}$, we have

$$\begin{aligned} k(s,x) &< e^{rs}\mathbb{E}\Bigg[\int_s^{\theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T} e^{-ru} c\, du \\ &\qquad + e^{-r(\theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T)} \psi\left(\theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T, X^{s,x}_{\theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T}\right)\Bigg] \\ &\le v_g(s,x). \end{aligned} \tag{4.15}$$

Set $\widetilde{\psi} = \psi \vee k$. Since $k(T,x) = \psi(T,x) \ge \gamma x$ for $0 \le x \le \frac{K}{\gamma}$ and $\psi(s,x) = \gamma x$ for $0 \le s < T$, $0 \le x \le \frac{K}{\gamma}$, we have $\widetilde{\psi}(s,x) = \max\{\gamma x, k(s,x)\}$ for $0 \le s \le T$, $0 \le x \le \frac{K}{\gamma}$. Being the maximum of two continuous functions, $\widetilde{\psi}$ is continuous. Also, $\psi \le \widetilde{\psi} \le v_g$.

According to the principle of dynamic programming,

$$v_g(s,x) = \sup_{\tau\in\mathcal{S}_T^{s,x},\tau\le\theta^{s,x}_{\frac{K}{\gamma}}} e^{rs}\mathbb{E}\left[\int_s^\tau e^{-ru}c\,du + e^{-r\tau}v_g(\tau, X^{s,x}_\tau)\right]. \tag{4.16}$$

Comparing (4.9) and (4.16), we see that

$$v_g(s,x) = \sup_{\tau\in\mathcal{S}_T^{s,x},\tau\le\theta^{s,x}_{\frac{K}{\gamma}}} e^{rs}\mathbb{E}\left[\int_s^\tau e^{-ru}c\,du + e^{-r\tau}\widetilde{\psi}(\tau, X^{s,x}_\tau)\right]. \tag{4.17}$$

We fix $(s,x)\in[0,T]\times\left[0,\frac{K}{\gamma}\right]$ and define

$$Y^{s,x}_t \triangleq e^{rs}\left[\int_s^t e^{-ru}c\,du + e^{-rt}v_g(t,X^{s,x}_t)\right] \text{ for } s\le t\le \theta^{s,x}_0\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge T.$$

We set $\overline{\tau}\triangleq\min\{t\in[s,T]: v_g(t,X^{s,x}_t)=\widetilde{\psi}(t,X^{s,x}_t)\}$. Since $v_g=\widetilde{\psi}$ on $\partial_p D_0$, we have $\overline{\tau}\le\theta^{s,x}_0\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge T$. According to the theory of optimal stopping applied to (4.17) (see, e.g., [25, Thms. D.12, D.13] or [35, pp. 124–127]), $Y^{s,x}_{t\wedge\theta^{s,x}_0\wedge\theta^{s,x}_{\frac{K}{\gamma}}}$ is a supermartingale and the stopped process $Y^{s,x}_{t\wedge\overline{\tau}}$ is a martingale.

Step 3: *Optimal strategies for the game.* From (4.15) and the fact that $\widetilde{\psi}=\psi\vee k$, we see that $\overline{\tau}=\min\{t\in[s,T]: v_g(t,X^{s,x}_t)=\psi(t,X^{s,x}_t)\}$ and

$$v_g(\overline{\tau},X^{s,x}_{\overline{\tau}})=\psi(\overline{\tau},X^{s,x}_{\overline{\tau}})=\begin{cases}\gamma X^{s,x}_{\overline{\tau}} & \text{if } \overline{\tau}<T,\\ (X^{s,x}_T\wedge L)\vee(\gamma X^{s,x}_T) & \text{if } \overline{\tau}=T.\end{cases} \tag{4.18}$$

Define $\tau^*=\infty$ if $\overline{\tau}=T$ and $0<X^{s,x}_T<\frac{L}{\gamma}$, and define $\tau^*=\overline{\tau}$ otherwise so that $\tau^*\in\mathcal{S}^{s,x}$ and $\overline{\tau}=\tau^*\wedge T$. We have

$$v_g(\tau^*,X^{s,x}_{\tau^*})=\gamma X^{s,x}_{\tau^*} \text{ if } \tau^*\le T. \tag{4.19}$$

For every $\rho\in\mathcal{S}^{s,x}_K$, (4.19), (4.13), and the fact that $v_g(T,x)=x\wedge L$ when $0\le x\le\frac{L}{\gamma}$ imply

$$\begin{aligned} &J_g(s,x;\rho,\tau^*)\\ &= e^{rs}\mathbb{E}\Bigg[\int_s^{\rho\wedge\tau^*\wedge T} e^{-ru}c\,du + e^{-r(\rho\wedge\tau^*\wedge T)}\big(\mathbb{I}_{\{\tau^*\le\rho\wedge T\}}\gamma X^{s,x}_{\tau^*}+\mathbb{I}_{\{\rho<\tau^*\}}K\\ &\qquad\qquad + \mathbb{I}_{\{\rho\wedge\tau^*=\infty\}}(X^{s,x}_T\wedge L)\big)\Bigg]\\ &\ge e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\tau^*\wedge T} e^{-ru}c\,du + e^{-r(\rho\wedge\tau^*\wedge T)}v_g(\rho\wedge\tau^*\wedge T, X^{s,x}_{\rho\wedge\tau^*\wedge T})\right]\\ &= \mathbb{E}Y^{s,x}_{\rho\wedge\overline{\tau}} = \mathbb{E}Y^{s,x}_s = v_g(s,x). \end{aligned} \tag{4.20}$$

This implies $\underline{v}_g(s,x)\ge v_g(s,x)$.

To show that $v_g(s,x) \geq \overline{v}_g(s,x)$, we set $\rho^* \triangleq \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x}$, which is in $\mathcal{S}_K^{s,x}$. For every $\tau \in \mathcal{S}^{s,x}$, we have $\rho^* \wedge \tau \wedge T \in \mathcal{S}_T^{s,x}$, and thus

$$\begin{aligned} & J_g(s,x;\rho^*,\tau) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\rho^*\wedge\tau\wedge T} e^{-ru}c\,du + e^{-r(\rho^*\wedge\tau\wedge T)}\big(\mathbb{I}_{\{\tau\leq\rho^*\wedge T\}}\gamma X_\tau^{s,x} + \mathbb{I}_{\{\rho^*<\tau\}}K\right. \\ &\qquad\qquad \left. + \mathbb{I}_{\{\rho^*\wedge\tau=\infty\}}(X_T^{s,x}\wedge L)\big)\right] \\ &\leq e^{rs}\mathbb{E}\left[\int_s^{\rho^*\wedge\tau\wedge T} e^{-ru}c\,du + e^{-r(\rho^*\wedge\tau\wedge T)}\psi(\rho^*\wedge\tau\wedge T, X_{\rho^*\wedge\tau\wedge T}^{s,x})\right] \\ &\leq v_g(s,x). \end{aligned} \tag{4.21}$$

This implies $\overline{v}_g(s,x) \leq v_g(s,x)$. We conclude that $v_g = \underline{v}_g = \overline{v}_g$.

Step 4: *Proof of* (4.10). With $\overline{\tau} \in \mathcal{S}_T^{s,x}$ as defined in Step 2, we have from (4.14) and (4.18) that for every $\rho \in \mathcal{S}_T^{s,x}$,

$$\begin{aligned} & \widetilde{J}_g(s,x;\rho,\overline{\tau}) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\overline{\tau}} e^{-ru}c\,du + e^{-r(\rho\wedge\overline{\tau})}\big(\mathbb{I}_{\{\overline{\tau}<\rho\}}\gamma X_{\overline{\tau}}^{s,x} + \mathbb{I}_{\{\rho\leq\overline{\tau}\}}\varphi(\rho, X_\rho^{s,x})\big)\right] \\ &\geq e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\overline{\tau}} e^{-ru}c\,du + e^{-r(\rho\wedge\overline{\tau})}v_g(\rho\wedge\overline{\tau}, X_{\rho\wedge\overline{\tau}}^{s,x})\right] \\ &= \mathbb{E}Y_{\rho\wedge\overline{\tau}}^{s,x} \;=\; \mathbb{E}Y_s^{s,x} \;=\; v_g(s,x). \end{aligned}$$

On the other hand, with ρ^* defined as in Step 3 and

$$\overline{\rho} \triangleq \rho^* \wedge T = \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x} \wedge T \in \mathcal{S}_T^{s,x}, \tag{4.22}$$

we have $\varphi(\overline{\rho}, X_{\overline{\rho}}^{s,x}) = \psi(\overline{\rho}, X_{\overline{\rho}}^{s,x})$. Thus, for every $\tau \in \mathcal{S}_T^{s,x}$,

$$\begin{aligned} & \widetilde{J}_g(s,x;\overline{\rho},\tau) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}\wedge\tau} e^{-ru}c\,du + e^{-r(\overline{\rho}\wedge\tau)}\big(\mathbb{I}_{\{\tau<\overline{\rho}\}}\gamma X_\tau^{s,x} + \mathbb{I}_{\{\overline{\rho}\leq\tau\}}\varphi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\big)\right] \\ &= e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}\wedge\tau} e^{-ru}c\,du + e^{-r(\overline{\rho}\wedge\tau)}\psi(\overline{\rho}\wedge\tau, X_{\overline{\rho}\wedge\tau}^{s,x})\right] \\ &\leq v_g(s,x). \end{aligned} \tag{4.23}$$

We complete the argument as in Step 3. ◻

PROPOSITION 4.6 (Case II). *Assume $\delta K \leq c$. In this case, we define*

$$v_g(s,x) \triangleq \inf_{\rho\in\mathcal{S}_T^{s,x},\rho\leq\theta_{\frac{K}{\gamma}}^{s,x}} e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru}c\,du + e^{-r\rho}\varphi(\rho, X_\rho^{s,x})\right] \tag{4.24}$$

for $(s,x) \in [0,T] \times \big[0, \frac{K}{\gamma}\big]$, and we define $v_g(s,x) \triangleq \gamma x$ for $(s,x) \in [0,T] \times (\frac{K}{\gamma}, \infty)$. Then $v_g = \underline{v}_g = \overline{v}_g$ on $[0,T] \times \big[0,\infty\big)$. Furthermore,

$$v_g(s,x) = \inf_{\rho\in\mathcal{S}_T^{s,x}} \sup_{\tau\in\mathcal{S}_T^{s,x}} \widetilde{J}_g(s,x;\rho,\tau) = \sup_{\tau\in\mathcal{S}_T^{s,x}} \inf_{\rho\in\mathcal{S}_T^{s,x}} \widetilde{J}_g(s,x;\rho,\tau) \tag{4.25}$$

for $0 \leq s \leq T$ and $x \geq 0$.

Proof. The claims about v_g for $x > \frac{K}{\gamma}$ follow immediately from Lemmas 4.1 and 4.2. We thus restrict our attention to $0 \le x \le \frac{K}{\gamma}$.

