LECTURE 7 EXERCISES

1 : Do exercises 4.1.29, 4.2.7 and 4.2.29 (For exercise 4.2.29 just do part *a*)) on pages 125, 129 and 136 respectively of the Dembo and Zeitouni book.

2: Let $\{A_{n,m}\}_{n,m\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_{n,m}\}_{n,m\in\mathbb{N}}$ be strictly positive real numbers such that

i) For all n, m, $A_{n,m} \leq B_n + C_{n,m}$.

ii) $\lim_{m\uparrow\infty} \limsup_{n\uparrow\infty} \frac{1}{n} \log C_{n,m} = -\infty$.

Prove that

$$\limsup_{m\uparrow\infty} \liminf_{n\uparrow\infty} \frac{1}{n} \log A_{n,m} \leq \liminf_{n\uparrow\infty} \frac{1}{n} \log B_n$$

3: Prove the following variant on Varadhan's integral lemma. Let $\mathcal{X}, \mathcal{Y}$ be two Polish spaces with respective distance functions $d_{\mathcal{X}}, d_{\mathcal{Y}}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{X_n\}_{n\in\mathbb{N}}$ be $\mathcal{X}$ valued random variables with distributions $\{\mu_n\}_{n\in\mathbb{N}}$ under $\mathbb{P}$. Assume $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies a LDP with good rate function I. Let $f : \mathcal{X} \mapsto \mathcal{Y}$ be continuous and for each n let $f_n : \mathcal{X} \mapsto \mathcal{Y}$ be measurable such that the random variables $f(X_n)$ and $f_n(X_n)$ are exponentially equivalent. Let $\phi : \mathcal{X} \mapsto \mathbb{R}$ and $\psi : \mathcal{Y} \mapsto \mathbb{R}$ be continuous functions.

Prove that if there exists a $\gamma > 1$ such that

$$\limsup_{n\uparrow\infty} \frac{1}{n} \log E_P \left[\exp\left(\gamma n \left(\psi(f_n(X_n)) + \phi(X_n)\right)\right)\right] < \infty$$

then

$$\lim_{n\uparrow\infty} \frac{1}{n} \log E_P \left[\exp\left(n \left(\psi(f_n(X_n)) + \phi(X_n)\right)\right)\right] = \sup_{x\in\mathcal{X}} \left(\psi(f(x)) + \phi(x) - I(x)\right)$$

Hints : the exponential equivalence of $f(X_n), f_n(X_n)$ yield the following statements

i) For any $x \in \mathcal{X}, r > 0, \delta > 0$

$$\liminf_{n\uparrow\infty} \frac{1}{n} \log P\left(d_{\mathcal{X}}(X_n, x) < r\right)$$
$$= \liminf_{n\uparrow\infty} \frac{1}{n} \log P\left(d_{\mathcal{X}}(X_n, x) < r, d_{\mathcal{Y}}(f_n(X_n), f(X_n)) \leq \delta\right)$$

ii) For any bounded measurable $G : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$, open $A \subset \mathcal{X}$ and $\delta > 0$

$$\limsup_{n\uparrow\infty} \frac{1}{n} \log E_P \left[\exp\left(nG(X_n, f_n(X_n))\right) 1_{\{X_n\in A\}}\right]$$
$$= \limsup_{n\uparrow\infty} \frac{1}{n} \log E_P \left[\exp\left(nG(X_n, f_n(X_n))\right) 1_{\{X_n\in A\}} 1_{\{d_{\mathcal{Y}}(f(X_n), f_n(X_n))<\delta\}}\right]$$

Use these two facts to mimick the proof of Varadhan's integral lemma.