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SENSOR ALLOCATION PROBLEMS ON THE REAL LINE

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Abstract

A large number n of sensors (finite connected intervals) are placed randomly on the real line so that the distances between the consecutive midpoints are independent random variables with expectation inversely proportional to n. In this work we address two fundamental sensor allocation problems. Interference problem tries to reallocate the sensors from their initial positions to eliminate overlaps. Coverage problem, on the other hand, allows overlaps, but tries to eliminate uncovered spaces between the originally placed sensors. Both problems seek to minimize the total sensor movement while reaching their respective goals.

Using tools from queueing theory, Skorokhod reflections and weak convergence, we investigate asymptotic behaviour of optimal costs as n increases to infinity. The introduced methodology is then used to address a more complicated, modified coverage problem, in which the overlaps between any two sensors can not exceed a certain parameter.

Keywords: sensor allocation, queueing theory, potential outflow, reflected random walks, Skorokhod maps, weak convergence

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1. Introduction

A large number n of sensors are distributed over the positive half-line. Each sensor is represented by a finite closed interval of the same length σ^n, and the distances between consecutive midpoints are i.i.d. random variables with expectation $1/n$. An arbitrary realization will possibly have some overlapping sensors, as well as sensors with gaps between them. The *coverage* and *interference* problems aim to *reallocate the sensors in an optimal way* to either eliminate the overlaps (interference problem) or rather to remove the gaps, securing uninterrupted coverage starting from the origin (coverage problem). In both problems the optimality criteria are chosen to be minimization of the total movement of sensors.

In this paper we consider both problems in the asymptotic setting where the number of sensors (n) increases to infinity and the length σ^n becomes infinitely small, roughly inversely proportional to n. Our main goal at this stage is *not to solve* the coverage or interference problems, but to *understand* orders of magnitude for optimal allocation costs and their dependence on the limiting behaviour of σ^n. The results are of particular importance in computer science, communications and networking; and later may be used as a starting point for finding an optimal solution. To the best of our knowledge, such an asymptotic setting has yet to be fully studied. Some initial results appear in the conference proceedings ([9], coverage) and ([10], interference) treating the special cases when the i.i.d. distances between sensor midpoints are either uniform [9] or exponential [10]. Both papers study performance bounds for some allocation algorithms, mostly using *straightforward* computations. The methods had their limitations - not all choices of σ^n could be treated, and the bounds were mostly one-sided. Besides, no coherent connection was established between the coverage and interference problems.

In what follows we introduce a unifying methodology to address both problems simultaneously. The spectrum of all values for the sensor length σ^n is decomposed into three different regions (or categories), characterized by whether the length is asymptotically greater than, less than or equal to the average original distance of $1/n$ between consecutive sensors' midpoints. The three categories will be referred to as *overloaded*, *underloaded* or *critical*; and each will produce distinct two-sided estimates for both coverage and interference optimal costs. The terminology clearly suggests an analogy with *queueing theory* and, indeed, quite a bit of it will be used. In particular, we use the so-called *potential outflow* function (3.1) and related functionals. Our main findings are summarized in Theorem 2.1. It is shown that the two costs behave quite the opposite in the underloaded and overloaded cases, while exhibiting the same order of magnitude in the critical case.

The resulting table may seem intuitive or even simple (we prefer *elegant*), yet the situation was far from being that clear before the 'queueing link' had been established. In fact, the orders of O(1) were not expected, for the *coverage* problem in particular. There, a certain Skorokhod reflection map (6.3) had to be introduced (the result may be of separate interest in queueing theory as it can be used for analyzing busy periods in $G/D/\infty$ models). In addition, to show the *almost sure* relation we could no longer rely on the methodology of asymptotic queueing analysis (very valuable to establish the tightness, for example). Instead we had to bound the reflection map by a certain discrete-time continuous-space Markov chain in the spirit of [13], making it possible to apply the classical 'Law of Large Numbers for Markov chains'.

The new methods are then exploited to study a more complicated, *mixed* problem (see (2.5) for the formulation, and Theorem 2.2). Once again, the O(1) estimates were the hardest to show, requiring a modified version (8.2) of the previously introduced Skorokhod map. After that, the map (8.2) had to be bounded from both sides by reflected random walks, amenable to analysis. Overall we believe that our methodology is robust, and can be applied to various related questions.

Related literature. Both coverage and interference problems have been studied before. [6] deals with covering of a two-dimensional domain by sensors. A specific and more efficient type of coverage is called *barrier coverage* and concerns covering the perimeter of a two dimensional object. For the barrier coverage problem the authors of [11] propose efficient algorithms to determine, after deploying the sensors, whether a region is k-barrier covered. For the problem of coverage on a line, the authors of [5] consider the complexity of the (total) displacement problem on a line and propose deterministic algorithms; similarly, the same authors in [4] analyze the complexity of the maximum displacement problem on a line and propose deterministic algorithms.

Interference in a network may be caused by node transmission power and proximity and affects the overall communication connectivity. In [2] the authors propose connectivity-preserving and spanner constructions that are interference-minimal while in [14] they study the problem of minimizing the average interference while still maintaining desired network properties, such as connectivity, point-to-point connections, or multicast trees.

Notation. We use $\mathcal{L}\{[a,b]\} = b-a$ for the length of the interval $[a,b]$ on the real line. For $x,y \in \mathbb{R}$, we use $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$. We also use $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$ for positive and negative parts of $x \in \mathbb{R}$, respectively. For a vector $a \in \mathbb{R}^n$ we use $\|a\| = \sum_{k=1}^n |a_k|$. For a function $f(\cdot)$ on $\mathbb{R}$ we use $f(u-) = \lim_{t\uparrow u} f(t)$ to denote a left-hand limit of f at point u.

For positive sequences $\{x_n, n \geq 1\}$ and $\{y_n, n \geq 1\}$ we say that $x_n \in \Theta(y_n)$ if there exist two constants $c_1 > 0$ and $c_2 > 0$, independent of n, such that $c_1 y_n \leq x_n \leq c_2 y_n$ for all n large enough. In addition, we say $x_n \in \mathrm{O}(y_n)$ (resp., $x_n \in \Omega(y_n)$), if there exists a constant $c > 0$ such that $x_n \leq c y_n$ (resp., $c y_n \leq x_n$) for n large enough.

2. Setting and the main results

The n sensors are represented by n finite closed intervals $[\tau_i^n, \tau_i^n + \sigma^n],\ 1 \leq i \leq n$, where we use τ_i^n for the left endpoint of the i^{th} sensor; and σ^n for its length. All sensors have the same length σ^n. The model is parameterized by $n \in \mathbb{N}$, the number of sensors under consideration. We start with a basic assumption.

Assumption 2.1. *For a given $n \in \mathbb{N}$, the left endpoints $\{\tau_i^n\}_{i=1}^n$ satisfy $\tau_1^n = \zeta_1/n$ and $\tau_i^n - \tau_{i-1}^n = \zeta_i/n,\ 2 \leq i \leq n$, where $\{\zeta_i\}_{i\geq 1}$ are i.i.d. positive, absolutely continuous random variables with $\mathbb{E}(\zeta_1) = 1$ and $\mathbb{E}(\zeta_1^2) < \infty$.*

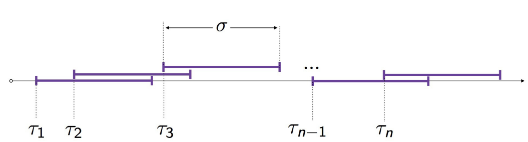

FIGURE 1: *Sensors (without superscripts, and lifted above each other to clarify the exposition).*

2.1. Interference and Coverage problems

This paper is about allocating (or, better, re-allocating) the sensors along the positive real half-line to achieve certain properties. A set $\pi^n = \{\pi_i^n\}_{i=1}^n \in \mathbb{R}^n$ will be called a *displacement policy*, each $\pi_i^n \in \mathbb{R}$ being a displacement of the i^{th} sensor from its *original* location $[\tau_i^n, \tau_i^n + \sigma^n]$ to a new location $[\tau_i^n - \pi_i^n, \tau_i^n - \pi_i^n + \sigma^n]$. The sensors must not exit the half-line, thus requiring $\pi_i^n \le \tau_i^n$. For a policy π^n we let $\|\pi^n\| = \sum_{i=1}^n |\pi_i^n|$ represent the total *displacement cost*.

• Problem 1. (Interference): minimize the total displacement cost needed to eliminate overlaps between the sensors. That is, find

$$C_{int}^n = \min_{\pi^n \in \Pi_1^n} \|\pi^n\|, \tag{2.1}$$

where $\pi^n \in \Pi_1^n$, if $(\tau_i^n - \pi_i^n, \tau_i^n - \pi_i^n + \sigma^n) \cap (\tau_j^n - \pi_j^n, \tau_j^n - \pi_j^n + \sigma^n) = \emptyset$ for $i \neq j$ and, in addition, $\pi_i^n \le \tau_i^n$ for each i.

• Problem 2. (Coverage): minimize the total displacement cost needed to eliminate gaps between the sensors, that is, find

$$C_{cov}^n = \min_{\pi^n \in \Pi_2^n} \|\pi^n\|, \tag{2.2}$$

where $\pi^n \in \Pi_2^n$, if $\bigcup_{i=1}^n [\tau_i^n - \pi_i^n, \tau_i^n - \pi_i^n + \sigma^n] = \Big[0, \max_{i=1,..,n}\{\tau_i^n - \pi_i^n + \sigma^n\}\Big]$.