Step 1: *Construction of bounds on* v_g. Define $h_3(t,x) \triangleq \gamma x$ and $h_2(t,x) \triangleq x$ for $0 \le t \le T$ and $0 \le x \le \frac{K}{\gamma}$, so that $h_3 \le \varphi \le h_2$. For $(t,x) \in [0,T] \times \big[0, \frac{K}{\gamma}\big]$, we have (4.12) and

$$\mathcal{L}_g h_3(t,x) = c\gamma + \delta\gamma\big(x - g(t,x)\big) \le c + (1-\gamma)(\delta\gamma x - c) \le c. \tag{4.26}$$

Let $\rho \in \mathcal{S}_T^{s,x}$ satisfy $\rho \le \theta^{s,x}_{\frac{K}{\gamma}}$, and apply (4.5) and (4.26) to conclude

$$\begin{aligned} h_3(s,x) &\le e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru} c\,du + e^{-r\rho} h_3(\rho, X_\rho^{s,x})\right] \\ &\le e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru} c\,du + e^{-r\rho} \varphi(\rho, X_\rho^{s,x})\right]. \end{aligned} \tag{4.27}$$

Taking the infimum over ρ, we obtain

$$\gamma x \le v_g(s,x) \text{ for } 0 \le s \le T,\ 0 \le x \le \frac{K}{\gamma}. \tag{4.28}$$

We repeat the above argument with h_2 and $\rho = \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}} \wedge T$, using (4.12) to reverse the first inequality and $\varphi \le h_2$ to reverse the second, to obtain

$$\begin{aligned} h_2(s,x) &\ge e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru} c\,du + e^{-r\rho} h_2(\rho, X_\rho^{s,x})\right] \\ &\ge e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru} c\,du + e^{-r\rho} \varphi(\rho, X_\rho^{s,x})\right] \\ &\ge v_g(s,x) \text{ for } 0 \le s \le T,\ 0 \le x \le \frac{K}{\gamma}. \end{aligned} \tag{4.29}$$

In fact, since for $(s,x) \in [0,T) \times \big(0, \frac{K}{\gamma}\big)$, with positive probability $X^{s,x}$ exits $[0,T] \times \big(0, \frac{K}{\gamma}\big)$ through the set $\{T\} \times \big(L, \frac{K}{\gamma}\big]$, where h_2 is strictly greater than φ, the second inequality in (4.29) is strict for such (s,x). This implies

$$v_g(s,x) < x \text{ for } 0 \le s < T,\ 0 < x < \frac{K}{\gamma}. \tag{4.30}$$

Step 2: *Optimal stopping time.* Let $\widetilde{c} \in (c, \infty)$ be given and let k be defined by (4.7). For $0 \le s < T$ and $0 < x < \frac{K}{\gamma}$, using the second part of (4.7), we have

$$\begin{aligned} k(s,x) > e^{rs}\mathbb{E}\Bigg[&\int_s^{\theta_0^{s,x}\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge T} e^{-ru} c\,du \\ &+ e^{-r(\theta_0^{s,x}\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge T)} \varphi\left(\theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T, X^{s,x}_{\theta_0^{s,x}\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge T}\right)\Bigg] \\ \ge v_g(s,x).& \end{aligned} \tag{4.31}$$

We set $\widetilde{\varphi} = \varphi \wedge k$. Because $\varphi(T,x) = k(T,x) \le x \wedge K$ for $0 \le x \le \frac{K}{\gamma}$, and $\varphi(t,x) = x \wedge K$ for $0 \le t < T$, $0 \le x \le \frac{K}{\gamma}$, we have

$$\widetilde{\varphi}(t,x) = (x \wedge K) \wedge k(t,x) \text{ for } 0 \le t \le T,\ 0 \le x \le \frac{K}{\gamma}.$$

This shows that $\widetilde{\varphi}$ is continuous. From (4.24) we have $v_g \le \varphi$, and hence $v_g \le \widetilde{\varphi} \le \varphi$. According to the principle of dynamic programming,

$$v_g(s,x) \triangleq \inf_{\rho \in \mathcal{S}_T^{s,x}, \rho \le \theta^{s,x}_{\frac{K}{\gamma}}} e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru}c\,du + e^{-r\rho}v_g(\rho, X_\rho^{s,x})\right]. \tag{4.32}$$

Comparing (4.24) and (4.32), we see that

$$v_g(s,x) \triangleq \inf_{\rho \in \mathcal{S}_T^{s,x}, \rho \le \theta^{s,x}_{\frac{K}{\gamma}}} e^{rs}\mathbb{E}\left[\int_s^\rho e^{-ru}c\,du + e^{-r\rho}\widetilde{\varphi}(\rho, X_\rho^{s,x})\right]. \tag{4.33}$$

We fix $(s,x) \in [0,T] \times \left[0, \frac{K}{\gamma}\right]$ and define

$$Z_t^{s,x} = e^{rs}\left[\int_s^t e^{-ru}c\,du + e^{-rt}v_g(t, X_t^{s,x})\right] \text{ for } s \le t \le \theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T.$$

We set

$$\overline{\rho} \triangleq \min\{t \in [s,T] : v_g(t, X_t^{s,x}) = \widetilde{\varphi}(t, X_t^{s,x})\}. \tag{4.34}$$

Since $v_g = \widetilde{\varphi}$ on $\partial_p D_0$, we have $\overline{\rho} \le \theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge T$. According to the theory of optimal stopping, $Z^{s,x}_{t \wedge \theta_0^{s,x} \wedge \theta^{s,x}_{\frac{K}{\gamma}}}$ is a submartingale and the stopped process $Z^{s,x}_{t\wedge\overline{\rho}}$ is a martingale.

Step 3: *Optimal strategies for the game.* Because of (4.31), we have that $\overline{\rho} = \min\{t \in [s,T] : v_g(t, X_t^{s,x}) = \varphi(t, X_t^{s,x})\}$. In particular, $v_g(\overline{\rho}, X_{\overline{\rho}}^{s,x}) = X_{\overline{\rho}}^{s,x} \wedge K$ on $\{\overline{\rho} < T\}$. Inequality (4.30) then implies

$$v_g(\overline{\rho}, X_{\overline{\rho}}^{s,x}) = K < X_{\overline{\rho}}^{s,x} \text{ on } \{\overline{\rho} < \theta_0^{s,x} \wedge T\}. \tag{4.35}$$

Define

$$\rho^* \triangleq \begin{cases} \infty & \text{if } \overline{\rho} = T \text{ and } X_{\overline{\rho}}^{s,x} > 0, \\ \overline{\rho} & \text{otherwise,} \end{cases}$$

so that $\rho^* \in \mathcal{S}_K^{s,x}$ and $\overline{\rho} = \rho^* \wedge T$. For every $\tau \in \mathcal{S}^{s,x}$, (4.28), (4.35), and the fact that $v_g(T,x) = \varphi(T,x) \ge x \wedge L$ when $0 \le x \le \frac{K}{\gamma}$ imply

$$\begin{aligned}
&J_g(s,x;\rho^*,\tau) \\
&= e^{rs}\mathbb{E}\Bigg[\int_s^{\rho^*\wedge\tau\wedge T} ce^{-ru}\,du + e^{-r(\rho^*\wedge\tau\wedge T)}\big(\mathbb{I}_{\{\tau\le\rho^*\wedge T\}}\gamma X_\tau^{s,x} + \mathbb{I}_{\{\rho^*<\tau\}}K \\
&\qquad + \mathbb{I}_{\{\rho^*\wedge\tau=\infty\}}(X_T^{s,x}\wedge L)\big)\Bigg] \\
&\le e^{rs}\mathbb{E}\left[\int_s^{\rho^*\wedge\tau\wedge T} ce^{-ru}\,du + e^{-r(\rho^*\wedge\tau\wedge T)}v_g(\rho^*\wedge\tau\wedge T, X^{s,x}_{\rho^*\wedge\tau\wedge T})\right] \\
&= \mathbb{E}Z^{s,x}_{\overline{\rho}\wedge\tau} \;=\; Z_s^{s,x} \;=\; v_g(s,x).
\end{aligned} \tag{4.36}$$

This implies $\overline{v}_g(s,x) \le v_g(s,x)$.

We set

$$\overline{\tau} \triangleq \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x} \wedge T, \tag{4.37}$$

$$\tau^* \triangleq \begin{cases} \overline{\tau} & \text{if } \overline{\tau} < T, \\ T & \text{if } \overline{\tau} = T,\ X_T^{s,x} \geq \frac{L}{\gamma} \text{ or if } \overline{\tau} = T,\ X_T^{s,x} = 0, \\ \infty & \text{if } \overline{\tau} = T, 0 < X_T^{s,x} < \frac{L}{\gamma}, \end{cases} \tag{4.38}$$

so that $\tau^* \in \mathcal{S}^{s,x}$. For every $\rho \in \mathcal{S}_K^{s,x}$, we have

$$\begin{aligned} &J_g(s,x;\rho,\tau^*) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\tau^*\wedge T} ce^{-ru}\,du + e^{-r(\rho\wedge\tau^*\wedge T)}\big(\mathbb{I}_{\{\tau^*\leq\rho\wedge T\}}\gamma X_\tau^{s,x} + \mathbb{I}_{\{\rho<\tau^*\}}K \right. \\ &\qquad\qquad \left. + \mathbb{I}_{\{\rho\wedge\tau^*=\infty\}}(X_T^{s,x}\wedge L)\big)\right] \\ &\geq e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\tau^*\wedge T} ce^{-ru}\,du + e^{-r(\rho\wedge\tau^*\wedge T)}\varphi(\rho^*\wedge\tau\wedge T, X_{\rho\wedge\tau^*\wedge T}^{s,x})\right] \\ &\geq v_g(s,x). \end{aligned} \tag{4.39}$$

This implies $\underline{v}_g(s,x) \geq v_g(s,x)$. We conclude that $v_g = \underline{v}_g = \overline{v}_g$.

Step 4. *Proof of* (4.25). With $\overline{\rho} \in \mathcal{S}^{s,x}$ given by (4.34), we have $v_g(\overline{\rho}, X_{\overline{\rho}}^{s,x}) = \varphi(\overline{\rho}\wedge\tau, X_{\overline{\rho}\wedge\tau}^{s,x})$, and (4.28) implies that for $\tau \in \mathcal{S}_T^{s,x}$,

$$\begin{aligned} &\widetilde{J}_g(s,x;\overline{\rho},\tau) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}\wedge\tau} e^{-ru}c\,du + e^{-r(\overline{\rho}\wedge\tau)}\big(\mathbb{I}_{\{\tau<\overline{\rho}\}}\gamma X_\tau^{s,x} + \mathbb{I}_{\{\overline{\rho}\leq\tau\}}\varphi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\big)\right] \\ &\leq e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}\wedge\tau} e^{-ru}c\,du + e^{-r(\overline{\rho}\wedge\tau)}v_g(\overline{\rho}\wedge\tau, X_{\overline{\rho}\wedge\tau}^{s,x})\right] \\ &= \mathbb{E}Z_{\overline{\rho}\wedge\tau}^{s,x} \;=\; Z_s^{s,x} \;=\; v_g(s,x). \end{aligned}$$

With $\overline{\tau} \in \mathcal{S}_T^{s,x}$ defined by (4.37), we have for every $\rho \in \mathcal{S}_T^{s,x}$,

$$\begin{aligned} &\widetilde{J}_g(s,x;\rho,\overline{\tau}) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\overline{\tau}} e^{-rs}c\,du + e^{-r(\rho\wedge\overline{\tau})}\big(\mathbb{I}_{\{\overline{\tau}<\rho\}}\gamma X_{\overline{\tau}}^{s,x} + \mathbb{I}_{\{\rho\leq\overline{\tau}\}}\varphi(\rho, X_\rho^{s,x})\big)\right] \\ &= e^{rs}\mathbb{E}\left[\int_s^{\rho\wedge\overline{\tau}} e^{-rs}c\,du + e^{-r(\rho\wedge\overline{\tau})}\varphi(\rho\wedge\overline{\tau}, X_{\rho\wedge\overline{\tau}}^{s,x})\right] \;\geq\; v_g(s,x). \end{aligned} \tag{4.40}$$

We complete the argument as in Step 3. ∎

Proposition 4.7 (overlapping case). *Assume $\delta K \leq c \leq rK$. In this case, v_g defined by* (4.9) *agrees with v_g defined by* (4.24), *and for $0 \leq s \leq T$ and $0 \leq x \leq \frac{K}{\gamma}$,*

$$\begin{aligned} &v_g(s,x) \\ &= e^{rs}\mathbb{E}\left[\int_s^{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T} e^{-ru}c\,du + e^{-r(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T)}\psi\left(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T, X_{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T}^{s,x}\right)\right] \\ &= e^{rs}\mathbb{E}\left[\int_s^{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T} e^{-ru}c\,du + e^{-r(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T)}\varphi\left(\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T, X_{\theta_0^{s,x}\wedge\theta_{\frac{K}{\gamma}}^{s,x}\wedge T}^{s,x}\right)\right]. \end{aligned} \tag{4.41}$$

Furthermore,

$$\gamma x < v_g(s,x) < x \wedge K \textit{ for } 0 \le s < T \textit{ and } 0 < x < \frac{K}{\gamma}. \tag{4.42}$$

Proof. The function v_g defined by (4.9) satisfies (4.10), the function v_g defined by (4.24) satisfies (4.25), and so these definitions of v_g coincide. With $\overline{\rho} \triangleq \overline{\tau}$ given by (4.22) and (4.37), inequalities (4.23) and (4.40) imply

$$\begin{aligned}
&e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}} e^{-ru} c\, du + e^{-r\overline{\rho}}\varphi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\right] \\
&= e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}} e^{-ru} c\, du + e^{-r\overline{\rho}}\psi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\right] \\
&\le v_g(s,x) \\
&\le e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}} e^{-ru} c\, du + e^{-r\overline{\rho}}\varphi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\right],
\end{aligned}$$

which gives us (4.41).

