EXAMPLE OF A WEAK BUT NOT FULL LDP

Let $\mu_n = \left(1 - \frac{1}{n}\right)\delta_0 + \frac{1}{n}\delta_n$, i.e. μ_n is the distribution of a Bernoulli random variable taking values at 0 and n with respective probabilities $1 - \frac{1}{n}$ and $\frac{1}{n}$. Let $I : \mathbb{R} \mapsto [0, \infty]$ be given by

$$I(x) = \begin{cases} 0 & x = 0 \\ \infty & else \end{cases}$$

Then $\{\mu_n\}_{n\in\mathbb{N}}$

i) Satisies the weak LDP with good rate function I and converges weakly to δ_0 the unit mass at 0.
ii) *Does not* satisfy the full LDP.

Proof of i). Clearly, I is a good rate function and μ_n converges weakly to δ_0. Let G be open. If $0 \notin G$ then

$$-\infty = \inf_{x\in G} I(x) \leq \liminf_{n\uparrow\infty} \frac{1}{n} \log \mu_n(G)$$

If $0 \in G$ then $\mu_n(G) \geq 1 - \frac{1}{n}$ and hence

$$0 = \inf_{x\in G} I(x) \leq \liminf_{n\uparrow\infty} \frac{1}{n} \log \mu_n(G)$$

Thus, the lower bound for open sets holds. Now, let K be a compact set. If $0 \notin K$ then for n large enough $\mu_n(K) = 0$ and hence

$$\limsup_{n\uparrow\infty} \frac{1}{n} \log \mu_n(K) = -\infty = -\inf_{x\in K} I(x)$$

If $0 \in K$ then for large enough n, $\mu_n(K) = 1 - \frac{1}{n}$ and hence

$$\limsup_{n\uparrow\infty} \frac{1}{n} \log \mu_n(K) = 0 = -\inf_{x\in K} I(x)$$

and hence the upper bound holds for compacts and the weak LDP follows. □

Proof of ii). Let $F = [1, \infty)$. Then for all $n \geq 2$, $\mu_n(F) = \frac{1}{n}$ and hence

$$\limsup_{n\uparrow\infty} \frac{1}{n} \log \mu_n(F) = 0$$

but $\inf_{x\in F} I(x) = \infty$ so the upper bound and hence the full LDP cannot hold.

□