The main objective of the paper is to understand the large n asymptotics of the cost of total movement. It appears, the results strongly depend on whether the sensor length σ^n is greater than, less than or asymptotically (as parameter $n \uparrow \infty$) equal to the average distance of $1/n$ between initially placed sensors.

Assumption 2.2. *The sensor length σ^n is deterministic and satisfies*

$$\alpha_\sigma := \lim_{n\to\infty} \frac{\sigma^n}{1/n} = \lim_{n\to\infty} n\sigma^n < \infty. \tag{2.3}$$

Definition 2.1. The set of sensors is of type **U** (underloaded) if $\alpha_\sigma < 1$; of type **O** (overloaded) if $\alpha_\sigma > 1$; of type **C** (critical) if $\alpha_\sigma = 1$ and $\lim\limits_{n\to\infty} \sqrt{n}(n\sigma^n - 1) \in (-\infty, \infty)$.

Theorem 2.1. (Interference and Coverage). *Let Assumptions 2.1 and 2.2 hold. The table below summarizes asymptotics (as $n \to \infty$) of the cost of optimal displacement for problems* (2.1) *and* (2.2)*, for each type of sensor sets.*

Regime	**Interference** C^n_{int}	**Coverage** C^n_{cov}	**Result holds in**
U	$\mathrm{O}(1)$	$\Theta(n)$	*Expectation/a.s.*
C	$\Theta(n^{1/2})$	$\Theta(n^{1/2})$	*Expectation*
O	$\Theta(n)$	$\mathrm{O}(1)$	*Expectation/a.s.*

Theorem 2.1 is proven in several stages, over the next five sections. Section 3 deals with some necessary prerequisites. The interference problem is treated in Sections 4 and 5: we introduce certain lower and upper bounds (Section 4), and analyze them asymptotically afterwards (Section 5). Similarly, the coverage problem is treated in Sections 6 and 7. The concluding Section 10 discusses possible alternatives to the characterization (Definition 2.1) of limiting regimes.

2.2. Mixed problem

We introduce a modification of the *coverage problem*. Let r^n be a deterministic sequence of positive numbers with $r^n \le \sigma^n$, and assume the existence of the limit

$$\alpha_r := \lim_{n\to\infty} nr^n < \infty. \tag{2.4}$$

• Problem 3. (Mixed): minimize the total displacement cost needed to eliminate gaps between the sensors, with an additional requirement that no two sensors can overlap by more than r^n. That is, we seek

$$C^n_{mix} = \min_{\Pi^n_2 \cap \Pi^n_3} \|\pi^n\|, \tag{2.5}$$

where $\pi^n \in \Pi^n_3$, if $\mathcal{L}\Big\{[\tau^n_i - \pi^n_i, \tau^n_i - \pi^n_i + \sigma^n] \cap [\tau^n_j - \pi^n_j, \tau^n_j - \pi^n_j + \sigma^n]\Big\} \le r^n, \quad i \ne j.$

For the mixed problem we will show that the *coverage* results of Theorem 2.1 remain the same for underloaded and critical sensor sets. On the other hand, when sensor set is overloaded, the results will depend on asymptotic behaviour of the difference $\sigma^n - r^n$.

Definition 2.2. Recall Definition 2.1. A sensor set of type **O** (overloaded) is said to be of type $\mathbf{O^{(u)}}$ if $\alpha_\sigma - \alpha_r < 1$; of type $\mathbf{O^{(o)}}$ if $\alpha_\sigma - \alpha_r > 1$; of type $\mathbf{O^{(c)}}$ if $\alpha_\sigma - \alpha_r = 1$ and $\lim_{n\to\infty} \sqrt{n}(n\sigma^n - nr^n - 1) \in (-\infty, \infty)$.

Theorem 2.2. (Mixed). *Let Assumptions 2.1 and 2.2 hold. The table below summarizes the asymptotics of C^n_{mix} for each type of sensor sets.*

Regimes	**Mixed** C^n_{mix}	**Result holds in**
$\mathbf{U}$, $\mathbf{O^{(o)}}$	$\Theta(n)$	*Expectation/a.s.*
$\mathbf{C}$, $\mathbf{O^{(c)}}$	$\Theta(n^{1/2})$	*Expectation*
$\mathbf{O^{(u)}}$	O(1)	*Expectation/a.s.*

Since all five categories $\mathbf{U}$, $\mathbf{C}$, $\mathbf{O^{(o)}}$, $\mathbf{O^{(u)}}$ and $\mathbf{O^{(c)}}$ are disjoint, the table actually says that $\Theta(n)$ estimate, for example, holds for both types $\mathbf{U}$ and $\mathbf{O^{(o)}}$; and so on. The theorem is treated in Sections 8 and 9. Once again, some bounds are introduced and then analyzed asymptotically. To some extent the mixed problem resembles the coverage problem.

3. Preliminary analysis

3.1. Potential outflow. Connection to queueing theory

The *potential outflow* of a counting process $N^n(t) := \sum_{i=1}^n 1_{\{\tau_i^n \le t\}}$, for $0 \le t \le \tau^n$, is defined as

$$W^n(t) = \sigma^n N^n(t) - t \,, \qquad 0 \le t \le \tau^n. \tag{3.1}$$

The notion comes from the $G/D/1$ queueing theory: the left endpoints $\{\tau_i^n\}$ represent arrival epochs with *Generally* distributed inter-arrival times, and the sensor's length σ^n stands for *Deterministic* service requirement. N^n is an *arrival* process. The potential outflow W^n will play a key role in construction and analysis of various displacement policies. Definition (3.1) implies (pay attention to the left-hand limit):

$$W^n(\tau_k^n-) = (k-1)\sigma^n - \tau_k^n, \qquad 1 \le k \le n. \tag{3.2}$$

3.2. Base coverage policy

Introduce a policy $\pi^n = \{\pi_k^n\}_{k=1}^n$ satisfying $\pi_k^n = -W^n(\tau_k^n-) = \tau_k^n - (k-1)\sigma^n$, which trivially places the k^{th} sensor in the interval $[(k-1)\sigma^n, k\sigma^n]$ for each $k = 1, ..., n$. The base coverage policy is clearly admissible for each of the problems (2.1), (2.2) and (2.5), and can be used as an upper bound for either C^n_{int}, C^n_{cov} or C^n_{mix} :

$$C^n_{int/cov/mix} \le \sum_{k=1}^n \Big| W^n(\tau_k^n-) \Big|. \tag{3.3}$$

3.3. Ordered policies

In the sequel, we will restrict the policy sets Π_1^n, Π_2^n, Π_3^n to policies satisfying (3.4), allowing to contain only the *ordered* (that is, according to their initial order) displacement of sensors. The claim is justified in Lemma 3.1 below, whose simple proof is omitted. Such a reduction of the policy sets is indeed crucial and will be used later to establish a connection to queueing theory. To keep it simple, we will abuse the notation and retain the letters Π_1^n, Π_2^n, Π_3^n.

Lemma 3.1. *An optimal displacement policy for each of the problems* (2.1)*,* (2.2) *and* (2.5) *necessarily satisfies*

$$\tau_i^n - \pi_i^n \le \tau_j^n - \pi_j^n \ , \qquad \textit{for} \qquad i < j. \tag{3.4}$$

4. Interference problem

4.1. Characterization of a policy, and a lower bound on performance

For a given initial allocation $\{\tau_1^n, \tau_2^n, \dots\}$ and a policy $\pi^n \in \Pi_1^n$, introduce $\{\xi\}_{i=1}^n$, recursively defined as $\xi_1 = \tau_1^n - \pi_1^n$ and

$$\xi_k = \tau_k^n - \pi_k^n - \sum_{i=1}^{k-1} (\sigma^n + \xi_i) \ , \qquad k = 2, ..., n. \tag{4.1}$$

Clearly $\xi_1 \ge 0$, and it is the distance from the origin to the 1^{st} shifted sensor. Likewise, for each k, the quantity $\xi_k \ge 0$ represents the length of the gap between *consecutive* shifted sensors $k-1$ and k (since $\sum_{i=1}^{k-1}(\sigma^n + \xi_i)$ is the *right endpoint* of the $(k-1)^{th}$ shifted sensor). Note that the shifted sensors must be consecutive, as required by (3.4).

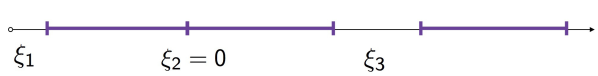

FIGURE *2: Gaps between shifted sensors (interference problem).*

Essentially, $\{\xi\}_{i=1}^n$ uniquely characterize the policy π^n due to (see also (3.2))

$$\pi_k^n = -\left(W^n(\tau_k^n -) + \sum_{i=1}^{k} \xi_i \right), \qquad 1 \le k \le n. \tag{4.2}$$

As a result, since $\xi_i \geq 0$, we can deduce the following lower bound

$$C^n_{int} = \sum_{k=1}^{n} \left| W^n(\tau^n_k -) + \sum_{i=1}^{k} \xi_i \right| \geq \sum_{k=1}^{n} \Big(W^n(\tau^n_k -) \Big)^+. \tag{4.3}$$

4.2. Right-shift policy and an upper bound on performance

RIGHT-SHIFT POLICY: the first sensor remains on its original location. After that, for $k \geq 1$: the $(k+1)^{th}$ sensor remains on its original location if it does not overlap with already re-allocated k^{th} sensor; otherwise, the $(k+1)^{th}$ sensor is shifted to the right by the minimum distance required for that overlap to be eliminated.