We return to (4.11), replacing τ with $\overline{\rho}$ and using the fact that when $0 \le s < T$ and $0 < x < \frac{K}{\gamma}$, there is a positive probability that $X^{s,x}$ exits $[0,T] \times \left[0, \frac{K}{\gamma}\right]$ through the set $\left\{(t,x) : t = T, 0 < x < \frac{K}{\gamma}\right\}$, where $h_1 = K$ is strictly larger than ψ. This implies

$$K > e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}} e^{-ru} c\, du + e^{-r\overline{\rho}}\psi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\right] = v_g(s,x).$$

The second inequality in (4.42) follows from this and (4.30). For the first inequality in (4.42), we replace ρ in (4.27) with $\overline{\rho}$ and use the fact that when $0 \le s < T$ and $0 < x < \frac{K}{\gamma}$, there is a positive probability that $X^{s,x}$ exits $[0,T] \times \left[0, \frac{K}{\gamma}\right]$ through the set $\left\{(t,x) : t = T, 0 < x < \frac{L}{\gamma}\right\}$, where $h_3 = \gamma x$ is strictly smaller than φ, to obtain

$$\gamma x < e^{rs}\mathbb{E}\left[\int_s^{\overline{\rho}} e^{-ru} c\, du + e^{-r\overline{\rho}}\varphi(\overline{\rho}, X_{\overline{\rho}}^{s,x})\right] = v_g(s,x). \qquad \square$$

4.5. Membership of v_g in $\mathcal{G}$. To show that $v_g \in \mathcal{G}$ whenever $g \in \mathcal{G}$, we must verify that v_g is continuous and satisfies (3.1)–(3.3). Property (3.1) is provided by Propositions 4.5 and 4.6. When $c \le rK$, we obtain the lower bound in (3.3) directly from (4.9) and the fact that $\psi \ge \gamma x$, and (4.13) provides the upper bound. When $\delta K \le C$, the upper bound in (3.3) comes from (4.24), the fact that $\varphi \le K$ on $[0,T] \times \left[0, \frac{K}{\gamma}\right]$, and (4.29). The lower bound comes from (4.28). It remains to verify that v_g is continuous and satisfies (3.2), which is the subject of this section.

LEMMA 4.8. *We have*

$$0 \le v_g(s,y) - v_g(s,x) \le y - x \textit{ for } 0 \le s \le T \textit{ and } 0 \le x \le y. \tag{4.43}$$

Proof. In Step 4 of the proofs of Propositions 4.5 and 4.6, we produced stopping times $\overline{\rho}, \overline{\tau} \in \mathcal{S}_T^{s,x}$ such that

$$\widetilde{J}_g(s,x;\overline{\rho},\tau) \le v_g(s,x) \le \widetilde{J}_g(s,x;\rho,\overline{\tau}) \text{ for all } \rho,\tau \in \mathcal{S}_T^{s,x}. \tag{4.44}$$

It follows from this that $v_g(s,x) = \widetilde{J}_g(s,x;\overline{\rho},\overline{\tau})$. Relation (4.44) was developed for $(s,x) \in [0,T] \times \big[0, \frac{K}{\gamma}\big]$, but in light of Lemma 4.2, it holds as well for $(s,x) \in [0,T] \times \big[\frac{K}{\gamma}, \infty\big)$ if we define $\overline{\rho} = \overline{\tau} = s$ in this case.

We note that ψ and φ satisfy (3.2), and we use the representations (4.10), (4.25) to show that v_g does as well. Without loss of generality, we consider only the case $s = 0$. We let $0 \le x \le y < \infty$ be given. Then $X_t^{0,x} \le X_t^{0,y}$ for $0 \le t \le T$, almost surely, and $\mathcal{S}_T^{0,x} \subset \mathcal{S}_T^{0,y}$.

Consider the nonnegative martingale $Z_t = e^{-\sigma W_t - \frac{1}{2}\sigma^2 t}$. We compute

$$\begin{aligned} d\big((X_t^{0,y} - X_t^{0,x})Z_t\big) &= (r-\sigma^2)(X_t^{0,y} - X_t^{0,x})Z_t\,dt \\ &\quad -\delta\big[(X_t^{0,y} - X_t^{0,x}) - \big(g(t,X_t^{0,y}) - g(t,X_t^{0,x})\big)\big]Z_t\,dt \\ &\le (r-\sigma^2)(X_t^{0,y} - X_t^{0,x})Z_t\,dt. \end{aligned}$$

Gronwall's inequality implies $(X_t^{0,y} - X_t^{0,x})Z_t \le (y-x)e^{(r-\sigma^2)t}$, or equivalently,

$$e^{-rt}\big(X_t^{0,y} - X_t^{0,x}\big) \le (y-x)e^{\sigma W_t - \frac{1}{2}\sigma^2 t},\ 0 \le t \le \tau_0^{0,x}.$$

Let $\overline{\tau}, \overline{\rho} \in \mathcal{S}_T^{0,x}$ be the stopping times appearing in (4.44) corresponding to the initial condition $(0,x)$. For every $\tau \in \mathcal{S}^{0,x}$, we have

$$\begin{aligned} &\widetilde{J}_g(0,x;\overline{\rho},\tau) \\ &= \mathbb{E}\left[\int_0^{\overline{\rho}\wedge\tau} ce^{-ru}\,du + e^{-r(\overline{\rho}\wedge\tau)}\big(\mathbb{I}_{\{\tau<\overline{\rho}\}}\psi(\tau, X_\tau^{0,x}) + \mathbb{I}_{\{\overline{\rho}\le\tau\}}\varphi(\overline{\rho}, X_{\overline{\rho}}^{0,x})\big)\right] \\ &= \widetilde{J}_g(0,y;\overline{\rho},\tau) - \mathbb{E}\Big[e^{-r(\overline{\rho}\wedge\tau)}\Big(\mathbb{I}_{\{\tau<\overline{\rho}\}}\big(\psi(X_\tau^{0,y}) - \psi(X_\tau^{0,x})\big) \\ &\qquad\qquad + \mathbb{I}_{\{\overline{\rho}\le\tau\}}\big(\varphi(\overline{\rho}, X_{\overline{\rho}}^{0,y}) - \varphi(\overline{\rho}, X_{\overline{\rho}}^{0,x})\big)\Big)\Big] \\ &\ge \widetilde{J}_g(0,y;\overline{\rho},\tau) - \mathbb{E}\Big[e^{-r(\overline{\rho}\wedge\tau)}(X_{\overline{\rho}\wedge\tau}^{0,y} - X_{\overline{\rho}\wedge\tau}^{0,x})\Big] \\ &\ge \widetilde{J}_g(0,y;\overline{\rho},\tau) - (y-x)\mathbb{E}e^{\sigma W(\overline{\rho}\wedge\tau) - \frac{1}{2}\sigma^2(\overline{\rho}\wedge\tau)} \\ &= \widetilde{J}_g(0,y;\overline{\rho},\tau) - (y-x). \end{aligned}$$

Furthermore, $\overline{\rho}\wedge\tau \in \mathcal{S}_T^{0,x}$ whenever $\tau \in \mathcal{S}_T^{0,y}$, and for $z = x$ and $z = y$, we have $\widetilde{J}_g(0,z;\overline{\rho},\tau) = \widetilde{J}_g(0,z;\overline{\rho},\overline{\rho}\wedge\tau)$. Therefore,

$$\begin{aligned} v_g(0,x) + y - x &= \widetilde{J}_g(0,x;\overline{\rho},\overline{\tau}) + y - x \\ &= \sup_{\tau\in\mathcal{S}_T^{0,x}} \widetilde{J}_g(0,x;\overline{\rho},\overline{\rho}\wedge\tau) + y - x \\ &\ge \sup_{\tau\in\mathcal{S}_T^{0,x}} \widetilde{J}_g(0,y;\overline{\rho},\overline{\rho}\wedge\tau) \\ &= \sup_{\tau\in\mathcal{S}_T^{0,y}} \widetilde{J}_g(0,y;\overline{\rho},\overline{\rho}\wedge\tau) \\ &= \sup_{\tau\in\mathcal{S}_T^{0,y}} \widetilde{J}_g(0,y;\overline{\rho},\tau) \\ &\ge \inf_{\rho\in\mathcal{S}_T^{0,y}} \sup_{\tau\in\mathcal{S}_T^{0,y}} \widetilde{J}_g(0,y;\rho,\tau) \\ &= v_g(0,y). \end{aligned}$$

This establishes the second inequality in (4.43).

The set of stopping times $\mathcal{S}_T^{0,x}$ is the set of all stopping times of the form $\tau \wedge \theta_0^{0,x}$, where τ is any stopping time in the set $\mathcal{S}_T$ of all stopping times satisfying $\tau \leq T$ almost surely. Therefore,

$$v_g(0,x) = \sup_{\tau \in \mathcal{S}_T} \inf_{\rho \in \mathcal{S}_T} \widetilde{J}_g(0,x;\rho \wedge \theta_0^{0,x}, \tau \wedge \theta_0^{0,x}),$$
$$v_g(0,y) = \sup_{\tau \in \mathcal{S}_T} \inf_{\rho \in \mathcal{S}_T} \widetilde{J}_g(0,y;\rho \wedge \theta_0^{0,y}, \tau \wedge \theta_0^{0,y}).$$

Thus, to prove the first inequality in (4.43), it suffices to show that

$$\widetilde{J}_g(0,x;\rho \wedge \theta_0^{0,x}, \tau \wedge \theta_0^{0,x}) \leq \widetilde{J}_g(0,y;\rho \wedge \theta_0^{0,y}, \tau \wedge \theta_0^{0,y})$$

for all $\rho, \tau \in \mathcal{S}_T$. This follows from the definition of $\widetilde{J}_g$ and $\theta_0^{0,x} \leq \theta_0^{0,y}$. ☐

The value functions of optimal stopping problems with continuous payoff functions are continuous (see [5]), and thus the representations (4.17) and (4.33) of v_g imply continuity of v_g. In this model, however, continuity can be proved without invoking the general theory. We have already shown in Lemma 4.8 that $v_g(s,x)$ is Lipschitz in $x \in [0,\infty)$, uniformly in $s \in [0,T]$. Given this, it is not difficult to show that $v_g(s,x)$ is jointly continuous in (s,x), and we do that here.

LEMMA 4.9. *The function v_g is continuous on $[0,T] \times [0,\infty)$.*

Proof. Because of Lemmas 4.1 and 4.8, we need only show for each fixed $x \in (0, \frac{K}{\gamma})$ that the function $s \mapsto v_g(s,x)$ is continuous. With x fixed, $s \in [0,T]$, $\epsilon > 0$, and $\delta > 0$, we define

$$A_{\epsilon,\delta}^{s,x} \triangleq \left\{ \max_{u \in [s, \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x} \wedge (s+\delta) \wedge T]} \left| X_u^{s,x} - x \right| \leq \epsilon \right\}.$$

Because g is bounded on $[0,T] \times [0, \frac{K}{\gamma}]$, (3.4) and (2.5) imply

$$\lim_{\delta \downarrow 0} \min_{s \in [0,T]} \mathbb{P}(A_{\epsilon,\delta}^{s,x}) = 1 \text{ for every } \epsilon > 0. \tag{4.45}$$

We proceed under the Case II assumption $\delta K \leq c$; the argument in Case I is similar. In Case II, the submartingale $Z_{t \wedge \theta_0^{s,x} \wedge \theta_{\frac{K}{\gamma}}^{s,x}}^{s,x}$ of Step 2 of the proof of Proposition 4.6 is a martingale when stopped at $\overline{\rho}$ given by (4.34). Let s and t satisfy $0 \leq s < t \leq (s+\delta) \wedge T$. Then

$$\begin{aligned} &v_g(s,x) \\ &= e^{rs} \mathbb{E}\left[\int_s^{\overline{\rho} \wedge t} e^{-rs} c\, du + e^{-r(\overline{\rho} \wedge t)} v_g(\overline{\rho} \wedge t, X_{\overline{\rho} \wedge t}^{s,x}) \right] \\ &\leq e^{rs} \mathbb{E}\left[\int_s^{\rho \wedge t} e^{-ru} c\, du + e^{-r\rho} v_g(\rho \wedge t, X_{\rho \wedge t}^{s,x}) \right] \text{ for } \rho \in \mathcal{S}_T^{s,x}, \rho \leq \theta_{\frac{K}{\gamma}}^{s,x}. \end{aligned} \tag{4.46}$$

