**EXAMPLE OF A WEAK BUT NOT FULL LDP**

Let $\mu_n = (1 - \frac{1}{n}) \delta_0 + \frac{1}{n} \delta_n$, i.e. $\mu_n$ is the distribution of a Bernoulli random variable taking values at 0 and $n$ with respective probabilities $1 - \frac{1}{n}$ and $\frac{1}{n}$. Let $I : \mathbb{R} \mapsto [0, \infty]$ be given by

$$I(x) = \begin{cases} 0 & x = 0 \\ \infty & \text{else} \end{cases}$$

Then $\{\mu_n\}_{n \in \mathbb{N}}$

i) Satisfies the weak LDP with good rate function $I$ and converges weakly to $\delta_0$ the unit mass at 0.

ii) Does not satisfy the full LDP.

*Proof of i).* Clearly, $I$ is a good rate function and $\mu_n$ converges weakly to $\delta_0$. Let $G$ be open. If $0 \notin G$ then

$$-\infty = \inf_{x \in G} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(G)$$

If $0 \in G$ then $\mu_n(G) \geq 1 - \frac{1}{n}$ and hence

$$0 = \inf_{x \in G} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(G)$$

Thus, the lower bound for open sets holds. Now, let $K$ be a compact set. If $0 \notin K$ then for $n$ large enough $\mu_n(K) = 0$ and hence

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K) = -\infty = -\inf_{x \in K} I(x)$$

If $0 \in K$ then for large enough $n$, $\mu_n(K) = 1 - \frac{1}{n}$ and hence

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K) = 0 = -\inf_{x \in K} I(x)$$

and hence the upper bound holds for compacts and the weak LDP follows. \(\square\)

*Proof of ii).* Let $F = [1, \infty)$. Then for all $n \geq 2$, $\mu_n(F) = \frac{1}{n}$ and hence

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(F) = 0$$

but $\inf_{x \in F} I(x) = \infty$ so the upper bound and hence the full LDP cannot hold. \(\square\)