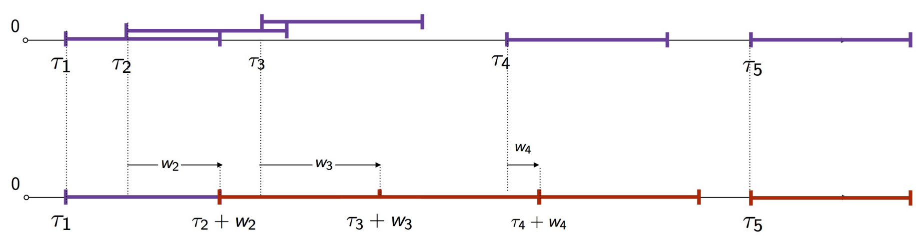

FIGURE 3: *An implementation of the right-shift policy:* 2^{nd}, 3^{rd} *and* 4^{th} *sensors are shifted to the right by* w_2, w_3 *and* w_4 *units, respectively. The* 1^{st} *and the* 5^{th} *sensors do not move. The corresponding policy* $\{\pi_1, \pi_2, ..\}$ *is then given as* $\pi_1 = \pi_5 = 0$, $\pi_i = -w_i$, *for* $i = 2, 3, 4$.

Under the right-shift policy, the gaps $\{\xi_i\}$ between shifted sensors must satisfy

$$\xi_k \;\; = \left(\sum_{i=1}^{k-1} (\sigma^n + \xi_i) - \tau^n_k \right)^-, \qquad 1 \leq k \leq n, \tag{4.4}$$

since $\sum_{i=1}^{k-1}(\sigma^n + \xi_i)$ is the *right endpoint* of the $(k-1)^{th}$ shifted sensor.

Referring to the queueing model from subsection 3.1 and its terminology, the gap's length ξ_k in (4.4) corresponds to the *idling time* between serving the $(k-1)^{th}$ and the k^{th} customers, in a standard $G/D/1$ queue under the First-Come-First-Serve discipline. Moreover, the quantity $\sum_{i=1}^{k-1}(\sigma^n + \xi_i)$ now represents the *departure time* of the $(k-1)^{th}$ customer after being served. Combining the latter observation with (4.1) and (4.4), we see that the absolute value of the k^{th} displacement

$$|\pi^n_k| = -\pi^n_k = \left(\sum_{i=1}^{k-1} (\sigma^n + \xi_i) - \tau^n_k \right)^+, \qquad 1 \leq k \leq n \tag{4.5}$$

represents the *waiting time* of the k^{th} arriving customer (See Fig. 3). As a result, the total displacement cost equals to the total waiting time among n customers and

$$C^n_{int} \le \sum_{k=1}^{n} w^n_k, \tag{4.6}$$

where w^n_k stands for the waiting time of the k^{th} customer in a corresponding queue.

5. Asymptotic analysis of Interference Problem

Proposition 5.1. (Overloaded regime). *For sensor sets of type* **O**, *as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{int}) = \Theta(n) \quad \textit{and} \quad C^n_{int} = \Theta(n) \quad \textit{a.s.}} \tag{5.1}$$

Proof. Combining (3.3) and (4.3), one gets

$$\sum_{k=1}^{n} \Big(W^n(\tau^n_k -)\Big)^+ \le C^n_{int} \le \sum_{k=1}^{n} \Big|W^n(\tau^n_k -)\Big|. \tag{5.2}$$

We start with the lower bound. Since $n\sigma^n \to \alpha_\sigma > 1$, there exist constants $\delta > 0$ and $c > 0$ so that for n large enough $\sigma^n - (1+\delta)/n \ge c/n$. From (3.2) and Assumption 2.1

$$W^n(\tau^n_k -) = -\sigma^n + \sum_{i=1}^{k} \left(\sigma^n - \frac{1+\delta}{n}\right) + \sum_{i=1}^{k} \left(\frac{1+\delta}{n} - \frac{\zeta_i}{n}\right).$$

Since $\mathbb{E}(\zeta_i) = 1$ and $\delta > 0$, the expectation of the last term is positive. In addition, by the Law of Large Numbers, we almost surely have $\sum_{k=1}^{n} \frac{k}{n} \left[\frac{1}{k} \sum_{i=1}^{k} (1+\delta-\zeta_i)\right] \ge 0$ for n large enough. This implies $\sum_{k=1}^{n} W^n(\tau^n_k -) \ge -n\sigma^n + (1/2)cn \ge -(1+\alpha_\sigma) + (1/2)cn \ge (1/3)cn$, for n large enough. Since $(W^n)^+ \ge W^n$, the lower bound in (5.2) together with the last inequality, imply $C^n_{int} = \Omega(n)$ a.s. and in expectation.

For the upper bound we use a crude estimate $|W^n(\tau^n_k -)| \le (k-1)\sigma^n + \tau^n_k$. By Definition 2.1, we have $\sum_{k=1}^{n}(k-1)\sigma^n = \mathrm{O}(n)$ so we just need to establish a bound for $\sum_{k=1}^{n} \tau^n_k$. We have (recall Assumption 2.1),

$$\sum_{k=1}^{n} \tau^n_k = \sum_{k=1}^{n} \frac{k}{n} + \sum_{k=1}^{n} \left(\tau^n_k - \frac{k}{n}\right) = \frac{n+1}{2} + \sum_{k=1}^{n} \frac{k}{n} \left[\frac{1}{k} \sum_{i=1}^{k} (\zeta_i - 1)\right].$$

The expectation of the last term above is trivially zero. Now, fix an arbitrary $\varepsilon > 0$. By the Law of Large Numbers, almost surely we have $(1/k) \sum_{i=1}^{k} (\zeta_i - 1) \le \varepsilon$ for all k large enough. Therefore, $\sum_{k=1}^{n} \tau^n_k \le n(1+\varepsilon)$ for all n large enough. Hence $\sum_{k=1}^{n} \tau^n_k = \mathrm{O}(n)$ both in expectation and almost surely, concluding the proof. □

Proposition 5.2. (Critical regime). *For sensor sets of type* **C**, *as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{int}) = \Theta(n^{1/2})} \tag{5.3}$$

Proof. Once again, we analyze the bounds from (5.2). Rewrite

$$\sum_{k=1}^{n} \Big(W^n(\tau^n_k -)\Big)^+ = \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ d[N^n(t) - a^n(t)] + \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ da^n(t) \tag{5.4}$$

where (see [7], p. 98) a^n is a compensator of N^n, hence the integrator $N^n(t) - a^n(t)$ in the first expression of (5.4) is a martingale. As a result, the process $M(\cdot) = \int_0^{\cdot} \Big(W^n(t-)\Big)^+ d[N^n(t) - a^n(t)]$ is a martingale (see Thm. 6.5.8 in [12], p. 88) with expectation zero, hence by the optional sampling theorem (see Thm. 3.22 in [8], p. 19) (since τ^n_n is a stopping time), we get that the first term in (5.4) has zero expectation. Regarding the second term of (5.4), the compensator must satisfy

$$\frac{da^n(t)}{dt} = n\ \frac{f_\zeta(nt - n\tau^n_{N^n(t-)})}{1 - F_\zeta(nt - n\tau^n_{N^n(t-)})} = n\ h_\zeta(nt - n\tau^n_{N^n(t-)}), \tag{5.5}$$

where h_ζ is a hazard rate of random variable ζ. We claim that

$$\int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ da^n(t) \geq c_1\ n \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ dt - c_2, \tag{5.6}$$

for some constants $c_1 > 0$ and $c_2 > 0$. Indeed, take some $c_1 > 0$ and let D be the set on which $h_\zeta(\cdot) \geq 2c_1$. Due to the "triangular" shape of W^n, we can ensure that c_1 is small enough to guarantee $\int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ \mathbb{I}_{D^c}(t)\ dt \leq \frac{1}{2} \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ dt + n\frac{(\sigma^n)^2}{2}$. Then $n \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ h_\zeta(nt - n\tau^n_{N^n(t-)})dt \geq 2c_1 n \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ \mathbb{I}_D(t)\ dt = 2c_1 n \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ dt - 2c_1 n \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ \mathbb{I}_{D^c}(t)\ dt \geq c_1 n \int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ dt - c_2$ for some $c_2 > 0$, because $(n\sigma^n)^2 = \Theta(1)$; and (5.6) follows. Combining (5.4) - (5.6),

$$\mathbb{E}\left[\sum_{k=1}^{n} \Big(W^n(\tau^n_k -)\Big)^+\right] \geq c_1\ n\ \mathbb{E}\left[\int_0^{\tau^n_n} \Big(W^n(t-)\Big)^+ dt\right] - c_2. \tag{5.7}$$

Introduce $B^n(t) := n^{-1/2}(N^n(t) - nt)$ and rewrite $W^n(t)$ as

$$W^n(t) = n^{-1/2} \Big[(n\sigma^n)B^n(t) + n^{1/2}(n\sigma^n - 1)t\Big] \tag{5.8}$$

It is well known that B^n converges to a standard Brownian motion (see Thm. 5.11 in [3], p. 110). In addition, we have $\mathbb{E}(\tau^n_n) = 1$. Next, we combine (5.8) and (5.7) with the lower bound inequality in (5.2), and use Fatou's Lemma, to get $\mathbb{E}(C^n_{int}) = \Omega(n^{1/2})$. For the upper bound, we use Lemma 2 from ([1], page p.1102) that claims $\mathbb{E}\left(|B^n(t)|\right) \le c_1(1+t^{c_2}),\ n \in \mathbb{N},\ t \in \mathbb{R}$ for some constants c_1 and c_2 independent of n and t. Applied to both (5.8) and the upper bound in (5.2), this yields $\mathbb{E}(C^n_{int}) = \mathrm{O}(n^{1/2})$ and (5.3). $\square$