If $\overline{\rho} < t$, then $v_g(\overline{\rho} \wedge t, X_{\overline{\rho} \wedge t}^{s,x}) = \widetilde{\varphi}(\overline{\rho}, X_{\overline{\rho}}^{s,x})$. But $\widetilde{\varphi} = \varphi \wedge k$, and (4.31) shows that for $(u,y) \in [0,T) \times \left(0, \frac{K}{\gamma}\right)$, we have $v_g(u,y) = \widetilde{\varphi}(u,y)$ if and only if $v_g(u,y) = \varphi(u,y) = y \wedge K$. This observation combined with (4.30) yields

$$v_g(u,y) = \widetilde{\varphi}(u,y) \;\Leftrightarrow\; v_g(u,y) = K \text{ for } u \in [0,T),\ y \in \left(0, \frac{K}{\gamma}\right). \tag{4.47}$$

We now choose $\epsilon > 0$ so that $0 < x - \epsilon < x + \epsilon < \frac{K}{\gamma}$. On the set $\{\overline{\rho} < t\} \cap A^{s,x}_{\epsilon,\delta}$ we have $0 < X^{s,x}_{\overline{\rho}\wedge t} = X^{s,x}_{\overline{\rho}} < \frac{K}{\gamma}$, and thus

$$v_g(\overline{\rho} \wedge t, X^{s,x}_{\overline{\rho}\wedge t}) = v_g(\overline{\rho}, X^{s,x}_{\overline{\rho}}) = K \geq v_g(t,x) - \epsilon.$$

On the set $\{\overline{\rho} \geq t\} \cap A^{s,x}_{\epsilon,\delta}$, we also have $v_g(\overline{\rho}\wedge t, X^{s,x}_{\overline{\rho}\wedge t}) \geq v_g(t,x) - \epsilon$, this time because of the Lipschitz continuity (4.43). The equality in (4.46) implies

$$v_g(s,x) \geq e^{rs}\mathbb{E}\big[e^{-r(\overline{\rho}\wedge t)} v_g(\overline{\rho} \wedge t, X^{s,x}_{\overline{\rho}\wedge t})\big] \geq e^{-r\delta}\mathbb{P}(A^{s,x}_{\epsilon,\delta})\big[v_g(t,x) - \epsilon\big]. \tag{4.48}$$

On the other hand, on the set $A^{s,x}_{\epsilon,\delta}$, we have $\theta^{s,x}_0 \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge t = t$, and the inequality in (4.46) implies

$$\begin{aligned}
&v_g(s,x) \\
&\leq e^{rs}\mathbb{E}\left[\int_s^{\theta^{s,x}_0\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge t} e^{-ru}c\,du + e^{-r(\theta^{s,x}_0\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge t)} v_g\left(\theta^{s,x}_0 \wedge \theta^{s,x}_{\frac{K}{\gamma}} \wedge t, X^{s,x}_{\theta^{s,x}_0\wedge\theta^{s,x}_{\frac{K}{\gamma}}\wedge t}\right)\right] \\
&\leq e^{rs}\int_s^{s+\delta} e^{-ru}c\,du + \big[1 - \mathbb{P}(A^{s,x}_{\epsilon,\delta})\big]K + \mathbb{E}\left[\mathbb{I}_{A^{s,x}_{\epsilon,\delta}} v_g(t, X^{s,x}_t)\right] \\
&\leq \frac{c}{r}(1 - e^{-r\delta}) + [1 - \mathbb{P}(A^{s,x}_{\epsilon,\delta})]K + \mathbb{P}(A^{s,x}_{\epsilon,\delta})\big(v_g(t,x) + \epsilon\big).
\end{aligned} \tag{4.49}$$

From (4.48) and (4.49), using the fact that $0 \leq v_g(t,x) \leq K$, we obtain

$$-[1 - e^{-r\delta}\mathbb{P}(A^{s,x}_{\epsilon,\delta})]K - \epsilon \leq v_g(s,x) - v_g(t,x) \leq \frac{c}{r}(1 - e^{-r\delta}) + [1 - \mathbb{P}(A^{s,x}_{\epsilon,\delta})]K + \epsilon.$$

Continuity of $s \mapsto v_g(s,x)$ follows from this and (4.45). □

5. Viscosity solution characterization of v_g. Propositions 4.5 and 4.6 establish (3.5). Except for the fact that we have fixed a function $g \in \mathcal{G}$ that may not satisfy the fixed point condition $v_g = g$, Proposition 4.5 says further that when $c \leq rK$, the convertible bond pricing problem reduces to the problem of optimal conversion in the region $[0,T] \times \big[0, \frac{K}{\gamma}\big]$. In particular, (4.20) and (4.21) show that the firm should use the call strategy $\rho^* = \theta^{s,x}_0 \wedge \theta^{s,x}_{\frac{K}{\gamma}}$. Proposition 4.6 shows that when $\delta K \leq c$, the convertible bond pricing problem reduces to the problem of optimal call. In particular, (4.36) and (4.39) show that the bondholder should use the conversion strategy τ^* of (4.38). Note that at maturity, τ^* mandates conversion if and only if the conversion value $\gamma X^{s,x}_T$ exceeds the par value L. These are the main assertions of Theorem 3.2.

In this section, we examine the versions of (3.8) and (3.9) appropriate for the situation with $g \in \mathcal{G}$ chosen a priori. These equations are

$$\min\{\mathcal{L}_g v - c, v - \gamma x\} = 0, \tag{5.1}$$

$$\max\{\mathcal{L}_g v - c, v - K\} = 0, \tag{5.2}$$

where $\mathcal{L}_g$ is given by (4.6). The proofs that the value function of the optimal stopping problem (4.9) satisfies (5.1) and that the value function of problem (4.24) satisfies (5.2), both in the viscosity sense on $(0,T) \times \big(0, \frac{K}{\gamma}\big)$ (see Definition 5.1 below), are standard and are omitted. Uniqueness of the continuous viscosity solutions of (5.1) and (5.2) subject to the boundary conditions (3.6) and (3.7) follows from Lemma 6.1 below; see Remark 6.2.

We refer the reader to [13] and [17] for a detailed development of the theory of second-order viscosity solutions for Hamilton–Jacobi–Bellman equations and to [34] for an application of this theory to optimal stopping.

DEFINITION 5.1. *Let v be a continuous function defined on* $(0,T)\times\big(0,\frac{K}{\gamma}\big)$.

(a) *The function v is a* viscosity subsolution *of* (5.1) (*respectively,* (5.2)) *if, for every point* $(t_0,x_0)\in(0,T)\times\big(0,\frac{K}{\gamma}\big)$ *and for every "test function"* $h\in C^{1,2}\big((0,T)\times\big(0,\frac{K}{\gamma}\big)\big)$ *satisfying* $v\le h$ *on* $(0,T)\times\big(0,\frac{K}{\gamma}\big)$ *and* $v(t_0,x_0)=h(t_0,x_0)$, *we have* $\min\{\mathcal{L}_g h(t_0,x_0)-c, h(t_0,x_0)-\gamma x_0\}\le 0$ (*respectively,* $\max\{\mathcal{L}_g h(t_0,x_0)-c, h(t_0,x_0)-K\}\le 0$).

(b) *The function v is a* viscosity supersolution *of* (5.1) (*respectively,* (5.2)) *if, for every point* $(t_0,x_0)\in(0,T)\times\big(0,\frac{K}{\gamma}\big)$ *and for every "test function"* $h\in C^{1,2}\big((0,T)\times\big(0,\frac{K}{\gamma}\big)\big)$ *satisfying* $v\ge h$ *on* $(0,T)\times\big(0,\frac{K}{\gamma}\big)$ *and* $v(t_0,x_0)=h(t_0,x_0)$, *we have* $\min\{\mathcal{L}_g h(t_0,x_0)-c, h(t_0,x_0)-\gamma x_0\}\ge 0$ (*respectively,* $\max\{\mathcal{L}_g h(t_0,x_0)-c, h(t_0,x_0)-K\}\ge 0$).

A function v is a viscosity solution *of one of these equations if it is both a viscosity subsolution and a viscosity supersolution.*

The pricing function for the convertible bond satisfies a variational inequality, and the solution of a variational inequality is often the solution to a *free boundary problem*, where the "free boundary" divides the region in which an action (conversion or call) should take place from a region in which no action should occur. We show in section 8 that the bond pricing function does indeed satisfy a free boundary problem, and we derive properties of the free boundary. To prepare for that analysis, we introduce the following sets.

In Case I ($c\le rK$) of Proposition 4.5, we define the *continuation set*

$$\begin{aligned}\mathcal{C}_T^I &\triangleq \left\{(t,x)\in(0,T)\times\left(0,\frac{K}{\gamma}\right): v_g(t,x)>\gamma x\right\}\\ &= \left\{(t,x)\in(0,T)\times\left(0,\frac{K}{\gamma}\right): v_g(t,x)>\widetilde{\psi}(t,x)\right\},\end{aligned} \tag{5.3}$$

where $\widetilde{\psi}(t,x)=\max\{\gamma x, k(t,x)\}$ is defined in Step 2 of the proof of Proposition 4.5. Because k satisfies (4.15), $v_g(t,x)>\gamma x$ if and only if $v_g(t,x)>\widetilde{\psi}(t,x)$. Because v_g and $\widetilde{\psi}$ are continuous, $\mathcal{C}_T^I$ is open. Define the *stopping set*

$$\begin{aligned}\mathcal{S}_T^I &\triangleq \left\{(t,x)\in[0,T]\times\left[0,\frac{K}{\gamma}\right]: v_g(t,x)=\psi(t,x)\right\}\\ &= \left\{(t,x)\in[0,T]\times\left[0,\frac{K}{\gamma}\right]: v_g(t,x)=\widetilde{\psi}(t,x)\right\}.\end{aligned}$$

The equality is justified by the same argument that justified the equality in (5.3) and the additional observation that $\psi(T,\cdot)=\widetilde{\psi}(T,\cdot)$. The set $\mathcal{S}_T^I$ is closed. Under the Case I assumption, v_g is a viscosity solution of (5.1), which is equivalent to the following three conditions:

(i) $v_g\ge\gamma x$ on $[0,T]\times\big[0,\frac{K}{\gamma}\big]$,

(ii) v_g is a viscosity supersolution of $\mathcal{L}_g v-c=0$ on $(0,T)\times\big(0,\frac{K}{\gamma}\big)$, and

(iii) v_g is a viscosity solution of $\mathcal{L}_g v - c = 0$ on $\mathcal{C}_T^I$.

In Case II ($\delta K \le c$) of Proposition 4.6, we define the *continuation set*

$$\mathcal{C}_T^{II} \triangleq \left\{(t,x) \in (0,T) \times \left(0, \frac{K}{\gamma}\right) : v_g(t,x) < K\right\}$$

$$= \left\{(t,x) \in (0,T) \times \left(0, \frac{K}{\gamma}\right) : v_g(t,x) < \widetilde{\varphi}(t,x)\right\}, \tag{5.4}$$

where $\widetilde{\varphi} = \varphi \wedge k$ is defined in Step 2 of the proof of Proposition 4.6, and the equality in (5.4) is justified by (4.47). The set $\mathcal{C}_T^{II}$ is open. Define the *stopping set*

$$\mathcal{S}_T^{II} \triangleq \left\{(t,x) \in [0,T] \times \left[0, \frac{K}{\gamma}\right] : v_g(t,x) = \varphi(t,x)\right\}$$

$$= \left\{(t,x) \in [0,T] \times \left[0, \frac{K}{\gamma}\right] : v_g(t,x) = \widetilde{\varphi}(t,x)\right\}.$$

The equality is justified by the argument that justified (5.4) and the additional observations that $\varphi(T,\cdot) = \widetilde{\varphi}(T,\cdot)$. The set $\mathcal{S}_T^{II}$ is closed. Under the Case II assumption, v_g is a viscosity solution of (5.2), which is equivalent to the following conditions:

(iv) $v_g \le K$ on $[0,T] \times \left[0, \frac{K}{\gamma}\right]$,

(v) v_g is a viscosity subsolution of $\mathcal{L}_g v - c = 0$ on $(0,T) \times \left(0, \frac{K}{\gamma}\right)$, and

(vi) v_g is a viscosity solution of $\mathcal{L}_g v - c = 0$ on $\mathcal{C}_T^{II}$.

Remark 5.2. In the overlapping case, $\delta K \le c \le rK$, we have from Proposition 4.7 that $\mathcal{C}_T^I = \mathcal{C}_T^{II} = (0,T) \times (0, \frac{K}{\gamma})$ and v_g is a viscosity solution of $\mathcal{L}_g u - c = 0$ on this set. Remark 4.4 applies in the overlapping case, which is why we require $\mathcal{L}_g v_g - c = 0$ to hold only in the viscosity sense.

