Lecture 4 Exercises

1: This is a (relatively easy) variation on Sanov’s Theorem. Let \( \{Y_i\}_{i=1}^{\infty} \) be i.i.d. real valued random variables with distribution \( \mu \) and cumulative distribution function \( F_\mu \). Fix a \( t \in \mathbb{R} \) and let \( X_t = 1_{(-\infty,t]}(Y_i) \). Let \( \hat{S}_n = \frac{1}{n} \sum_{i=1}^{n} X_t^i \) and let \( \mu_n \) be the distribution of \( \hat{S}_n \).

   i) Do the \( \{\mu_n\}_{n=1}^{\infty} \) satisfy a LDP? If so, identify the rate function \( \Lambda^{x,t} \).
   ii) For each \( t \), identify the set \( A_t = \{x \in \mathbb{R} : \Lambda^{x,t}(x) = 0\} \). As a function of \( t \) what is this set?

2: This is a (harder) variation on Sanov’s Theorem. Let \( \{Y_i\}_{i=1}^{\infty} \) be as in exercise 1. Fix a partition \( -\infty = t_0 < t_1 < t_2 ... < t_n < t_{n+1} = \infty \) of the real line such that \( F_j - F_{j-1} \equiv F(t_j) - F(t_{j-1}) > 0 \) for \( j = 1, ..., n+1 \) (assuming of course that \( \mu \) is such that this can be done). Let \( X_i = (1_{(-\infty,t_1]}(Y_i), 1_{(-\infty,t_2]}(Y_i), ..., 1_{(-\infty,t_n]}(Y_i)) \)

   i) Show directly that \( \Lambda^*(x) = \infty \) for \( x \not\in A \) where
      \[ A = \{x \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \cdots \leq x_n \leq 1\} \]
   ii) For \( x = (x_1, ..., x_n) \in A \) let \( x_0 = 0 \) and \( x_{n+1} = 1 \). Show that
      \[ \Lambda^*(x) = \sum_{i=1}^{n+1} (x_i - x_{i-1}) \log \left( \frac{x_i - x_{i-1}}{f_i - f_{i-1}} \right) \]

3: In this exercise, Cramér’s Theorem for i.i.d. random variables taking values in a finite set will be derived from Sanov’s Theorem. Namely, let \( \{X_i\}_{i=1}^{\infty} \) be i.i.d random variables taking values in a finite set \( \Sigma \) (which we identify as \( \Sigma = \{1, 2, ..., N\} \)) with distribution \( \mu \). Let \( \hat{S}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \) and let \( \mu_n \) be the distribution of \( \hat{S}_n \).

   i) Show that \( \hat{S}_n = \sum_{j=1}^{N} j L_n^X(j) \)
   ii) For a given \( \nu \in M_1(\Sigma) \) (i.e. \( \nu \in \Delta^N \)), let \( m_\nu = \sum_{j=1}^{N} j \nu_j \). Show for any \( \Gamma \subset \mathbb{R} \)
      \[ \hat{S}_n \in \Gamma \iff L_n^X \in \tilde{\Gamma} \]
      where \( \tilde{\Gamma} = \{\nu \in M_1(\Gamma) : m_\nu \in \Gamma\} \) and \( \tilde{\Gamma} \) is open or closed according to whether \( \Gamma \) is open or closed. Here, the distance between two measures \( \mu \) and \( \nu \) is just the Euclidean distance between the associated vectors in \( \mathbb{R}^N \).
   iii) Using the LDP for \( \{L_n^X\}_{n=1}^{\infty} \) we have
      \[ -\inf_{x \in \tilde{\Gamma}} \frac{1}{n} \log \mu_n(\Gamma) \leq \inf_{x \in \tilde{\Gamma}} \frac{1}{n} \log \mu_n(\Gamma) \leq \inf_{x \in \tilde{\Gamma}} \frac{1}{n} \log \mu_n(\Gamma) \leq -\inf_{x \in \tilde{\Gamma}} \frac{1}{n} \log \mu_n(\Gamma) \]
      Therefore, deduce that \( \{\mu_n\} \) satisfy a LDP with rate function \( \Lambda^* \) by showing for any set \( \Gamma \)
      \[ \inf_{\nu \in \tilde{\Gamma}} \frac{1}{n} \log H(\nu|x) = \inf_{x \in \tilde{\Gamma}} \frac{1}{n} \log H(\nu|x) = \inf_{x \in \tilde{\Gamma}} \frac{1}{n} \log \Lambda^*(x) \]