Proposition 5.3. (Underloaded regime). *For sensor sets of type* $\mathbf{U}$*, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{int}) = \mathrm{O}(1) \quad \textit{and} \quad C^n_{int} = \mathrm{O}(1) \quad \textit{a.s.}} \tag{5.9}$$

Proof. Consider (4.6) with w^n_k - the waiting time of the k^{th} customer in a $G/D/1$ queue. Recall that the times between arrivals are i.i.d. random variables, distributed as ζ/n, and the service is deterministic σ^n. From Lindley's relations (Sec. 4.20 in [16]) we have $w_1 = 0$ and $w^n_{k+1} = \left(w^n_k + \sigma^n - \zeta_{k+1}/n\right)^+$ for $k \ge 1$. Introduce $\phi^n_k = nw^n_{k+1}$ and $\widetilde{\zeta}_k = \zeta_{k+1}$ for $k \ge 0$. The sequence $\{\phi^n_k\}_{k\ge 0}$ satisfies $\phi^n_0 = 0$ and $\phi^n_{k+1} = \left(\phi^n_k + (n\sigma^n - \widetilde{\zeta}_{k+1})\right)^+$, defining a random walk on a positive half-plane, as introduced in Appendix A, with the *increment* $x^n_k = n\sigma^n - \widetilde{\zeta}_k$. To eliminate dependence on the parameter n in the increment, take some $\mathring{\sigma}$ satisfying $n\sigma^n < \mathring{\sigma} < 1$, (the existence of such $\mathring{\sigma}$ for n large enough is guaranteed by having $n\sigma^n \to \alpha_\sigma < 1$). Let $y_k = \mathring{\sigma} - \widetilde{\zeta}_k$, and define a sequence $\{\phi^y_k, k \ge 0\}$ as $\phi^y_0 = 0$ and $\phi^y_{k+1} = \left(\phi^y_k + y_{k+1}\right)^+,\ k \ge 0$. By the choice of $\mathring{\sigma}$, we must have $x^n_k \le y_k$. Next, we use the comparison Lemma A.2 to get $\phi^n_k \le \phi^y_k$ for all $k \ge 0$. Finally, by applying $\phi^n_k = nw^n_{k+1} \le \phi^y_k$ to (4.6)

$$C^n_{int} \le \frac{1}{n}\sum_{k=1}^{n-1}\phi^n_k \quad \le \quad \frac{1}{n}\sum_{k=1}^{n}\phi^y_k, \tag{5.10}$$

and we are ready to apply Lemma A.1 to $\mathbf{\Phi} := \{\phi^y_k, k \ge 0\}$, with i.i.d. increments $\{y_1, y_2, ...\}$, satisfying $\mathbb{E}(y_1) < 0$, to get (5.9). This completes the proof. $\square$

We point out the clear advantage of the *queueing* bound (4.6) over the upper *workload* bound (5.2), which would still be at least $\Omega(\sqrt{n})$ (see (5.3)). At the same time, the lower bound in (5.2) will *undergo* a change in the regime and become $\mathrm{O}(1)$, since in the underloaded case W^n tends to be negative.

6. Coverage problem

6.1. A trivial lower bound on performance

Let $\pi^n \in \Pi_2^n$ (also satisfying (3.4)). If $\tau_k^n \geq (k-1)\sigma^n$ for some $k \geq 1$ then, obviously, π_k^n must satisfy $\pi_k^n \geq \tau_k^n - (k-1)\sigma^n$. This implies $|\pi_k^n| \geq (\tau_k^n - (k-1)\sigma^n)^+ = (-W^n(\tau_k^n-))^+ = (W^n(\tau_k^n-))^-$. And we have a lower bound in terms of

$$C_{cov}^n \geq \sum_{k=1}^{n} \Big(W^n(\tau_k^n-)\Big)^- . \tag{6.1}$$

6.2. Left-shift policy and an upper bound on performance

Left-shift policy: the first sensor is shifted left to the origin. After that, for $k > 1$: the $(k+1)^{th}$ sensor remains on its original location if it overlaps with already re-allocated (possibly shifted) k^{th} sensor; otherwise, the $(k+1)^{th}$ sensor shifts left to close the gap with the re-allocated k^{th} sensor, without overlapping it.

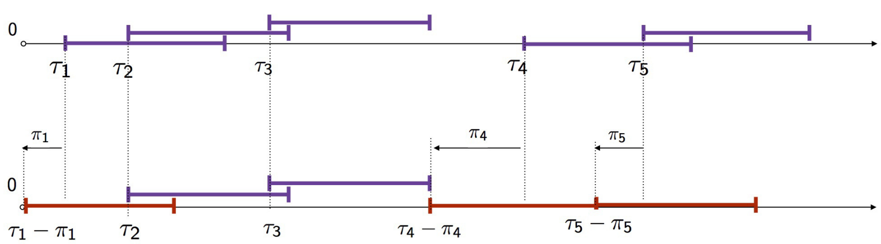

Figure 4: *An implementation of the left-shift policy:* 1^{st}, 4^{th} *and* 5^{th} *sensors are shifted left by* π_1, π_4 *and* π_5 *units respectively. The* 2^{nd} *and the* 3^{rd} *sensors do not move.*

Lemma 6.1. *The left-shift policy is characterized by the displacement*

$$\pi_k^n = \Big(L^n(\tau_k^n-)\Big)^- , \qquad k = 1,..,n, \tag{6.2}$$

where

$$L^n(t) = W^n(t) - \sup_{s\in[0,t]} (W^n(s) - \sigma^n)^+ \tag{6.3}$$

is a Skorokhod map (see Fig. 5 below), reflecting W^n *to* $(-\infty, \sigma^n]$.

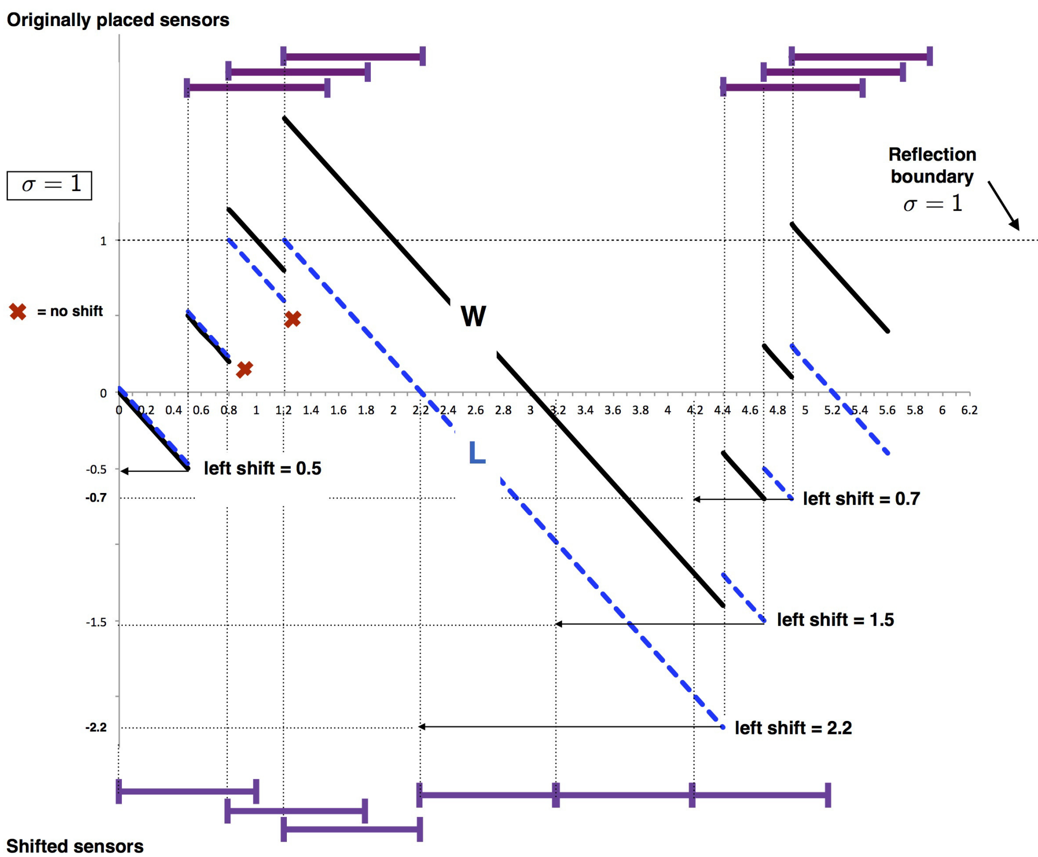

FIGURE 5: *Realization of the left-shift policy, governed by the map L. The length $\sigma = 1$ and the original left endpoints are given as $\tau_1 = 0.5$, $\tau_2 = 0.8$, $\tau_3 = 1.2$, $\tau_4 = 4.4$, $\tau_5 = 4.7$ and $\tau_6 = 4.9$. The function L is a Skorokhod reflection of the potential outflow W to $(-\infty, 1]$. The left shifts are given as $\pi_k = L(\tau_k-)^-$, implying $\pi_1 = L(0.5-)^- = 0.5$, $\pi_2 = \pi_3 = 0$, (because $L(0.8-) > 0$ and $L(1.2-) > 0$), $\pi_4 = 2.2$, $\pi_5 = 1.5$ and $\pi_6 = 0.7$.*

Lemma 6.1 implies the following upper bound for problem (2.2)

$$C^n_{cov} \leq \sum_{k=1}^{n} \Big(L^n(\tau^n_k-) \Big)^-. \tag{6.4}$$

Proof of Lemma 6.1. For $k = 2, ..., n$, define η_k to be the length of the overlap between the $(k-1)^{th}$ and k^{th} sensors, *after* the left-shift policy had been performed. Assume $\eta_1 = 0$.