6. Proof of Theorem 3.1. In this section we prove Theorem 3.1 and also prove that the continuous viscosity solutions of (5.1) and (5.2) with boundary conditions (3.6) and (3.7) are unique. In light of Propositions 4.5 and 4.6 and the discussion of section 5, this provides the final step in the proof of Theorem 3.2.

For $\epsilon \in \left[0, \frac{K}{\gamma}\right)$, we define the sets

$$D_\epsilon \triangleq [0,T] \times \left[\epsilon, \frac{K}{\gamma}\right], \quad \widetilde{D}_\epsilon \triangleq [0,T] \times \left[\log \epsilon, \log \frac{K}{\gamma}\right],$$

their parabolic boundaries

$$\partial_p D_\epsilon \triangleq \left([0,T] \times \left\{\epsilon, \frac{K}{\gamma}\right\}\right) \cup \left(\{T\} \times \left(\epsilon, \frac{K}{\gamma}\right)\right)$$

$$\partial_p \widetilde{D}_\epsilon \triangleq \left([0,T] \times \left\{\log \epsilon, \log \frac{K}{\gamma}\right\}\right) \cup \left(\{T\} \times \left(\log \epsilon, \log \frac{K}{\gamma}\right)\right),$$

and their topological boundaries

$$\partial D_\epsilon \triangleq \partial_p D_\epsilon \cup \left(\{0\} \times \left(\epsilon, \frac{K}{\gamma}\right)\right), \quad \partial \widetilde{D}_\epsilon \triangleq \partial_p \widetilde{D}_\epsilon \cup \left(\{0\} \times \left(\log \epsilon, \log \frac{K}{\gamma}\right)\right).$$

In the above definitions, we use the convention $\log 0 = -\infty$, so $\widetilde{D}_0$, $\partial_p \widetilde{D}_0$, and $\partial \widetilde{D}_0$ are subsets of the extended real numbers. The following comparison lemma is a modification of Theorem 8.2 of [13], differing by the fact that the functions u and v satisfy different equations rather than the same equation.

LEMMA 6.1 (comparison). *Let f,g in $C(D_0)$ be given. Let $u,v \in C(D_0)$ be respective viscosity sub- and supersolutions on $D_0 \setminus \partial D_0$ of the equations*

$$\min\{\mathcal{L}_f u - c, u - \gamma x\} = 0, \tag{6.1}$$

$$\min\{\mathcal{L}_g v - c, v - \gamma x\} = 0. \tag{6.2}$$

Alternatively, let u,v be respective viscosity sub- and supersolutions of the equations

$$\max\{\mathcal{L}_f u - c, u - K\} = 0, \tag{6.3}$$

$$\max\{\mathcal{L}_g v - c, v - K\} = 0. \tag{6.4}$$

Assume further that one of the functions u or v (let us say u) satisfies

$$0 \le u(t,y) - u(t,x) \le y - x \text{ for } (t,x),(t,y) \in D_0. \tag{6.5}$$

Then for every $\lambda \ge 0$, we have

$$\begin{aligned} &\max_{(t,x)\in D_0} e^{\lambda t}\big(u(t,x) - v(t,x)\big)^+ \\ &\le \max\left\{\frac{\delta}{r+\lambda} \max_{(t,x)\in D_0} e^{\lambda t}\big(f(t,x) - g(t,x)\big)^+, \max_{(t,x)\in \partial_p D_0} e^{\lambda t}\big(u(t,x) - v(t,x)\big)^+\right\}. \end{aligned} \tag{6.6}$$

Proof. We provide the proof under the assumptions that u is a subsolution of (6.1) and v is a supersolution of (6.2). Because f, g, u, and v are continuous, it suffices to prove

$$\begin{aligned} &\max_{(t,x)\in D_\epsilon} e^{\lambda t}\big(u(t,x) - v(t,x)\big)^+ \\ &\le \max\left\{\frac{\delta}{r+\lambda} \max_{(t,x)\in D_\epsilon} e^{\lambda t}\big(f(t,x) - g(t,x)\big)^+, \max_{(t,x)\in \partial_p D_\epsilon} e^{\lambda t}\big(u(t,x) - v(t,x)\big)^+\right\} \end{aligned}$$

for every $\epsilon \in \big(0, \frac{K}{\gamma}\big)$. To do this, we define $\widetilde{u}(t,\xi) \triangleq e^{\lambda t} u\big(t, e^\xi\big)$, $\widetilde{v}(t,\xi) \triangleq e^{\lambda t} v\big(t, e^\xi\big)$, $\widetilde{f}(t,\xi) \triangleq e^{\lambda t} f\big(t, e^\xi\big)$, and $\widetilde{g}(t,\xi) \triangleq e^{\lambda t} g\big(t, e^\xi\big)$. In terms of these functions, we need to prove that for every $\epsilon \in \big(0, \frac{K}{\gamma}\big)$,

$$\begin{aligned} &\max_{(t,\xi)\in \widetilde{D}_\epsilon} \big(\widetilde{u}(t,\xi) - \widetilde{v}(t,\xi)\big)^+ \\ &\le \max\left\{\frac{\delta}{r+\lambda} \max_{(t,\xi)\in \widetilde{D}_\epsilon} \big(\widetilde{f}(t,\xi) - \widetilde{g}(t,\xi)\big)^+, \max_{(t,\xi)\in \partial_p \widetilde{D}_\epsilon} \big(\widetilde{u}(t,\xi) - v(t,\xi)\big)^+\right\}. \end{aligned} \tag{6.7}$$

For $\eta > 0$, we define $\widetilde{u}^\eta(t,\xi) \triangleq \widetilde{u}(t,\xi) - \frac{\eta}{t}$, so that $\lim_{t\downarrow 0} \widetilde{u}^\eta(t,\xi) = -\infty$ uniformly in ξ. We will show for all $\epsilon \in \big(0, \frac{K}{\gamma}\big)$ that

$$\begin{aligned} &\max_{(t,\xi)\in \widetilde{D}_\epsilon} \big(\widetilde{u}^\eta(t,\xi) - \widetilde{v}(t,\xi)\big)^+ \\ &\le \max\left\{\frac{\delta}{r+\lambda} \max_{(t,\xi)\in \widetilde{D}_\epsilon} \big(\widetilde{f}(t,\xi) - \widetilde{g}(t,\xi)\big)^+, \max_{(t,\xi)\in \partial_p \widetilde{D}_\epsilon} \big(\widetilde{u}^\eta(t,\xi) - v(t,\xi)\big)^+\right\}. \end{aligned} \tag{6.8}$$

We can then let $\eta \downarrow 0$ in (6.8) to obtain (6.7) and conclude the proof.

The change of variable transforms (6.1) and (6.2) into

$$\min\Bigg\{ -\widetilde{u}_t + (r+\lambda)\widetilde{u} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\widetilde{u}_\xi - \delta e^{-\lambda t - \xi}\widetilde{f}(t,\xi)\widetilde{u}_\xi + ce^{-\xi}\widetilde{u}_\xi - \frac{1}{2}\sigma^2\widetilde{u}_{\xi\xi} - e^{\lambda t}c, \widetilde{u} - \gamma e^{\lambda t + \xi}\Bigg\} = 0,$$

$$\min\Bigg\{ -\widetilde{v}_t + (r+\lambda)\widetilde{v} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\widetilde{v}_\xi - \delta e^{-\lambda t - \xi}\widetilde{g}(t,\xi)\widetilde{v}_\xi + ce^{-\xi}\widetilde{v}_\xi - \frac{1}{2}\sigma^2\widetilde{v}_{\xi\xi} - e^{\lambda t}c, \widetilde{v} - \gamma e^{\lambda t + \xi}\Bigg\} = 0.$$

On the set

$$\widetilde{\mathcal{C}}_{\widetilde{u}} \triangleq \left\{(t,\xi) \in \widetilde{D}_0 \setminus \partial\widetilde{D}_0 : \widetilde{u}(t,\xi) > \gamma e^{\lambda t + \xi}\right\},$$

the function $\widetilde{u}$ is a viscosity subsolution of

$$-\widetilde{u}_t + (r+\lambda)\widetilde{u} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\widetilde{u}_\xi - \frac{1}{2}\sigma^2\widetilde{u}_{\xi\xi} - e^{\lambda t}c = \delta e^{-\lambda t - \xi}\widetilde{f}(t,\xi)\widetilde{u}_\xi - ce^{-\xi}\widetilde{u}_\xi, \tag{6.9}$$

and so for $\eta > 0$, the function $\widetilde{u}^\eta$ is also a viscosity subsolution of this equation on $\widetilde{\mathcal{C}}_{\widetilde{u}}$. On $\widetilde{D}_0 \setminus \partial\widetilde{D}_0$, $\widetilde{v}(t,\xi) \geq \gamma e^{\lambda t + \xi}$, and $\widetilde{v}$ is a viscosity supersolution of

$$-\widetilde{v}_t + (r+\lambda)\widetilde{v} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\widetilde{v}_\xi - \frac{1}{2}\sigma^2\widetilde{v}_{\xi\xi} - e^{\lambda t}c = \delta e^{-\lambda t - \xi}\widetilde{g}(t,\xi)\widetilde{v}_\xi - ce^{-\xi}\widetilde{v}_\xi. \tag{6.10}$$

Let us assume that (6.8) is violated for some $\eta > 0$ and $\epsilon \in \left(0, \frac{K}{\gamma}\right)$. This means that

$$\max_{(t,x)\in\widetilde{D}_\epsilon} \left(\widetilde{u}^\eta(t,\xi) - \widetilde{v}(t,\xi)\right)^+ > \frac{\delta}{r+\lambda} \max_{(t,\xi)\in\widetilde{D}_\epsilon} \left(\widetilde{f}(t,\xi) - \widetilde{g}(t,\xi)\right)^+. \tag{6.11}$$

Let $\alpha > 0$ be given, and set

$$M_\alpha \triangleq \max_{(t,\xi),(t,\zeta)\in\widetilde{D}_\epsilon} \left(\widetilde{u}^\eta(t,\xi) - \widetilde{v}(t,\zeta) - \frac{\alpha}{2}|\xi - \zeta|^2\right).$$

The maximum is attained at some point $(t_\alpha, \xi_\alpha, \zeta_\alpha)$. According to a slight variant of Lemma 3.1 of [13],

$$\lim_{\alpha\to\infty} \alpha|\xi_\alpha - \zeta_\alpha|^2 = 0 \text{ and } \lim_{\alpha\to\infty} M_\alpha = \max_{(t,\xi)\in\widetilde{D}_\epsilon} \left(\widetilde{u}^\eta(t,\xi) - \widetilde{v}(t,\xi)\right). \tag{6.12}$$

Violation of (6.8) implies that for large α, the points (t_α, ξ_α) and (t_α, ζ_α) are bounded away from the parabolic boundary $\partial_p\widetilde{D}_\epsilon$. Furthermore, because $\lim_{t\downarrow 0} \widetilde{u}^\eta(t,\xi) = -\infty$, these points are bounded away from the topological boundary $\partial\widetilde{D}_\epsilon$ as well.

