Portfolio Optimization Under Fixed Transaction Costs

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The model

Market with two securities:

- $b(t)$ - bond without interest rate
- $p(t)$ - stock, an Ito process

$$\begin{cases} db(t) &= 0 \\ dp(t) &= p(t)(\mu dt + \sigma dW_t) \end{cases}$$

with some $b(0) > 0$ and $p(0) > 0$;

- $\mu > 0$ - drift-mean rate of return
- $\sigma > 0$ -volatility
- W_t-standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

Denote

- $B(s)$ - current holdings in bond account
- $S(s)$ - current holdings in stocks
- $X(s) = B(s) + S(s)$ - total wealth at time s

So the total wealth satisfy

$$dX(s) = S(s)(\mu dt + \sigma dW_s)$$

and of course

$$X(s) \geq 0, \quad \text{for} \;\; t \leq s \leq T$$

for some fixed terminal time T and initial t.

At any instance the investor may rebalance his portfolio by moving capital from stocks to bond and vice versa.

$S(t) = S(t, X(t))$ can be taken as control.

The aim is to maximize an expected utility at some terminal time T , i.e. to find

$$w(x,t) = \sup_{\mathcal{A}} E\{U(X_T) \mid X_t = x\}$$

where $\mathcal{A}$ - is the set of all admissible controls.

The control $S(t) = S(t, X(t))$ is *Markovian* - the value we choose at time t depends only on the state of the system at this time.

With such control the process $X(t)$ is still a Markov Process.

Merton model (70's).No transactions case.

The solution $w(x,t)$ in this case is well known and is given by the Stochastic Control Methods - HJB equation:

$$\begin{cases} w_t + \sup_s \left[\frac{1}{2}\sigma^2 s(t,x)^2 w_{xx} + \mu s(t,x) w_x\right] = 0 \\ \\ w(0,t) = 0 \;\text{ for } t \in [0,T) \\ \\ w(x,T) = \frac{x^\alpha}{\alpha} \end{cases}$$

Such that:

$$w(x,t) = \frac{x^\alpha}{\alpha} e^{\lambda(T-t)} \quad \text{, where} \quad \lambda = \frac{\mu^2 \alpha}{2\sigma^2(1-\alpha)}$$

And the optimal investment strategy is

$$S^*(t, X_t) = \frac{\mu}{\sigma^2(1-\alpha)} X_t$$

So the Merton optimal strategy dictates that it is optimal to keep a *fixed proportion* of the total wealth invested in stocks.

Proportional transaction costs
Norman and Davis (1990)
Shreve and Soner (1994)

- Models with consumption.
- The local time type of the strategy

The policy leads to instantaneous controls, where, one trades "continuously" and with infinitesimal amounts when the risky proportion process is at boundary.

This is not the case in the real world.

Fixed transactions costs.

At every intervention time τ_i (the time where the investor rebalances his portfolio position) the investor has to pay fixed transaction cost K. Usually he pays the fee from the bond account.

Now we have to choose a sequence of intervention times and to trade only at these times and NOT at every instant as it was before, because the fixed component in transaction fee can lead such a policies to bankruptcy.

Definition of control

Impulse control for the process is a sequence

$$v = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), ...)_{j \le N}; \quad N \le \infty$$

- $\tau_1 < \tau_2 < ...$ are F_t - stopping times

- ζ_j are F_{τ_j} - measurable random variables representing the corresponding impulses.

τ_1, $\tau_2, ...$ are the *intervention times* - the times when we decide to intervene and give the system the impulses ζ_1, $\zeta_2, ..$ respectively.

- $N \le \infty$ is the number of interventions.

We are looking for

$$w(t, S, B) = \max_{v \in I} E(U(S_T + B_T) | S_t = S, B_t = B)$$

over all impulse control policies.

If $v = ((\tau_1, \zeta_1), (\tau_2, \zeta_2), ...)$ is applied to the process $X(t)$, it behaves according to :

$$\begin{cases} dX(t) = S(t)(\mu dt + \sigma dW_t), & \tau_{j-1} \leq t < \tau_j \leq T \\ X(\tau_j) = X(\tau_j^-) - K, & j = 1, 2, ...; \;\; \tau_j \leq T \\ B(\tau_j) = B(\tau_j^-) - K - \zeta_j \\ S(\tau_j) = S(\tau_j^-) + \zeta_j \end{cases}$$

where we put $\tau_0 = 0$.

Define *Intervention or maximum operator*

$$Mw(t,B,S) = \max_{\zeta}\{w(t,B-\zeta-K,S+\zeta)\}$$

It represents the value of the strategy, that consists of doing the best immediate action and to behave optimal afterwards.