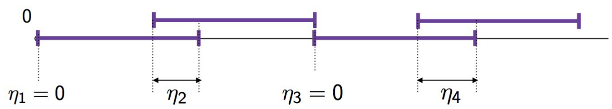

FIGURE 6: *Overlaps between shifted sensors (coverage by the left-shift policy).*

It is easy to see that the quantities $\{\eta_k\}$ satisfy the following recursive relation

$$\eta_k = \left(\sum_{i=1}^{k-1} (\sigma^n - \eta_i) - \tau_k^n \right)^+ = \left(W^n(\tau_k^n -) - \sum_{i=1}^{k-1} \eta_i \right)^+, \tag{6.5}$$

where we use the convention $\sum_1^0 = 0$. Indeed, the relation (6.5) is true because $\sum_{i=1}^{k-1}(\sigma^n - \eta_i)$ is the coordinate of the *right endpoint* of the $(k-1)^{th}$ sensor after the *left-shift* allocation. Consequently, we have

$$\pi_k^n = \left(\tau_k^n - \sum_{i=1}^{k-1} (\sigma^n - \eta_i) \right)^+ = \left(W^n(\tau_k^n -) - \sum_{i=1}^{k-1} \eta_i \right)^-. \tag{6.6}$$

From (6.3), since W^n is decreasing on $[\tau_{k-1}^n, \tau_k^n)$, we have $L^n(\tau_k^n -) = W^n(\tau_k^n -) - \sup_{s\in[0,\tau_{k-1}^n]}(W^n(s) - \sigma^n)^+$, and (6.2) will follow from (6.6) once we show that

$$\sum_{i=1}^{k-1} \eta_i = \sup_{s\in[0,\tau_{k-1}^n]} (W^n(s) - \sigma^n)^+. \tag{6.7}$$

Introduce $R(t) = \sup_{s\in[0,t]}(W^n(s) - \sigma^n)^+$. The function R is non-decreasing, piecewise constant function, with jumps possible only at the points $\{\tau_k^n\}_{k=1}^n$:

$$R(t) = \sum_{k\,:\,\tau_k^n \le t} \Delta R(\tau_k^n), \quad t \ge 0, \tag{6.8}$$

where $\Delta R(\tau_k^n) := R(\tau_k^n) - R(\tau_k^n -)$ is a jump of R at the point τ_k^n. Next we show that

$$\eta_k = \Delta R(\tau_k^n), \quad 1 \le k \le n, \tag{6.9}$$

which will in turn imply (6.7) and conclude the lemma. For $k = 1$ the relation (6.9) holds by the definition, since $\sup_{s\in[0,\tau_1^n]}(W^n(s) - \sigma^n)^+ = 0$ (see (3.1)). At the same time, the quantities $\{\Delta R(\tau_k^n)\}$ satisfy, for $k \ge 2$

$$\begin{aligned}
\Delta R(\tau_k^n) &= \sup_{s\in[0,\tau_k^n]} (W^n(s) - \sigma^n)^+ - \sup_{s\in[0,\tau_{k-1}^n]} (W^n(s) - \sigma^n)^+ \\
&= \left(W^n(\tau_k^n) - \sigma^n - \sup_{s\in[0,\tau_{k-1}^n]} (W^n(s) - \sigma^n)^+ \right)^+ = \left(W^n(\tau_k^n -) - \sum_{i=1}^{k-1} \Delta R(\tau_i^n) \right)^+,
\end{aligned}$$

which is the same recursive relation as (6.5) for quantities $\{\eta_k\}$. Together with $\eta_1 = \Delta R(\tau_1^n)$, the recursion implies (6.9), concluding the lemma. □

7. Asymptotic analysis of Coverage Problem

Proposition 7.1. (Underloaded regime). *For sensor sets of type* **U***, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{cov}) = \Theta(n) \quad and \quad C^n_{cov} = \Theta(n) \quad a.s.} \tag{7.1}$$

Proof. Apply the lower bound (6.1) together with the upper bound (3.3) to get

$$\sum_{k=1}^{n} \Big(W^n(\tau_k^n-)\Big)^- \le C^n_{cov} \le \sum_{k=1}^{n} \Big|W^n(\tau_k^n-)\Big|. \tag{7.2}$$

We follow the proof of Proposition 5.1. The upper bound treatment is identical. As for the lower bound, there must exist constants $\delta > 0$ and $c > 0$ so that for n large enough $(1-\delta)/n - \sigma^n \ge c/n$. Next, consider

$$-W^n(\tau_k^n-) = \sigma^n + \sum_{i=1}^{k}\left(\frac{1-\delta}{n} - \sigma^n\right) + \sum_{i=1}^{k}\left(\frac{\zeta_i}{n} - \frac{1-\delta}{n}\right). \tag{7.3}$$

Since $\mathbb{E}(\zeta_i) = 1$ and $\delta > 0$, the expectation of the last term is trivially positive. Then proceed as in the lower bound treatment in Proposition 5.1. □

Proposition 7.2. (Critical Regime). *For sensor sets of type* **C***, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{cov}) = \Theta(n^{1/2})} \tag{7.4}$$

Proof. We use bounds (7.2) together with similar analysis from Proposition 5.2. □

Proposition 7.3. (Overloaded Regime). *For sensor sets of type* **O***, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{cov}) = \mathrm{O}(1) \quad and \quad C^n_{cov} = \mathrm{O}(1) \quad a.s.} \tag{7.5}$$

Proof. Our job will be to show the boundedness (in expectation and almost surely) of (6.4). It would be easier to operate with τ_k^n instead of τ_k^n-; and the definition (6.3) and Fig. 5 suggest an obvious inequality

$$(L^n(\tau_k^n-))^- \le (L^n(\tau_k^n))^- + \sigma^n. \tag{7.6}$$

Since $n\sigma^n < \infty$, the boundedness of (6.4) will follow once we can show the boundedness of $\sum_{k=1}^n (L^n(\tau_k^n))^-$. To simplify the notation, let $L_0^n = 0$, $W_0^n = 0$ and $L_k^n = L^n(\tau_k^n)$,

$W_k^n = W^n(\tau_k^n)$ for $k \geq 1$. Since $W^n(\cdot)$ can only increase at points $\{\tau_k^n\}$, we can rewrite (6.3) as

$$L_k^n = W_k^n - \max_{r \in \{0,..,k\}} (W_r^n - \sigma^n)^+ \ , \quad k \geq 1,$$

Define $M_k^n = W_k^n - \max_{r \in \{0,..,k\}} W_r^n$. Since $W_0^n = 0$, we have $\max_{r \in \{0,..,k\}} W_r^n = \max_{r \in \{0,..,k\}} (W_r^n)^+ \geq \max_{r \in \{0,..,k\}} (W_r^n - \sigma^n)^+$, yielding $L_k^n \geq M_k^n$ and, in particular, due to non-positivity of M^n,

$$(L_k^n)^- \leq (M_k^n)^-. \tag{7.7}$$

Let $\psi_k^n = nM_k^n$. Together, the relations (6.2), (6.4), (7.6) and (7.7) imply

$$C_{cov}^n \leq n\sigma^n + \frac{1}{n} \sum_{k=1}^{n} (\psi_k^n)^-. \tag{7.8}$$

From the definition of M^n, we have $\psi_0^n = 0$ and

$$\psi_k^n = nW_k^n - \max_{r \in \{0,..,k\}} nW_r^n \ , \qquad k \geq 1, \tag{7.9}$$

where, using (3.1) and Assumption 2.1,

$$nW_k^n = \sum_{i=1}^{k} (n\sigma^n - n[\tau_i^n - \tau_{i-1}^n]) = \sum_{i=1}^{k} (n\sigma^n - \zeta_i). \tag{7.10}$$

The relations (7.9) - (7.10) define $\mathbf{\Psi}^n = \{\psi_k^n \ , \ k \geq 0\}$ - a random walk on the negative half-plane, as described in Appendix B, with i.i.d. increments $\{n\sigma^n - \zeta_k \ , \ k \geq 1\}$.

To properly analyze (7.8) we first need to eliminate dependence on n in the increment $n\sigma^n - \zeta_k$. Take some $\mathring{\sigma}$, satisfying $1 < \mathring{\sigma} < n\sigma^n$ (the existence of such $\mathring{\sigma}$ for n large enough is guaranteed by having $n\sigma^n \to \alpha_\sigma > 1$). Define $z_k = \mathring{\sigma} - \zeta_k$ and introduce $\mathbf{\Psi}^z = \{\psi_k^z \ , \ k \geq 0\}$ defined as $\psi_0^z = 0$ and $\psi_{k+1}^z = -(\psi_k^z + z_{k+1})^- \ , \ k \geq 0$. Once again, as described in Appendix B, $\mathbf{\Psi}^z$ is a random walk on the negative half-plane, with i.i.d. increments $\{z_k \ , \ k \geq 1\}$. By the choice of $\mathring{\sigma}$, we must have $z_k \leq n\sigma^n - \zeta_k$, hence one can apply a comparison result of Lemma B.2 to get $(\mathbf{\Psi}^n)^- \leq (\mathbf{\Psi}^z)^-$. Applying the latter to (7.8), together with Lemma B.1 (valid due to $\mathbb{E}(z_1) > 0$), we get

$$C_{cov}^n \leq n\sigma^n + \frac{1}{n} \sum_{k=1}^{n} (\psi_z^n)^- < \infty, \tag{7.11}$$

in expectation and almost surely. This concludes the proposition. □

8. Mixed Problem

Mixed policy (MP): the first sensor is shifted left to the origin. After that, for every $k \geq 1$: the $(k+1)^{th}$ sensor remains on its original location if it overlaps with already re-allocated k^{th} sensor by no more than r^n; in the case the overlap is greater than r^n, the $(k+1)^{th}$ sensor shifts right to make the overlap equal r^n. Lastly, if there is no overlap between the $(k+1)^{th}$ sensor and the re-allocated k^{th}, the $(k+1)^{th}$ sensor shifts left until it touches the k^{th} sensor, without overlapping.