There are two cases to consider. In the first case, $(t_\alpha, \xi_\alpha) \notin \widetilde{\mathcal{C}}_{\widetilde{u}}$, and so $\widetilde{u}^\eta(t_\alpha, \xi_\alpha) = \gamma e^{\lambda t_\alpha + \xi_\alpha} - \frac{\eta}{t_\alpha} \leq \gamma e^{\lambda t_\alpha + \xi_\alpha}$. We have

$$M_\alpha \leq \widetilde{u}^\eta(t_\alpha, \xi_\alpha) - \widetilde{v}(t_\alpha, \zeta_\alpha) \leq \gamma e^{\lambda t_\alpha}\left(e^{\xi_\alpha} - e^{\zeta_\alpha}\right). \tag{6.13}$$

In the other case, (t_α, ξ_α) is in $\widetilde{\mathcal{C}}_{\tilde u}$. Because $\widetilde{u}^\eta$ is a subsolution of (6.9) in a neighborhood of (t_α, ξ_α), $\widetilde{v}$ is a supersolution of (6.10) in a neighborhood of (t_α, ζ_α), and these points are bounded away from $\partial \widetilde{D}_\epsilon$, condition (8.5) of Theorem 8.3 of [13] is satisfied (our time variable is reversed from that of [13]). That theorem with $\epsilon = \frac{1}{\alpha}$ implies the existence of numbers b, X, and Y such that $X \le Y$ and[1]

$$(b, \alpha(\xi_\alpha - \zeta_\alpha), X) \in \overline{\mathcal{P}}^{2,+} \widetilde{u}^\eta(t_\alpha, \xi_\alpha) \text{ and } (b, \alpha(\xi_\alpha - \zeta_\alpha), Y) \in \overline{\mathcal{P}}^{2,-} \widetilde{v}(t_\alpha, \zeta_\alpha)$$

(see the use of Theorem 8.3 on page 50 in [13]; see also page 17). Note that (t_α, ξ_α) and (t_α, ζ_α) are in the open set $\widetilde{D}_\epsilon \setminus \partial \widetilde{D}_\epsilon$. Moreover, they provide terms that can replace the time derivative, the spatial derivative, and the second spatial derivative in the subsolution and supersolution inequalities for (6.9) and (6.10):

$$\begin{aligned} &-b + (r+\lambda)\widetilde{u}^\eta(t_\alpha, \xi_\alpha) - \left(r - \delta - \frac{1}{2}\sigma^2\right)\alpha(\xi_\alpha - \zeta_\alpha) - \frac{1}{2}\sigma^2 X - e^{\lambda t_\alpha} c \\ &\qquad \le \delta e^{-\lambda t_\alpha - \xi_\alpha} \widetilde{f}(t_\alpha, \xi_\alpha)\alpha(\xi_\alpha - \zeta_\alpha) - c e^{-\xi_\alpha}\alpha(\xi_\alpha - \zeta_\alpha), \end{aligned} \tag{6.14}$$

$$\begin{aligned} &-b + (r+\lambda)\widetilde{v}(t_\alpha, \zeta_\alpha) - \left(r - \delta - \frac{1}{2}\sigma^2\right)\alpha(\xi_\alpha - \zeta_\alpha) - \frac{1}{2}\sigma^2 Y - e^{\lambda t_\alpha} c \\ &\qquad \ge \delta e^{-\lambda t_\alpha - \zeta_\alpha} \widetilde{g}(t_\alpha, \zeta_\alpha)\alpha(\xi_\alpha - \zeta_\alpha) - c e^{-\zeta_\alpha}\alpha(\xi_\alpha - \zeta_\alpha). \end{aligned} \tag{6.15}$$

Subtracting (6.15) from (6.14) and using $\sup_{\xi,\zeta \ge \log\epsilon, \xi\ne\zeta} \left| \frac{e^{-\xi} - e^{-\zeta}}{\xi - \zeta} \right| = \frac{1}{\epsilon}$, we obtain

$$\begin{aligned} M_\alpha &\le \widetilde{u}^\eta(t_\alpha, \xi_\alpha) - \widetilde{v}(t_\alpha, \zeta_\alpha) \\ &\le \frac{\delta}{r+\lambda} e^{-\lambda t_\alpha - \xi_\alpha}\big(\widetilde{f}(t_\alpha, \xi_\alpha) - \widetilde{g}(t_\alpha, \zeta_\alpha)\big)\alpha(\xi_\alpha - \zeta_\alpha) \\ &\quad + \frac{\delta}{\epsilon(r+\lambda)} e^{-\lambda t_\alpha} |\widetilde{g}(t_\alpha, \zeta_\alpha)| \alpha(\xi_\alpha - \zeta_\alpha)^2 + \frac{c}{\epsilon(r+\lambda)}\alpha(\xi_\alpha - \zeta_\alpha)^2. \end{aligned} \tag{6.16}$$

But also, (6.5) implies, at least formally, that $0 \le u_x(t,x) \le 1$, or equivalently, $0 \le \widetilde{u}_\xi(t,\xi) \le e^{\lambda t + \xi}$. Of course u_x and $\widetilde{u}_\xi$ may not exist, but (6.5) implies that $\alpha(\xi_\alpha - \zeta_\alpha)$, the surrogate for $\widetilde{u}_\xi(t_\alpha, \xi_\alpha)$, must satisfy $0 \le \alpha(\xi_\alpha - \zeta_\alpha) \le e^{\lambda t_\alpha + \xi_\alpha}$. Using this inequality in (6.16), we obtain

$$\begin{aligned} M_\alpha &\le \frac{\delta}{r+\lambda}\big(\widetilde{f}(t_\alpha, \xi_\alpha) - \widetilde{g}(t_\alpha, \zeta_\alpha)\big) + O\big(\alpha(\xi_\alpha - \zeta_\alpha)^2\big) \\ &= \frac{\delta}{r+\lambda}\big(\widetilde{f}(t_\alpha, \xi_\alpha) - \widetilde{g}(t_\alpha, \xi_\alpha)\big) + \frac{\delta}{r+\lambda}\big(\widetilde{g}(t_\alpha, \xi_\alpha) - \widetilde{g}(t_\alpha, \zeta_\alpha)\big) \\ &\quad + O\big(\alpha(\xi_\alpha - \zeta_\alpha)^2\big) \\ &\le \frac{\delta}{r+\lambda} \max_{(t,\xi)\in\widetilde{D}_\epsilon} \big(\widetilde{f}(t,\xi) - \widetilde{g}(t,\xi)\big)^+ + \frac{\delta}{r+\lambda}\big(\widetilde{g}(t_\alpha, \xi_\alpha) - \widetilde{g}(t_\alpha, \zeta_\alpha)\big) \\ &\quad + O\big((\xi_\alpha - \zeta_\alpha)^2\big). \end{aligned} \tag{6.17}$$

[1] Following [13], for a real-valued function u defined on a domain $D \subset \mathbb{R}^2$ and for (t,ξ) in the interior of D, we set $\mathcal{P}^{2,+}u(t,\xi) \triangleq \{(b,\alpha,X) \in \mathbb{R}^3 : u(s,y) \le u(t,\xi) + b(s-t) + \alpha(y-\xi) + \frac{1}{2}X(y-\xi)^2 + o(s-t) + o((y-x)^2) \text{ as } (s,y) \to (t,\xi)\}$ and define $\mathcal{P}^{2,-}u(t,\xi)$ analogously to $\mathcal{P}^{2,+}$ but with the inequality in the definition reversed. We next define $\overline{\mathcal{P}}^{2,\pm}u(t,\xi) \triangleq \{(b,\alpha,X) \in \mathbb{R}^3 : \text{there exists } (t_n,\xi_n,b_n,\alpha_n,X_n) \in D \times \mathbb{R}^3 \text{ such that } (b_n,\alpha_n,X_n) \in \mathcal{P}^{2,\pm}u(t_n,\xi_n) \text{ and } (t_n,\xi_n,u(t_n,\xi_n),b_n,\alpha_n,X_n) \to (t,\xi,u(t,\xi),b,\alpha,X)\}$.

Letting $\alpha \to \infty$ in (6.13) and (6.17), using (6.12), and using the uniform continuity of $\widetilde{g}$ on $\widetilde{D}_\epsilon$, we contradict (6.11). $\square$

Proof of Theorem 3.1. Set $\lambda \triangleq \delta + 1$ and endow $\mathcal{G}$ with the metric

$$d(f,g) \triangleq \max_{(t,x)\in D_0} e^{\lambda t}|f(t,x) - g(t,x)| \text{ for all } f,g \in \mathcal{G}. \tag{6.18}$$

Under this metric, $\mathcal{G}$ is complete. Let $f,g \in \mathcal{G}$ be given, and define $u = \mathcal{T}f$ and $v = \mathcal{T}g$. According to subsection 4.5, u and v are in $\mathcal{G}$. In particular, (6.5) is satisfied. We apply Lemma 6.1, noting that u and v are viscosity solutions of (6.1) and (6.2), respectively, or viscosity solutions of (6.3) and (6.4), respectively, and they agree on $\partial_p D_0$, to conclude that

$$d(u,v) \le \frac{\delta}{r+\lambda} \max_{(t,x)\in D_0} e^{\lambda t}\big(f(t,x) - g(t,x)\big)^+.$$

Reversing the roles of f and g, we obtain the contraction property for $\mathcal{T}$. $\square$

Remark 6.2. Uniqueness of the continuous viscosity solution of (6.1) or (6.3) with boundary conditions (3.6) and (3.7) follows from Lemma 6.1 with $f = g$.

7. Asymptotic behavior. We relate the problem of this paper to the perpetual convertible bond. To do this, we reverse time, denoting by $u_L(t,x)$ the price of the bond for fixed par value $L \in [0,K]$ when the time to maturity is t and the firm value is x. This section requires standing assumption (2.6). We have the following variation of Lemma 6.1.

LEMMA 7.1. *Fix $T > 0$ and let g_1 and g_2 in $C\big([0,T] \times \big[0, \frac{K}{\gamma}\big]\big)$ be a viscosity subsolution and a viscosity supersolution, respectively, of*

$$\min\{g_t + \mathcal{N}g - c, g - \gamma x\} = 0 \text{ on } (0,T) \times \left(0, \frac{K}{\gamma}\right) \tag{7.1}$$

or a viscosity subsolution and viscosity supersolution, respectively, of

$$\max\{g_t + \mathcal{N}g - c, g - K\} = 0 \text{ on } (0,T) \times \left(0, \frac{K}{\gamma}\right), \tag{7.2}$$

where $\mathcal{N}$ is the nonlinear operator

$$\begin{aligned}\mathcal{N}g(t,x) \triangleq{}& rg(t,x) - (rx - c)g_x(t,x)\\ &+\delta\big(x - g(t,x)\big)g_x(t,x) - \frac{1}{2}\sigma^2x^2g_{xx}(t,x).\end{aligned}$$

Assume that either g_1 or g_2 satisfies (3.2). *If $g_1(0,\cdot) \le g_2(0,\cdot)$ and*

$$g_1(t,0) \le g_2(t,0),\ g_1\Big(t, \frac{K}{\gamma}\Big) \le g_2\Big(t, \frac{K}{\gamma}\Big), \quad 0 \le t \le T, \tag{7.3}$$

then $g_1 \le g_2$. In particular, if g_1 and g_2 are viscosity solutions of (7.1) *or* (7.2), *$g_1(0,\cdot) = g_2(0,\cdot)$, and equality holds in both parts of* (7.3), *then $g_1 = g_2$.*

Proof. Apply the time-reversed version of Lemma 6.1 with $\lambda = 0$, $u = f = g_1$, and $v = g = g_2$ to conclude that

$$\max_{(t,x)\in[0,T]\times[0,\frac{K}{\gamma}]} \big(g_1(t,x) - g_2(t,x)\big)^+ \le \frac{\delta}{r} \max_{(t,x)\in[0,T]\times[0,\frac{K}{\gamma}]} \big(g_1(t,x) - g_2(t,x)\big)^+.$$

Since $\delta < r$, we have $g_1 \le g_2$. ∎

Regardless of the initial time to maturity, as a function of the firm value and remaining time to maturity, the convertible bond price must satisfy one (or both) of (7.1) and (7.2), depending on whether $c \le rK$ or $\delta K \le c$. The uniqueness assertion in Lemma 7.1 guarantees that while the bond price does depend on the time to maturity, it does not depend on the time when the bond was issued, or equivalently, it does not depend on the *initial* time to maturity.

A perpetual convertible bond never matures, and hence the time variable and the par value are irrelevant. Its price $p(x)$ is a function of the underlying firm value alone. The following result is proved in [36].

THEOREM 7.2. *The perpetual convertible bond price function p is continuous on $[0,\infty)$, continuously differentiable on $\big(0,\frac{K}{\gamma}\big)$, and satisfies $0 \le p'(x) \le 1$ for $0 < x < \frac{K}{\gamma}$ and $p(x) = \gamma x$ for $x \ge \frac{K}{\gamma}$.*

If $c \le rK$, then p, regarded as a function of (t,x) with $p_t = 0$, is a continuous viscosity solution of (7.1) *satisfying*

$$p(0) = 0, \quad p\left(\frac{K}{\gamma}\right) = K. \tag{7.4}$$

Furthermore, there exists $C_o^* \in \big(0,\frac{K}{\gamma}\big]$ such that p restricted to $(0, C_o^*)$ is strictly greater than γx and is a classical solution of $\mathcal{N}p = c$, whereas $p(x) = \gamma x$ for $x \ge C_o^*$.

If $\delta K \le c$, then p is a continuous viscosity solution of (7.2) *satisfying* (7.4). *Furthermore, there exists $C_a^* \in \big(0,\frac{K}{\gamma}\big]$ such that p restricted to $(0, C_a^*)$ is strictly less than K and is a classical solution of $\mathcal{N}p = c$, whereas $p(x) = K$ for $C_a^* \le x \le \frac{K}{\gamma}$.*

Uniqueness of p in [36] is proved only in the class of functions that are smooth in the continuation region, not within the class of all continuous functions. We upgrade the uniqueness result to the larger class here.