It's clear that in general holds:

$$w(t,B,S) \geq Mw(t,B,S),$$

because it's not always optimal to trade at time t.
But when it's optimal, then

$$w(t,B,S) = Mw(t,B,S),$$

Recall the variational inequalities for optimal stopping problem:

Suppose that, the process $Y(t) \in V \subset R^k$ satisfies stochastic differential equation of the form

$$dY(t) = b(Y(t)) + \sigma(Y(t))dW(t) \ ; \ \ Y(0) = y$$

Define the differential operator L

$$L\phi = \sum_{i=1}^{k} b_i(y)\frac{\partial \phi}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{k}(\sigma\sigma^{tr})_{ij}(y)\frac{\partial^2 \phi}{\partial y_i \partial y_j}$$

Find $\Phi(y)$ and a stopping time $\tau < T$ such that

$$\Phi(y) = \sup_{\tau < T} E^y[g(Y_\tau)]$$

for some continuous function g

The main ideas

Suppose we can find a sufficiently smooth function ϕ, such that

- $\phi \geq g$ on V
- Define the continuation region

$$D = \{x \in V : \phi(x) > g(x)\}$$

- $L\phi \leq 0$ on $V \setminus \bar{D}$
- $L\phi = 0$ on D

then $\tau_D = \inf\{t > 0 : \ Y_t \notin D\}$
is the optimal stopping time for this problem and

$$\phi(y) = E^y[g(Y_{\tau_D})] = \Phi(y) = \sup_{\tau < T} E^y[g(Y_\tau)]$$

The impulse control (main ideas)

We are looking for a function w, such that:

- $w \geq Mw$ on V
- The *continuation region*

$$D = \{(t, B, S) : w > Mw\}$$

- $Lw \leq 0$ on $V \setminus \bar{D}$
- $Lw = 0$ on D

The intervention times are exit times from D.

QVI verification theorem.

We say that a sufficiently smooth function $w = w(t, B, S)$ is a solution of the quasi-variational inequalities (QVI) if

$$\begin{cases} Lw \leq 0 \\ \\ w \geq \mathsf{M}w \quad \text{and} \quad w = \mathsf{M}w \ \text{ on } \ D^c \\ \\ (w - \mathsf{M}w)Lw = 0 \\ \\ w(T, B, S) = U(B, S) \quad - \text{Utility function} \end{cases}$$

and the *impulse control policy* is defined as:

$$(\tau_0, \zeta_0) := (0, 0)$$

$$\tau_i := \inf\{t \geq \tau_{i-1} : \ w(t, B_{t^-}, S_{t^-}) = Mw(t, B_{t^-}, S_{t^-})\}$$

$$\zeta_i = \arg\max_{\zeta}\{w(t, B_{t^-} - \zeta - K, S_{t^-} + \zeta)\}$$

Theorem (Eastham and Hastings (1988))
The impulse control, defined above is optimal.

"...QVI's are typically very hard to solve explicitly. This is especially the case when there is a fixed finite time horizon T.
...I do not know of any general solution method. You are quite right that you have to do the same fantastic work for every problem. In fact, it has been said that these problems are so difficult that "every explicit solution is a triumph over nature"!

Good luck!
Best regards,
Bernt Oksendal"

works about fixed transaction costs

- Eastham and Hastings (1988) :
linear utility function, finite interval.

- Atkinson and Willmott (1997), Korn (1998)
approximation methods.

- Oksendal and Sulem(2002) :
the extension of **Shreve and Soner** work for case when fixed component is added to transaction costs. Numerical methods

- Pliska (2002) :
has solved explicitly the problem of optimally tracking a target mix of asset categories when there are fixed transaction costs.
Infinite interval.

Approximation scheme

Assume that the transaction cost is small

$$K = R\varepsilon^4, \quad \text{for some const } R$$

Recall that in the no transaction case the optimal amount of money, invested in bond is given as

$$B^*(t) = (1-u)X(t), \text{ with } u = \frac{\mu}{\sigma^2(1-\alpha)}$$

Rescale our variables by introducing the new variable $Y = Y(X_t, t)$:

$$B(t) = B^*(t) + \varepsilon Y(X_t, t) = (1-u)X(t) + \varepsilon Y$$

$$S(t) = X(t) - B(t) = uX(t) - \varepsilon Y$$

assumptions

• the no-transaction region (NT) has the form

$$D = \{(t, x, Y) : \widehat{Y}^- < Y < \widehat{Y}^+\}$$

• the upper and the lower boundaries of NT are

$$B^* + \varepsilon\widehat{Y}^+ \quad \text{and} \quad B^* + \varepsilon\widehat{Y}^-$$

• the upper and the lower optimal restarting lines are:

$$B^* + \varepsilon\widehat{Y}^- \quad \text{and} \quad B^* + \varepsilon Y^-$$

(the signs of $\widehat{Y}^-$and Y^- are negative).

• all $Y's$ can depend on $X(t)$

Also holds

$$\widehat{Y}^- \leq Y^- \leq 0 \leq Y^+ \leq \widehat{Y}^+$$

The optimal policy is:

1. Don't do anything in the NT region

2. If (X, Y) reaches either upper or lower boundary $(X, \widehat{Y}^{+})$ or $(X, \widehat{Y}^{-})$ the investor has to make a transaction and to move it back to NT
i.e. to corresponding restarting lines (X, Y^{+}) or (X, Y^{-}).