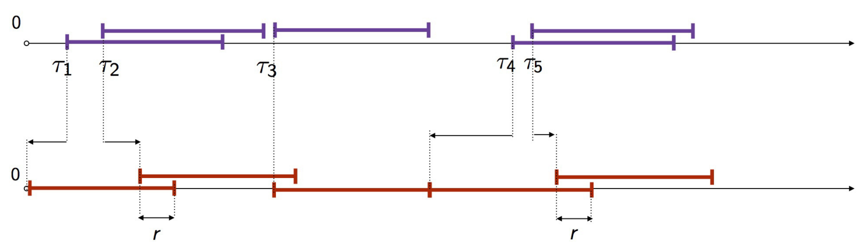

Figure 7: *An implementation of the mixed policy: the 1^{st} and the 4^{th} sensors are shifted to the left to close the gap; the 2^{nd} and the 5^{th} sensors (and it is the main difference from the left-shift policy) are shifted to the right, to ensure that the overlap does not exceed r; the 3^{rd} sensor stays on its original location, since the overlap is already smaller than r.*

Lemma 8.1. *The MP policy is characterized by the displacement*

$$\pi_k^n = \Big(P^n(\tau_k^n-)\Big)^- - \Big(P^n(\tau_k^n-) - r^n\Big)^+, \tag{8.1}$$

where P is a modified Skorokhod reflection map

$$P^n(t) = W^n(t) - \sum_{k:\, \tau_k^n \leq t} \Big(P^n(\tau_k^n-)\Big)^+ \wedge r^n. \tag{8.2}$$

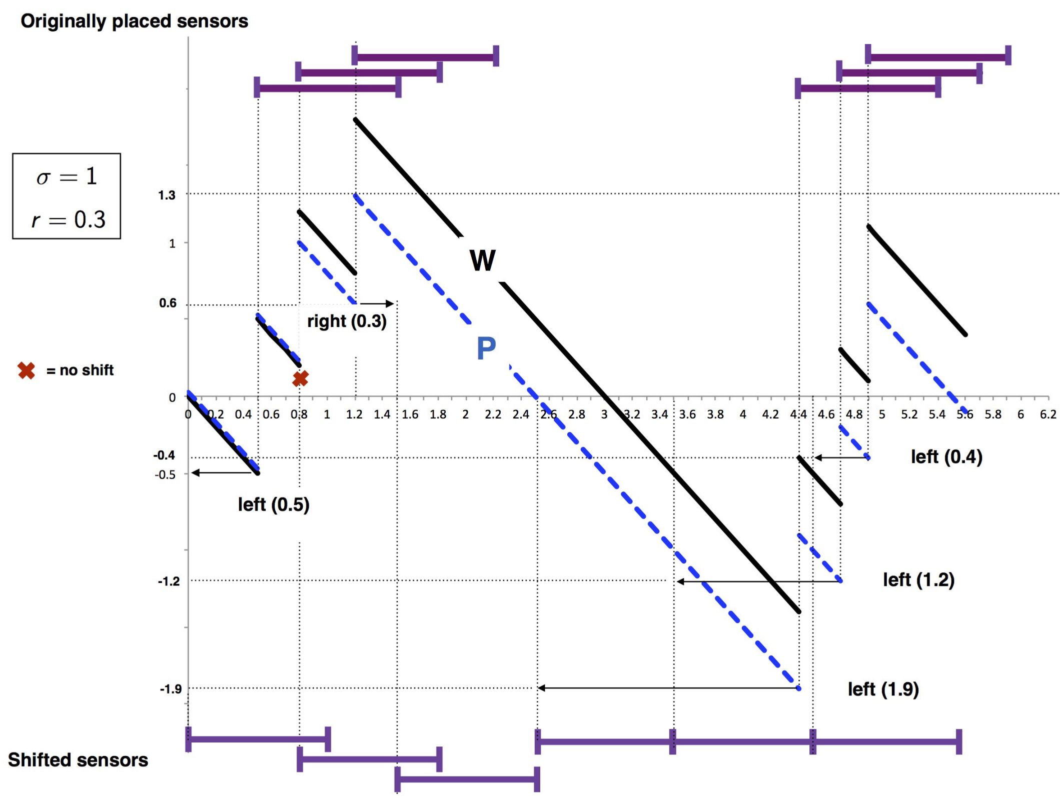

FIGURE 8: *Mixed mapping for sensors displacement. Here we have the same set of original sensors as in Fig. 5. Both functions P and L (reflected mapping from Fig. 5) behave identically up to the jump at the point* 0.8*, and this is due to the fact that* $P(0.8-) = 0.2 < 0.3$*, in which case P can not jump above the level $\sigma = 1$. On the other hand, at the jump point* 1.2 *the function P does go above the level $\sigma = 1$, (since* $P(1.2-) = 0.6 > 0.3$*) although its jump is now shortened by* $r = 0.3$*, and equals to* $1 - 0.3 = 0.7$.

Discussion of Lemma 8.1. The policy says that (recall Section 2.1), the k^{th} sensor moves right if $P^n(\tau_k^n-) > r^n$; moves left if $P^n(\tau_k^n-) < 0$; and remains in its original place if $0 \le P^n(\tau_k^n-) < r^n$ (see Fig. 8).

Looking closer, we can see that (8.2) has quite a lot of similarities with (6.3). Indeed, from its definition and Fig. 5 it is easy to see that $L^n(t) = W^n(t) - \sum_{k:\, \tau_k^n \le t} (L^n(\tau_k^n-))^+$. Therefore, the processes L^n and P^n would be identical if not for the r^n - restriction.

Naturally, the MP policy produces an upper bound

$$C^n_{mix} \le \sum_{k=1}^{n} \left[\Big(P^n(\tau^n_k-) \Big)^- + \Big(P^n(\tau^n_k-) - r^n \Big)^+ \right]. \tag{8.3}$$

Proof of Lemma 8.1. Similarly to the proof of Lemma 6.1 (see Fig. 6), let $\{\tilde{\eta}_k\}_{k=2}^n$ be the *overlaps* between $(k-1)^{th}$ and k^{th} sensors in the final placement according to the MP policy. Assume $\tilde{\eta}_1 = 0$. One can show that the MP policy implies the following recursive relation

$$\tilde{\eta}_k = \left(\sum_{i=1}^{k-1} (\sigma^n - \tilde{\eta}_i) - \tau^n_k \right)^+ \wedge r^n = \left(W^n(\tau^n_k-) - \sum_{i=1}^{k-1} \tilde{\eta}_i \right)^+ \wedge r^n. \tag{8.4}$$

Again, relation (8.4) is true because $\sum_{i=1}^{k-1}(\sigma^n - \tilde{\eta}_i)$ is the coordinate of the *right endpoint* of the $(k-1)^{th}$ sensor after the *left-shift* allocation. Consequently, we have

$$\pi^n_k = \left(\tau^n_k - \sum_{i=1}^{k-1} (\sigma^n - \tilde{\eta}_i) \right)^+ - \left(\sum_{i=1}^{k-1} (\sigma^n - \tilde{\eta}_i) - \tau^n_k - r^n \right)^+ \tag{8.5}$$

$$= \left(W^n(\tau^n_k-) - \sum_{i=1}^{k-1} \tilde{\eta}_i \right)^- - \left(W^n(\tau^n_k-) - \sum_{i=1}^{k-1} \tilde{\eta}_i - r^n \right)^+. \tag{8.6}$$

From (8.2), $P^n(\tau^n_k-) = W^n(\tau^n_k-) - \sum_{i=1}^{k-1} \Big(P^n(\tau^n_i-) \Big)^+ \wedge r^n$ and the relation (8.1) will follow once we show that

$$\tilde{\eta}_k = \Big(P^n(\tau^n_k-) \Big)^+ \wedge r^n \quad , \qquad 1 \le k \le n. \tag{8.7}$$

Since $P^n(\tau^n_1-) < 0$, the relation (8.7) holds trivially for $k = 1$. Next, proceed by induction. For $k \ge 2$, assume that (8.7) holds for $1, 2, \ldots, k-1$. From (8.4)

$$\tilde{\eta}_k = \left(W^n(\tau^n_k-) - \sum_{i=1}^{k-1} \Big(P^n(\tau^n_i-) \Big)^+ \wedge r^n \right)^+ \wedge r^n = \Big(P^n(\tau^n_k-) \Big)^+ \wedge r^n, \tag{8.8}$$

as desired. The proof is now complete. □

9. Asymptotic analysis of Mixed Problem

We shall need the following simple consequence of the definitions in (2.2) and (2.5)

$$C^n_{mix} \ge C^n_{cov} \tag{9.1}$$

Proposition 9.1. (Underloaded regime). *For sensor sets of type* $\mathbf{U}$*, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{mix}) = \Theta(n).} \tag{9.2}$$

Proof. Due to (9.1), (7.2) and an upper bound (3.3) from the base coverage policy

$$\sum_{k=1}^{n} \Big(W^n(\tau^n_k -)\Big)^- \le C^n_{mix} \le \sum_{k=1}^{n} \Big|W^n(\tau^n_k -)\Big| \tag{9.3}$$

and the arguments from the proof of Proposition 7.1 for $n\sigma^n \to \alpha_\sigma < 1$ imply (9.2). □

Proposition 9.2. (Critical regime). *For sensor sets of type* $\mathbf{C}$*, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{mix}) = \Theta(n^{1/2})} \tag{9.4}$$

Proof. We use the bounds (9.3) and the considerations from Proposition 5.2. □

Note that in obtaining both estimates (9.2) and (9.4) we did not use any information about r^n. It comes into play in the last, considerably more involved, asymptotic regime for this problem. Each of the three different sub-regimes will be treated separately.