LEMMA 7.3. *Let p be the perpetual convertible bond price function. If $c \le rK$, then p is the unique viscosity solution of* (7.1) *on $\big(0,\frac{K}{\gamma}\big)$ that is continuous on $\big[0,\frac{K}{\gamma}\big]$ and satisfies* (7.4). *If $\delta K \le c$, then p is the unique viscosity solution of* (7.2) *on $\big(0,\frac{K}{\gamma}\big)$ that is continuous on $\big[0,\frac{K}{\gamma}\big]$ and satisfies* (7.4).

Proof. We provide the proof for the case $c \le rK$. In the second case, $\delta K \le c$, a similar proof is possible.

Let $q \in C\big[0,\frac{K}{\gamma}\big]$ be a viscosity solution of (7.1) on $\big(0,\frac{K}{\gamma}\big)$ satisfying (7.4). Assume

$$\max_{x\in[0,\frac{K}{\gamma}]} \big(p(x) - q(x)\big) = p(x_0) - q(x_0) > 0. \tag{7.5}$$

Then $x_0 \in \big(0,\frac{K}{\gamma}\big)$ and $p(x_0) > q(x_0) \ge \gamma x_0$, so $x_0 \in (0, C_o^*)$. Because p is twice continuously differentiable in $(0, C_o^*)$, we can use $p + q(x_0) - p(x_0)$ as a test function for the viscosity supersolution q to obtain

$$rq(x_0) - (rx_0 - c)p'(x_0) + \delta\big(x_0 - q(x_0)\big)p'(x_0) - \frac{1}{2}\sigma^2 x_0^2 p''(x_0) \ge c.$$

But p satisfies $\mathcal{N}p(x_0) = c$, so

$$rp(x_0) - (rx_0 - c)p'(x_0) + \delta\big(x_0 - p(x_0)\big)p'(x_0) - \frac{1}{2}\sigma^2 x_0^2 p''(x_0) = c.$$

Subtracting these relations, we obtain

$$r\big(p(x_0) - q(x_0)\big) \le \delta\big(p(x_0) - q(x_0)\big)p'(x_0).$$

But $0 \le p'(x_0) \le 1$ and $0 \le \delta < r$, so we have a contradiction to (7.5).

Assume on the other hand that

$$\max_{x\in[0,\frac{K}{\gamma}]} \big(q(x) - p(x)\big) = q(x_0) - p(x_0) > 0. \tag{7.6}$$

Then $x_0 \in \big(0, \frac{K}{\gamma}\big)$ and $q(x_0) > p(x_0) \ge \gamma x_0$. We have $q \le p + q(x_0) - p(x_0)$, and if $x_0 \ne C_o^*$, so that p is twice continuously differentiable in a neighborhood of x_0, we can use $p + q(x_0) - p(x_0)$ as a test function for the viscosity subsolution q to obtain

$$rq(x_0) - (rx_0 - c)p'(x_0) + \delta\big(x_0 - q(x_0)\big)p'(x_0) - \frac{1}{2}\sigma^2 x_0^2 p''(x_0) \le c. \tag{7.7}$$

But $\mathcal{N}p(x_0) \ge c$ means that

$$rp(x_0) - (rx_0 - c)p'(x_0) + \delta\big(x_0 - p(x_0)\big)p'(x_0) - \frac{1}{2}\sigma^2 x_0^2 p''(x_0) \ge c. \tag{7.8}$$

Subtracting these relations, we obtain

$$r\big(q(x_0) - p(x_0)\big) \le \delta\big(q(x_0) - p(x_0)\big)p'(x_0), \tag{7.9}$$

and we conclude as before.

The only other possibility is that (7.6) holds and $x_0 = C_o^* \in \big(0, \frac{K}{\gamma}\big)$. According to Theorem 7.2, p' is defined on $\big(0, \frac{K}{\gamma}\big)$, and using the equations $\mathcal{N}p = c$ to the left of C_o^* and $p(x) = \gamma x$ to the right of x_0, we see that the left- and right-hand second derivatives $p''(x_0-)$ and $p''(x_0+) = 0$ exist. Furthermore, $p(x) - \gamma x$ attains its minimum value of 0 at x_0, so $p''(x_0-) \ge 0$. We need only rule out the case $p''(x_0-) > 0$, for in the event $p''(x_0-) = 0$, the function p is twice continuously differentiable at x_0 and we can use $p + q(x_0) - p(x_0)$ as a test function as above.

Suppose $p''(x_0-) > 0 = p''(x_0+)$. Let $\underline{p}$ be the solution in $\big(0, \frac{K}{\gamma}\big]$ of the ordinary differential equation $\mathcal{N}\underline{p} = c$ satisfying $\underline{p}(x_0) = \gamma x_0$ and $\underline{p}'(x_0) = \gamma$. On $(0, x_0]$, p is a solution to this terminal value problem and hence agrees with $\underline{p}$. In particular, $\underline{p}''(x_0) = p''(x_0-) > 0$, and this implies $\underline{p}(x) > \gamma x = p(x)$ for x in some interval $(x_0, x_0+\epsilon)$, where $\epsilon > 0$. The function $q-\underline{p}$ attains a local maximum at x_0 because $q-p$ does, and we can use $\underline{p} + q(x_0) - p(x_0)$ as a test function for the viscosity subsolution q as above. This leads to (7.7), with $p''(x_0-)$ replacing $p''(x_0)$. Inequality (7.8) holds for all $x \in (0, x_0)$, and letting $x \uparrow x_0$, we obtain (7.8), with $p''(x_0-)$ replacing $p''(x_0)$ as well. This implies (7.9), and (7.6) is contradicted. ∎

Proof of Theorem 3.6. The terminal condition (3.6), with time reversed, states that for $0 \le L \le K$, we have

$$\gamma x = u_0(0, x) \le u_L(0, x) \le u_K(0, x) = x \wedge K, \quad 0 \le x \le \frac{K}{\gamma}.$$

The functions u_0, u_L, and u_K are continuous viscosity solutions of (7.1) or (7.2), depending on whether $c \le rK$ or $\delta K \le c$. Lemma 7.1 and the membership of u_0 and u_K in $\mathcal{G}$ (see, in particular, (3.3)) imply that for $0 \le L \le K$,

$$\gamma x \le u_0(t, x) \le u_L(t, x) \le u_K(t, x) \le x \wedge K, \quad t \ge 0,\ 0 \le x \le \frac{K}{\gamma}. \tag{7.10}$$

For $0 \le t_1 \le t_2$, we have $u_0(0, \cdot) = \gamma x \le u_0(t_2 - t_1, \cdot)$, and we can apply Lemma 7.1 with $g_1(0, \cdot) = u_0(0, \cdot)$ and $g_2(0, \cdot) = u_0(t_2 - t_1, \cdot)$ to conclude that $u_0(t_1, \cdot) \le u_0(t_2, \cdot)$.

In other words, $u_0(t,x)$ is nondecreasing in t for each fixed x. On the other hand, $u_K(0,x) = x \wedge K \geq u_K(t_2 - t_1, x)$, and this leads to the conclusion that $u_K(t,x)$ is nonincreasing in t for each fixed x. Both $u_0(t,\cdot)$ and $u_K(t,\cdot)$ are Lipschitz continuous with constant 1, and the Arzela–Ascoli theorem implies that they converge uniformly on $\left[0, \frac{K}{\gamma}\right]$ to Lipschitz continuous limits $u_-(\cdot)$ and $u_+(\cdot)$, respectively, as $t \to \infty$. Uniform convergence preserves the viscosity solution property (see [13]), and so u_- and u_+ are also continuous viscosity solutions of either (7.1) or (7.2). Lemma 7.3 implies $u_- = p = u_+$. Relation (7.10) then implies $\lim_{t\to\infty} u_L(t,x) = p(x)$ for all $x \in \left[0, \frac{K}{\gamma}\right]$, and the convergence is uniform in x. Of course, for $x \geq \frac{K}{\gamma}$, $u_L(t,x) = p(x) = \gamma x$. □

Remark 7.4. The proof of Theorem 3.6 shows that for all $t \geq 0$,

$$u_0(t,x) \leq \lim_{s\to\infty} u_0(s,x) = p(x) = \lim_{s\to\infty} u_K(s,x) \leq u_K(t,x) \tag{7.11}$$

and the convergence is uniform in $x \in \left(0, \frac{K}{\gamma}\right)$.

8. Continuation and stopping sets. We continue with the time reversal introduced in section 7, denoting by $u_L(t,x)$ the price of the convertible bond when time to maturity is t and the underlying firm value is x. Following section 5, in Case I ($c \leq rK$) of Proposition 4.5, we define

$$\mathcal{C}_L^I \triangleq \left\{(t,x) \in (0,\infty) \times \left(0, \frac{K}{\gamma}\right) : u_L(t,x) > \gamma x\right\}, \tag{8.1}$$

$$\mathcal{S}_L^I \triangleq \left\{(t,x) \in (0,\infty) \times \left(0, \frac{K}{\gamma}\right] : u_L(t,x) = \gamma x\right\}. \tag{8.2}$$

In Case II ($\delta K \leq c$) of Proposition 4.6, we define

$$\mathcal{C}_L^{II} \triangleq \left\{(t,x) \in (0,\infty) \times \left(0, \frac{K}{\gamma}\right) : u_L(t,x) < K\right\}, \tag{8.3}$$

$$\mathcal{S}_L^{II} \triangleq \left\{(t,x) \in (0,\infty) \times \left(0, \frac{K}{\gamma}\right] : u_L(t,x) = K\right\}. \tag{8.4}$$

To relate (8.4) to the definition of $\mathcal{S}_T^{II}$ in section 5, recall (4.47). The present section provides some information about the nature of the sets in (8.1)–(8.4).

LEMMA 8.1. *For all $t \geq 0$, the mapping $x \mapsto \frac{1}{x} u_L(t,x)$ is nonincreasing.*

Proof. We rescale u_L. Let $\ell > 0$ be given and define $\overline{u} : [0,\infty) \times \left[0, \frac{\ell K}{\gamma}\right] \to [0,\infty)$ by $\overline{u}(t,x) \triangleq \ell u\left(t, \frac{x}{\ell}\right)$. Because we have formally that $0 \leq u_x(t,x) \leq 1$, we also have formally that $0 \leq \overline{u}_x(t,x) \leq 1$. Furthermore,

$$\overline{u}_t(t,x) + \mathcal{N}\overline{u}(t,x) = \ell\left[u_t\left(t, \frac{x}{\ell}\right) + \mathcal{N}u\left(t, \frac{x}{\ell}\right)\right] + c(1-\ell)\overline{u}_x(t,x). \tag{8.5}$$

In Case I ($c \leq rK$) we let $0 < a < b < \frac{K}{\gamma}$ be given and set $\ell = \frac{b}{a} > 1$. Because $u_t + \mathcal{N}u \geq c$, (8.5) implies

$$\overline{u}_t(t,x) + \mathcal{N}\overline{u}(t,x) \geq \ell c + c(1-\ell)\overline{u}_x(t,x) \geq c. \tag{8.6}$$

In other words, $\overline{u}(t,x)$ is a viscosity supersolution of $\overline{u}_t + \mathcal{N}\overline{u} \geq c$ on $(0,\infty) \times \left(0, \frac{\ell K}{\gamma}\right)$. But also, $\overline{u}(t,x) \geq \gamma x$ for $0 \leq x \leq \frac{\ell K}{\gamma}$ because $u(t,x) \geq \gamma x$ for $0 \leq x \leq \frac{K}{\gamma}$. It follows that for every $T > 0$, $\overline{u}$ is defined and continuous on $[0,T] \times \left[0, \frac{K}{\gamma}\right]$ and is a supersolution of (7.1). Furthermore, $\overline{u}(0,\cdot) \geq u(0,\cdot)$, $\overline{u}(t,0) = u(t,0) = 0$, and $\overline{u}\left(t, \frac{K}{\gamma}\right) \geq K = u\left(t, \frac{K}{\gamma}\right)$. Lemma 7.1 implies $\overline{u} \geq u$ on $[0,T] \times \left[0, \frac{K}{\gamma}\right]$ for every $T > 0$. In particular, $\frac{b}{a}u(t,a) = \overline{u}(t,b) \geq u(t,b)$, which yields the desired result.