So in what follows we find all the boundaries and restarting lines.

So actually now we have a different process

$$(t, B, S) \to (t, X, Y)$$

with another characteristic operator L

$$L\varphi = \varphi_t + \mu(uX - \varepsilon Y)[\varphi_X + \frac{u-1}{\varepsilon}\varphi_Y] +$$

$$\frac{1}{2}\sigma^2(uX - \varepsilon Y)^2[\varphi_{XX} + 2\frac{u-1}{\varepsilon}\varphi_{XY} + \frac{(u-1)^2}{\varepsilon^2}\varphi_{YY}]$$

and the value function has changed as well

$$w(t, B, S) \to Q(t, X, Y)$$

More generalized version of QVI is needed
Oksendal lecture notes (2002)

In the NT we expand Q as:

$$Q(t,x,Y) = H_0(t,x) + \varepsilon^2 H_2(t,x) + \varepsilon^4 G(t,x,Y) + ..$$

Substitute it into $LQ = 0$ then order the terms by the powers of ε.

Taking only $O(1)$ and $O(\varepsilon^2)$ equations

and using smoothness conditions $(t < T)$:

$$\begin{cases} Q(t,x,Y) = Q(t,x-k,Y^+) \quad \forall\, Y \geq \widehat{Y}^+ \\ \\ Q_Y(t,x,Y) = Q_Y(t,x-k,Y^+) \quad \forall\, Y \geq \widehat{Y}^+ \end{cases}$$

as well as *optimality of transaction* condition

$$\frac{d}{d(\Delta y)} Q(t,x-k,\widehat{Y}^+ - \Delta y)|_{\Delta y = \widehat{Y}^+ - Y^+} = 0$$

we are able to obtain $\widehat{Y}^+$ and Y^+.

Just the same work is done for the lower bound.

We are free to choose some comfortable form of the Q-function, for instance -the power form

$$Q(t,x,Y) =$$

$$\frac{1}{\alpha}(x + \varepsilon^2 H(t,x) + \varepsilon^4 G(t,x,Y) + ...)^{\alpha} e^{\lambda(T-t)}$$

with $\lambda = \frac{\mu^2\alpha}{2\sigma^2(1-\alpha)}$

Results :

$$Y^{+} = 0, \ \ Y^{-} = 0 \quad \text{and}$$

$$\widehat{Y}^{+} = \sqrt[4]{\frac{12Ru^2(u-1)^2}{1-\alpha}} \ x^{\frac{3}{4}}$$

$$\text{also } \widehat{Y}^{-} = -\widehat{Y}^{+}$$

Maximizing logarithmic utility.

The new value function is

$$w(x,t) = \sup_{\mathcal{A}} E[\,\log X_T \mid X_t = x]$$

The optimal solution is always to keep the same proportion of the whole wealth in stocks.

$$u = \frac{b}{\sigma^2}$$

and the value function is given as

$$w(t,B,S) = w(t,x) = \ln x + \lambda(T-t), \quad \text{where} \quad \lambda = \frac{b^2}{2\sigma^2}$$

The solution of this problem in the no-transactions case makes us able to approach it now. We get

$$\widehat{Y}^+ = \sqrt[4]{12Ru^2(1-u)^2}\ \ x^{\frac{3}{4}}$$

Notice that this policy can be derived as the limit power utility policy taking $\alpha \downarrow 0$.

Goal function

Another kind of problems, arising in portfolio optimization context are so called **optimal goal problems**.

$$w(t,x) = \sup_{A} P(X_T = 1 \mid X_t = x)$$

The **no transaction case** solution was first derived by martingale methods by Heath (1993), Kulldorff (1993) and is as follows:

the optimal amount of money held in stock at time s is

$$S_s^* = \frac{1}{\sqrt{T-s}}\varphi(\Phi^{-1}(X_s)), \quad t \le s \le T$$

where $\Phi(\cdot)$ is the standard normal distribution function and $\varphi(\cdot)$ is the density of the standard normal distribution.

And the value function in this case is given by

$$w(t,x) = \Phi(\Phi^{-1}(x) + \mu\sqrt{T-t}).$$

Try to apply the above approximation method to the goal function problem.

Rescale our variables by introducing the new variable $Y = Y(t, X_t)$:

$$S_t = S_t^* + \frac{\varepsilon Y(X_t, t)}{\sqrt{T-t}} =$$

$$\frac{1}{\sqrt{T-t}}[\varphi(\Phi^{-1}(X_t)) + \varepsilon Y(t, X_t)]$$

As a result we get

$$\widehat{Y}^{+}(X_t, t) =$$

$$\sqrt[4]{\frac{12R}{\mu}\varphi^3(\Phi^{-1}(X_t))[\sqrt{T-t} + \Phi^{-1}(X_t)]^2}\ \sqrt[8]{T-t}$$