Proposition 9.3. ($\mathbf{O^{(o)}}$ regime). *For sensor sets of type* $\mathbf{O^{(o)}}$*, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{mix}) = \Theta(n) \quad \textit{and} \quad C^n_{mix} = \Theta(n) \quad \textit{a.s.}} \tag{9.5}$$

Proof. Start with the lower bound. Due to Lemma 3.1, a policy in Π^n_3 must satisfy

$$\tau^n_i - \pi^n_i + \sigma^n - r^n \le \tau^n_{i+1} - \pi^n_{i+1}, \quad \text{for each } 1 \le i \le n-1. \tag{9.6}$$

The above condition can be interpreted as that, once "shortened" by r^n *from the right*, the modified sensors no longer interfere. That is, by allocating original sensors to satisfy (9.6), we automatically allocate "shortened" sensors to avoid interference. Naturally, the relation does not work in another direction, since one still needs to achieve coverage - that is, a policy must satisfy the requirements of Π^n_2. As a result,

$$C^n_{mix} \ge C^{n,\sigma^n - r^n}_{int}, \tag{9.7}$$

with $C^{n,\sigma^n - r^n}_{int}$ being the analog of C^n_{int} from (2.1) for the case when all sensors are shortened by r^n from the right. Hence, applying to (9.7) the modified version of (4.3),

$$C^n_{mix} \ge \sum_{k=1}^{n} \Big(W^n_{\sigma^n - r^n}(\tau^n_k -)\Big)^+, \tag{9.8}$$

with $W^n_{\sigma^n - r^n}$ being the analog of (3.1)-(3.2), accommodating the shortened length of $\sigma^n - r^n$, that is $W^n_{\sigma^n - r^n}(t) = (\sigma^n - r^n)N^n(t) - t$. For the upper bound we follow the previous discussion and use the base coverage (3.3) but for sensors *shortened by* r^n from the right (obviously, such base coverage satisfies the requirements of the mixed problem). Combined with (9.8), we now have

$$\sum_{k=1}^{n} \Big(W^n_{\sigma^n - r^n}(\tau^n_k -) \Big)^+ \le C^n_{mix} \le \sum_{k=1}^{n} \Big| W^n_{\sigma^n - r^n}(\tau^n_k -) \Big|, \tag{9.9}$$

and the assertion (9.5) can be shown as in Proposition 5.1. □

Proposition 9.4. ($\mathbf{O^{(c)}}$ regime). *For sensor sets of type* $\mathbf{O^{(c)}}$*, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{mix}) = \Theta(n^{1/2})} \tag{9.10}$$

Proof. Follows from (9.9) and Proposition 5.2 after replacing σ^n by $\sigma^n - r^n$. □

Proposition 9.5. ($\mathbf{O^{(u)}}$ regime). *For sensor sets of type* $\mathbf{O^{(u)}}$*, as* $n \to \infty$

$$\boxed{\mathbb{E}(C^n_{mix}) = \mathrm{O}(1) \quad \textit{and} \quad C^n_{mix} = \mathrm{O}(1) \quad \textit{a.s.}} \tag{9.11}$$

Proof. Let π^n be the MP policy from (8.1). The statement of the proposition will follow once we show that

$$\sup_n \|\pi^n\| < \infty\ , \quad a.s. \qquad\qquad \sup_n \mathbb{E}\left(\|\pi^n\|\right) < \infty \tag{9.12}$$

Define $P^n_k := P^n(\tau^n_k -)$, $Z^n_k = \sigma^n - \zeta_k/n$,$1 \le k \le n$. From (8.2), (3.2) and Assumption 2.1 we have $P^n_1 = -\zeta_1/n < 0$ and

$$P^n_{k+1} = P^n_k + Z^n_{k+1} - (P^n_k)^+ \wedge r^n\ , \qquad 1 \le k \le n-1. \tag{9.13}$$

Lemma 9.1. *For the sequence* $\{P^n_k\}$ *from* (9.13) *and a random walk* $\{X^n_k\ ,\ k \ge 1\}$*, defined as* $X^n_1 = P^n_1$ *and* $X^n_{k+1} = -(X^n_k + Z^n_{k+1})^-$*, we have*

$$(P^n_k)^- \le (X^n_k)^-\ , \qquad 1 \le k \le n. \tag{9.14}$$

Proof of Lemma 9.1. Consider an auxiliary sequence $\{M^n_k\ ,\ k \ge 1\}$, defined recursively as $M^n_1 = P^n_1$ and $M^n_{k+1} = M^n_k + Z^n_{k+1} - (M^n_k)^+$. It is easy to see that $M^n_k \le P^n_k$ for all $k \ge 1$. The next step is to show that $X^n_k \le M^n_k$, by induction. Indeed, the relation holds for $k = 1$. Assume now it holds for k. Consider three cases. If $0 \le X_k^n \le M_k^n$ then $X_k^n = 0$ and $X_{k+1}^n = -(Z_{k+1}^n)^-$. On the other hand $M_{k+1}^n = Z_{k+1}^n \ge -(Z_{k+1}^n)^- = X_{k+1}^n$. Next, if $X_k^n \le 0 \le M_k^n$, then $M_{k+1}^n = Z_{k+1}^n$ and the relation $X_{k+1}^n \le M_{k+1}^n$ trivially holds for $Z_{k+1}^n \le 0$ since in that case $X_{k+1}^n = X_k^n + Z_{k+1}^n$. If $Z_{k+1}^n > 0$ then $X_{k+1}^n \le M_{k+1}^n = Z_{k+1}^n$ is obvious since $X_{k+1}^n \le 0$. Lastly, if $X_k^n \le M_k^n \le 0$ then $M_{k+1}^n = M_k^n + Z_{k+1}^n \ge X_k^n + Z_{k+1}^n \ge -(X_k^n + Z_{k+1}^n)^- = X_{k+1}^n$. Combining $X_k^n \le M_k^n$ and $M_k^n \le P_k^n$, we have $X_k^n \le P_k^n$ and (9.14) as a result. □

Lemma 9.2. *Let $U_k^n = Z_k^n - r^n$ for $1 \le k \le n$ and define a random walk $\{Y_k^n\ ,\ k \ge 1\}$, as follows: $Y_1^n = 0$ and $Y_{k+1}^n = (Y_k^n + U_{k+1}^n)^+$. Then*

$$(P_k^n - r^n)^+ \le Y_k^n \ , \qquad 1 \le k \le n. \tag{9.15}$$

Proof of Lemma 9.2. Since $Y_k^n \ge 0$, it would be enough to show $P_k^n - r^n \le Y_k^n$ for all $k \ge 1$. We will use the induction argument. The relation clearly holds for $k = 1$. Now, assume that $P_k^n - r^n \le Y_k^n$ holds for some $k \ge 1$, and show that $P_{k+1}^n - r^n \le Y_{k+1}^n$.

Case 1: $P_k^n - r^n \le Y_k^n = 0$. Then $Y_{k+1}^n = (U_{k+1}^n)^+$. At the same time

$$P_{k+1}^n - r^n = \begin{cases} Z_{k+1}^n - r^n \ , & \text{if} \quad 0 \le P_k^n \le r^n, \\ P_k^n + Z_{k+1}^n - r^n \ , & \text{if} \quad P_k^n < 0, \end{cases} \tag{9.16}$$

and we have $P_{k+1}^n - r^n \le (Z_{k+1}^n - r^n)^+ = (U_{k+1}^n)^+ = Y_{k+1}^n$.

Case 2: $P_k^n - r^n \le 0 \le Y_k^n$. In this case the relation follows once again from (9.16) since $P_{k+1}^n - r^n \le (Z_{k+1}^n - r^n)^+ = (U_{k+1}^n)^+ \le (Y_k^n + U_{k+1}^n)^+ = Y_{k+1}^n$.