In Case II ($\delta K \leq c$) we again let $0 < a < b < \frac{K}{\gamma}$ be given, but now set $\ell = \frac{a}{b} < 1$. In this case, $u_t + \mathcal{N}u \leq c$ and both inequalities in (8.6) are reversed. But also, $\overline{u} \leq \ell K \leq K$. It follow that $\overline{u}$ is a subsolution of (7.2), but it is defined only on the set $[0,\infty) \times \left[0, \frac{\ell K}{\gamma}\right] \subset [0,\infty) \times \left[0, \frac{K}{\gamma}\right]$. However, on the upper boundary $[0,\infty) \times \left\{\frac{\ell K}{\gamma}\right\}$ of this set, $\overline{u} = \ell K$ and $u\left(t, \frac{\ell K}{\gamma}\right) \geq \ell K$. The function $u(0,\cdot)$ also dominates $\overline{u}(0,\cdot)$. We fix an arbitrary $T > 0$ and apply Lemma 7.1 on the smaller domain $[0,T] \times \left[0, \frac{\ell K}{\gamma}\right]$ (just take γ in Lemma 7.1 to be $\frac{\gamma}{\ell}$) to conclude that $\overline{u} \leq u$ on this domain. In particular, $u(t,a) \geq \overline{u}(t,a) = \frac{a}{b}u(t,b)$, which yields the desired result. ∎

In Case I, we define the free boundary

$$c_L(t) \triangleq \inf\left\{x \in \left(0, \frac{K}{\gamma}\right] : u_L(t,x) = \gamma x\right\}, \quad t > 0, \tag{8.7}$$

and in Case II, we define the free boundary

$$d_L(t) \triangleq \inf\left\{x \in \left(0, \frac{K}{\gamma}\right] : u_L(t,x) = K\right\}, \quad t > 0. \tag{8.8}$$

In the overlapping case $\delta K \leq c \leq rK$, Remark 5.2 says that $c_L(t) = d_L(t) = \frac{K}{\gamma}$ for all $t > 0$. We see in Remark 8.4 below that $c_L(t)$ and $d_L(t)$ are positive, so inf could be replaced by min in (8.7) and (8.8).

Remark 8.2. Because $u_L(t,x)$ is nondecreasing in L, we have $\mathcal{S}_K^I \subset \mathcal{S}_L^I \subset \mathcal{S}_0^I$ in Case I and $\mathcal{S}_0^{II} \subset \mathcal{S}_L^{II} \subset \mathcal{S}_K^{II}$ in Case II. This implies that $c_L(t)$ is nondecreasing in L in Case I and $d_L(t)$ is nonincreasing in L in Case II. In the proof of Theorem 3.6 at the end of section 7, we saw $u_0(t,x)$ is nondecreasing in t and $u_K(t,x)$ is nonincreasing in t. This implies in Case I that $c_0(t)$ is nondecreasing and $c_K(t)$ is nonincreasing, while in Case II, $d_0(t)$ is nonincreasing and $d_K(t)$ is nondecreasing.

The following theorem asserts that the continuation set and stopping set are divided by the free boundary $c_L(\cdot)$ in Case I and $d_L(\cdot)$ in Case II.

THEOREM 8.3. *In Case* I ($c \leq rK$) *we have*

$$\mathcal{S}_L^I = \left\{(t,x) \in (0,\infty) \times \left(0, \frac{K}{\gamma}\right] : c_L(t) \leq x \leq \frac{K}{\gamma}\right\}.$$

In Case II ($\delta K \leq c$) *we have*

$$\mathcal{S}_L^{II} = \left\{(t,x) \in (0,\infty) \times \left(0, \frac{K}{\gamma}\right] : d_L(t) \leq x \leq \frac{K}{\gamma}\right\}.$$

Proof. In Case I, we must show that if $u_L(t,x) = \gamma x$ for some $x \in \left(0, \frac{K}{\gamma}\right)$, then $u_L(t,y) = \gamma y$ for all $y \in \left[x, \frac{K}{\gamma}\right]$. This follows immediately from Lemma 8.1. In Case II, the result follows from the nondecrease in x of $u_L(t,x)$. ∎

Remark 8.4. Consider Case I. If $(t_0,x_0) \in \mathcal{S}_L^I$, then $u_L(t_0,x_0) = \gamma x_0$. Because u_L is a viscosity solution of $\min\{u_t + \mathcal{N}u - c, u - \gamma x\} = 0$, we may use $h(x) = \gamma x$ as a "test function" at the point (t_0,x_0) for the viscosity supersolution property to obtain

$\mathcal{N}h(x_0) \geq c$, or equivalently, $x_0 \geq \frac{c}{\delta\gamma}$. It follows that $\min\left\{\frac{K}{\gamma}, \frac{c}{\delta\gamma}\right\} \leq c_L(t) \leq \frac{K}{\gamma}$ for every $t > 0$. In Case II, we have $K = u_L(t, d_L(t)) \leq d_L(t) \leq \frac{K}{\gamma}$.

THEOREM 8.5. *Let C_o^* and C_a^* be as in Theorem 7.2. In Case* I*, we have* $\lim_{t\to\infty} c_L(t) = C_o^*$. *In Case* II*, we have* $\lim_{t\to\infty} d_L(t) = C_a^*$.

Proof. In light of Remark 8.2, it suffices to prove the theorem for the limiting cases $L = 0$ and $L = \frac{K}{\gamma}$. We treat Case I only.

Since c_0 is nondecreasing and c_K is nonincreasing, these functions have limits $c_0(\infty)$, and $c_K(\infty)$ in $\left(0, \frac{K}{\gamma}\right]$ as $t \to \infty$, and we must show $c_0(\infty) = C_o^* = c_K(\infty)$. Because $C_o^* = \min\left\{x \in \left(0, \frac{K}{\gamma}\right] : p(x) = \gamma x\right\}$, we have from the first inequality in (7.11) that $c_0(t) \leq C_o^*$ and hence $c_0(\infty) \leq C_o^*$. But $u_0(t, c_0(t)) = \gamma c_0(t)$ and $u_0(t, \cdot)$ converges uniformly to $p(\cdot)$, so $c_0(\infty) \geq C_o^*$.

Using the second inequality in (7.11), we obtain $c_K(t) \geq C_o^*$, and hence $c_K(\infty) \geq C_o^*$. Assume $C_o^* < c_K(\infty) \leq \frac{K}{\gamma}$. Because $c_K(\cdot)$ is nonincreasing, u_K is a viscosity solution of $u_t + \mathcal{N}u = c$ on $(0, \infty) \times (0, c_K(\infty))$. Hence the limit $p(\cdot)$ is a viscosity solution of this equation on $(0, c_K(\infty))$. But $p(x) = \gamma x$ for $x \in (C_o^*, c_K(\infty))$, and this does not satisfy $\mathcal{N}p = c$. This contradiction implies $c_K(\infty) = C_0^*$. ∎

Acknowledgments. We benefited from discussions with H. Mete Soner and Luc Tartar about the partial differential equations in this work, and we thank them for their assistance.

REFERENCES

[1] L. Andersen and D. Buffum, *Calibration and implementation of convertible bond models*, J. Comput. Finance, 2 (2003–2004), pp. 1–34.
[2] P. Asquith, *Convertible bonds are not called late*, J. Finance, 50 (1995), pp. 1275–1289.
[3] P. Asquith and D. Mullins, Jr., *Convertible debt: Corporate call policy and voluntary conversion*, J. Finance, 46 (1991), pp. 1273–1289.
[4] E. Ayache, P. A. Forsyth, and K. R. Vetzal, *The valuation of convertible bonds with credit risk*, J. Derivatives, 7 (2003), pp. 9–29.
[5] B. Bassan and C. Ceci, *Optimal stopping problems with discontinuous reward: Regularity of the value function and viscosity solutions*, Stoch. Stoch. Rep., 72 (2002), pp. 55–77.
[6] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley, New York, 1999.
[7] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Political Economy, 81 (1973), pp. 637–659.
[8] M. Brennan and E. Schwartz, *Convertible bonds: Valuation and optimal strategies for call and conversion*, J. Finance, 32 (1977), pp. 1699–1715.
[9] M. Brennan and E. Schwartz, *Analyzing convertible bonds*, J. Financial Quant. Anal., 15 (1980), pp. 907–929.
[10] W. Cheung and I. Nelken, *Costing converts*, Risk, 7 (1994), pp. 47–49.
[11] G. Constantinides, *Warrant exercise and bond conversion in competitive markets*, J. Financial Econom., 13 (1984), pp. 371–397.
[12] G. Constantinides and R. Rosenthal, *Strategic analysis of the competitive exercise of certain financial options*, J. Econom. Theory, 32 (1984), pp. 128–138.
[13] M. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second-order partial differential equations*, Bull. Amer. Math. Soc., 27 (1992), pp. 1–67.
[14] M. H. A. Davis and F. Lischka, *Convertible bonds with market risk and credit risk*, in Applied Probability, R. Chan, Y.-K. Kwok, D. Yao, and Q. Zhang, eds., AMS/IP Stud. Adv. Math. 26, Amer. Math. Soc., Providence, RI, 2002, pp. 45–58.
[15] F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann., 300 (1994), pp. 463–520.
[16] K. Dunn and K. Eades, *Voluntary conversion of convertible securities and the optimal call strategy*, J. Financial Econom., 23 (1984), pp. 273–301.
[17] W. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, 1993.
[18] J. M. Harrison and S. Pliska, *Martingales and stochastic integrals in the theory of continuous*

trading, Stoch. Process. Appl., 11 (1981), pp. 215–260.

[19] M. Harris and A. Raviv, *A sequential model of convertible debt call policy*, J. Finance, 40 (1985), pp. 1263–1282.

[20] C. Hennessy and Y. Tserlukevich, *Analyzing Callable and Convertible Bonds when the Modigliani-Miller Assumptions are Violated*, EFA Moscow Meetings Paper, Haas School, University California at Berkeley, Berkeley, CA, 2005.

[21] T. Ho and M. Pteffer, *Convertible bonds: Model, value, attribution and analytics*, Financial Analysts J., 52 (1996), pp. 35–44.

[22] J. E. Ingersoll, *A contingent-claims valuation of convertible securities*, J. Financial Econom., 4 (1977), pp. 289–322.

[23] J. E. Ingersoll, *An examination of corporate call policies on convertible securities*, J. Finance, 32 (1977), pp. 463–478.

[24] J. Kallsen and C. Kühn, *Convertible bonds: Financial derivatives of game type*, in Exotic Option Pricing and Advanced Lévy Models, A. Kyprianov, W. Schoutems, and P. Willmott, eds., Wiley, New York, 2005, pp. 277–292.

[25] I. Karatzas and S. Shreve, *Methods of Mathematical Finance*, Springer, New York, 1998.

[26] I. Kifer, *Game options*, Finance Stoch., 4 (2002), pp. 443–463.

[27] B. Loschak, *The Valuation of Defaultable Convertible Bonds under Stochastic Interest Rates*, Ph.D. dissertation, Krannert Graduate School of Management, Purdue University, West Lafayette, IN, 1996.

[28] J. J. McConnell and E. S. Schwartz, *LYON taming*, J. Finance, 41 (1986), pp. 561–577.

[29] A. McNee, *The challenge of equity derivatives*, Risk, Technology Supplement, Aug. (1999), pp. S29–S31.

[30] R. C. Merton, *Theory of rational option pricing*, Bell J. Econom. Manag. Sci., 4 (1973), pp. 141–183.

[31] R. C. Merton, *On the pricing of corporate debt: The risk structure of interest rates*, J. Finance, 29 (1974), pp. 449–470.

[32] M. Miller and F. Modigliani, *The cost of capital, corporation finance, and the theory of investment*, Amer. Econ. Rev., 48 (1958), pp. 261–297.

[33] M. Miller and F. Modigliani, *Dividend policy, growth and the valuation of shares*, J. Business, 34 (1961), pp. 411–433.

[34] B. Øksendal and K. Reikvam, *Viscosity solutions of optimal stopping problems*, Stoch. Stoch. Rep., 62 (1998), pp. 285–301.

[35] A. N. Shiryayev, *Optimal Stopping Rules*, Springer, Berlin, 1977.

[36] M. Sîrbu, I. Pikovsky, and S. Shreve, *Perpetual convertible bonds*, SIAM J. Control Optim., 43 (2004), pp. 58–85.

[37] D. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer, New York, 1979.

[38] A. Takahashi, T. Kobayashi, and N. Nakagawa, *Pricing convertible bonds with default risk*, J. Fixed Income, 11 (2001), pp. 20–29.

[39] K. Tsiveriotis and C. Fernandes, *Valuing convertible bonds with credit risk*, J. Fixed Income, 8 (1998), pp. 95–102.

[40] A. B. Yigitbasioglu, *Pricing Convertible Bonds with Interest Rate, Equity, Credit and FX Risk*, EFMA London Meeting Paper, ISMA Center, University of Reading, Reading, UK, 2002.