Case 3: $0 \le P_k^n - r^n \le Y_k^n$. In this case (recall (9.13))

$$\begin{aligned} P_{k+1}^n - r^n &= P_k^n - r^n + Z_{k+1}^n - r^n \le Y_k^n \quad + Z_{k+1}^n - r^n \qquad (9.17) \\ &\le (Y_k^n + Z_{k+1}^n - r^n)^+ = (Y_k^n + U_{k+1}^n)^+ = Y_{k+1}^n. \end{aligned}$$

□

We are about to finalize the statement (9.12). From (8.3), (9.14) and (9.15)

$$\|\pi^n\| = \sum_{k=1}^n |\pi_k^n| \le \sum_{k=1}^n (X_k^n)^- + \sum_{k=1}^n Y_k^n = \frac{1}{n} \sum_{k=1}^n \Big[(\psi_k^n)^- + \phi_k^n \Big], \tag{9.18}$$

where $\psi_k^n = nX_k^n$ and $\phi_k^n = nY_k^n$ satisfy $\psi_1^n = -\zeta_1$, $\phi_1^n = 0$, and $\psi_{k+1}^n = -(\psi_k^n + n\sigma^n - \zeta_{k+1})^-$, $\phi_{k+1}^n = (\phi_k^n + n(\sigma^n - r^n) - \zeta_{k+1})^+$ for $k \ge 2$. The last step would be to apply Lemma A.1 for $\{\phi_n^y\ ,\ k \ge 1\}$ and Lemma B.1 for $\{\psi_k^n\ ,\ k \ge 1\}$, but not before both sequences are modified to eliminate dependence on the parameter n (as was done in Propositions 5.3 and 7.3). We omit the details. □

10. Concluding remarks

The paper presents a new *queueing* approach to study asymptotic behaviour of sensor allocation problems. The method is robust and can be expanded in different directions - we can introduce new allocation costs, allow sensor length to be random, or try to find asymptotically optical solutions. In addition, one can further investigate the reflection mappings (6.3), (8.2) and their applicability in related generalizations.

We lastly comment that Assumption 2.2 and Definition 2.1 can be relaxed. In fact, the results of Theorem 2.1 will still hold for underloaded regime **U** characterized by $0 \le \liminf n\sigma^n \le \limsup n\sigma^n < 1$, or overloaded regime **O** characterized by $1 < \liminf n\sigma^n$. For the critical **C** case when $n\sigma^n \to 1$, one can introduce a *lower intermediate* regime $n\sigma^n \approx 1 - \delta n^{-\beta}$ for $\delta > 0$ and $\beta \in (0, 1/2)$, bridging the critical regime (corresponding to $\beta = 1/2$) down to the underloaded ($\beta = 0$); as well as the *upper intermediate* regime $n\sigma^n \approx 1 + \delta n^{-\beta}$ for $\delta > 0$ and $\beta \in (0, 1/2)$, connecting the critical ($\beta = 1/2$) and overloaded ($\beta = 0$) regimes. The estimates for the interference problem can be easily adjusted to include $\Theta(n^\beta)$ for the lower intermediate, and $\Theta(n^{1-\beta})$ for the upper intermediate regimes. A similar situation, but with $\Theta(n^\beta)$ and $\Theta(n^{1-\beta})$ interchanged, we believe holds for the coverage problem. In addition, a *super-overloaded* regime $n\sigma^n \approx \gamma_n$ for some sequence $\gamma_n \uparrow \infty$ will imply the estimate of $\Theta(n\gamma_n)$ for the interference problem; the coverage problem will remain O(1).

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Appendix A. Random walk on a positive half-plane

For a sequence of real numbers $\{x_k\ ,\ k\geq 1\}$ define a new sequence $\{\phi_k\ ,\ k\geq 0\}$ as

$$\phi_0=0\ ,\qquad \phi_{k+1}=(\phi_k+x_{k+1})^+\ ,\qquad n\geq 0. \tag{A.1}$$

By induction (Lemma B1, p.197, [15]), ϕ_k satisfies for $k\geq 1$

$$\phi_k=s_k-\min_{0\leq r\leq k} s_r\ \ ,\ \ \text{where}\quad s_0=0\ ,\ s_k=x_1+\cdots+x_k. \tag{A.2}$$

In fact, both representations (A.1) and (A.2) are equivalent.

Lemma A.1. *Assume $\{x_1,\ldots\}$ are i.i.d. random variables with $\mathbb{E}(x_1)<0$ and $\mathbb{E}(x_1^2)<\infty$. Then the random process $\mathbf{\Phi}=\{\phi_n\ ,n\geq 0\}$ satisfies*

$$\sup_n\left(\frac{1}{n}\sum_{k=1}^n\phi_k\right)<\infty\ ,\quad a.s.\qquad\qquad \sup_n\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^n\phi_k\right)<\infty \tag{A.3}$$

Proof. The first statement would follow from Theorem 17.0.1 of ([13], p. 422) showing the result for a *positive Harris recurrent* chain with *finite invariant probability* that has at least *one moment*. Let us make sure that all the conditions hold. The recurrence of $\mathbf{\Phi}$ has been established in Prop. 8.5.1. of ([13], p. 193), the Harris recurrence (see the definition in [13], p. 199) follows from ψ-irreducibility of $\mathbf{\Phi}$ (see [13], Prop. 4.3.1 and the last paragraph of p. 87), and the Harris recurrence of the *petite set / atom* at $\{0\}$ ([13], Prop. 9.1.7 on p. 205), since, with probability one, the chain visits $\{0\}$ infinitely often. The latter claim, though very intuitive, follows from the boundedness of the expected *busy period* -time between successive visits to $\{0\}$, which can be found in ([15], Thm. 7 on p. 27, using their notation $\mathbb{E}(\bar{N})<\infty$ if $\alpha<0$). This last reference also implies the existence of a *finite invariant measure* (see [13], Thm. 10.2.1.(iii), p. 234). The properly normalized invariant measure then becomes a *stationary probability measure.* Finally, the existence of moments for the stationary probability is guaranteed by ([13], Prop 14.4.1, p. 352).

For the second statement in (A.3), since $\phi_0 = 0$, we can almost surely (path-wise) bound $\mathbf{\Phi}$ from above by stationary $\mathbf{\Phi}^\infty = \{\phi_k^\infty\ ,\ k \geq 0\}$, such that ϕ_0^∞ is already stationary distributed. As a result, for each $k \geq 1$ we have $\mathbb{E}(\phi_k) \leq \mathbb{E}(\phi_k^\infty) < \infty$ (since the stationary distribution has at least one moment), completing (A.3). $\square$

Lemma A.2. (comparison) *Assume we are given two sets of numbers* $\{x_k\ ,\ k \geq 1\}$ *and* $\{y_k\ ,\ k \geq 1\}$ *satisfying* $x_k \leq y_k$ *for all* $k \geq 1$. *Let* $\mathbf{\Phi}^x = \{\phi_k^x\ ,\ n \geq 0\}$ *be defined as in* (A.1) *and* $\mathbf{\Phi}^y = \{\phi_n^y\ ,\ k \geq 0\}$ *be defined as* $\phi_0^y = 0\ ,\ \phi_{k+1}^y = (\phi_k^y + y_{k+1})^+\ ,\ k \geq 0$. *Then* $\mathbf{\Phi}^x \leq \mathbf{\Phi}^y$, *that is* $\phi_k^x \leq \phi_k^y$ *for all* $k \geq 0$.

Proof. The case $k = 0$ is trivial. After that we apply the induction argument, because having $\phi_{k-1}^x \leq \phi_{k-1}^y$ for a certain $k \geq 1$ will necessarily imply $\phi_{k-1}^x + x_k \leq \phi_{k-1}^y + y_k$, which immediately yields $\phi_k^x = (\phi_{k-1}^x + x_k)^+ \leq (\phi_{k-1}^y + y_k)^+ = \phi_k^y$. $\square$

Appendix B. Random walk on a negative half-plane

Alternatively, for $\{x_k\ ,\ k \geq 1\}$ define $\mathbf{\Psi} = \{\psi_k\ ,\ k \geq 0\}$ as follows

$$\psi_0 = 0\ , \qquad \psi_{k+1} = -(\psi_k + x_{k+1})^-\ , \qquad k \geq 0. \tag{B.1}$$

Note that $\psi_k \leq 0$ for all $k \geq 0$. The above representation is equivalent to (see (A.2))

$$\psi_k = s_k - \max_{0 \leq r \leq k} s_r\ , \qquad k \geq 0. \tag{B.2}$$

Lemma B.1. *Assume* $\{x_1, \ldots\}$ *are i.i.d. random variables with* $\mathbb{E}(x_1) > 0$ *and* $\mathbb{E}(x_1^2) < \infty$. *Then the relations* (A.3) *hold for* $\mathbf{\Phi} = -\mathbf{\Psi} = (\mathbf{\Psi})^-$. $\square$

Proof. Note that $-\mathbf{\Psi} = (\mathbf{\Psi})^-$ due to non-positivity of $\mathbf{\Psi}$. Now, let $\mathbf{\Phi} = -\mathbf{\Psi}$. Then

$$\phi_k = -\psi_k = -s_k + \max_{0 \leq r \leq k} s_r = \widetilde{s}_k - \min_{0 \leq r \leq k} \{\widetilde{s}_r\}, \tag{B.3}$$

where $\widetilde{s}_k = -s_k = (-x_1) + \cdots + (-x_k)$. Relation (B.3) defines a random walk on the positive half-plane, as described in Appendix A, with the i.i.d. increments $\{-x_i, i = 1, \ldots\}$. Since $\mathbb{E}(-x_1) < 0$, the statement of the lemma follows from Lemma A.1. $\square$

Lemma B.2. (comparison) *For two sets* $\{x_k\ ,\ k \geq 1\}$ *and* $\{z_k\ ,\ k \geq 1\}$ *assume* $z_k \leq x_k$ *for all* $k \geq 1$. *Let* $\mathbf{\Psi}^x = \{\psi_k^x\ ,\ n \geq 0\}$ *be defined as in* (B.1) *and* $\mathbf{\Psi}^z = \{\psi_k^z\ ,\ n \geq 0\}$ *be defined as* $\psi_0^z = 0\ ,\ \psi_{k+1}^z = -(\psi_k^z + z_{k+1})^-\ ,\ k \geq 0$. *Then* $\mathbf{\Psi}^z \leq \mathbf{\Psi}^x$ *or* $(\mathbf{\Psi}^x)^- \leq (\mathbf{\Psi}^z)^-$. $\square$