LECTURE 4 EXERCISES

1 : This is a (relatively easy) variation on Sanov's Theorem. Let $\{Y_i\}_{i=1}^{\infty}$ be i.i.d. real valued random variables with distribution μ and cummulative distribution function F_μ. Fix a $t \in \mathbb{R}$ and let $X_i^t = 1_{(-\infty,t]}(Y_i)$. Let $\hat{S}_n^t = \frac{1}{n}\sum_{i=1}^n X_i^t$ and let μ_n^t be the distribution of $\hat{S}_n^t$.

i) Do the $\{\mu_n^t\}_{n=1}^{\infty}$ satisfy a LDP? If so, identify the rate function $\Lambda^{*,t}$.

ii) For each t, identify the set $A_t = \{x \in \mathbb{R} : \Lambda^{*,t}(x) = 0\}$. As a function of t what is this set?

2: This is a (harder) variation on Sanov's Theorem. Let $\{Y_i\}_{i=1}^{\infty}$ be as in exercise 1. Fix a partition $-\infty = t_0 < t_1 < t_2 ... < t_n < t_{n+1} = \infty$ of the real line such that $F_j - F_{j-1} \equiv F(t_j) - F(t_{j-1}) > 0$ for $j = 1, ..., n+1$ (assuming of course that μ is such that this can be done). Let

$$X_i = \left(1_{(-\infty,t_1]}(Y_i), 1_{(-\infty,t_2]}(Y_i), \ldots, 1_{(-\infty,t_n]}(Y_i)\right)$$

i) Show directly that $\Lambda^*(x) = \infty$ for $x \notin A$ where

$$A = \{x \in \mathbb{R}^n : 0 \le x_1 \le x_2 \cdots \le x_n \le 1\}$$

ii) For $x = (x_1, \ldots, x_n) \in A$ let $x_0 = 0$ and $x_{n+1} = 1$. Show that

$$\Lambda^*(x) = \sum_{i=1}^{n+1} (x_i - x_{i-1}) \log\left(\frac{x_i - x_{i-1}}{f_i - f_{i-1}}\right)$$

3 : In this exercise, Cramér's Theorem for i.i.d. random variables taking values in a finite set will be derived from Sanov's Theorem. Namely, let $\{X_i\}_{i=1}^{\infty}$ be i.i.d random variables taking values in a finite set Σ (which we identify as $\Sigma = \{1, 2, ..., N\}$) with distribution μ. Let $\hat{S}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and let μ_n be the distribution of $\hat{S}_n$.

i) Show that $\hat{S}_n = \sum_{j=1}^N jL_n^X(j)$

ii) For a given $\nu \in M_1(\Sigma)$ (i.e. $\nu \in \Delta^N$), let $m_\nu = \sum_{j=1}^N j\nu_j$. Show for any $\Gamma \subset \mathbb{R}$

$$\hat{S}_n \in \Gamma \Leftrightarrow L_n^X \in \tilde{\Gamma}$$

where $\tilde{\Gamma} = \{\nu \in M_1(\Gamma) : m_\nu \in \Gamma\}$ and $\tilde{\Gamma}$ is open or closed according to whether Γ is open or closed. Here, the distance between two measures μ and ν is just the Euclidean distance between the associated vectors in $\mathbb{R}^N$.

iii) Using the LDP for $\left\{L_n^X\right\}_{n=1}^{\infty}$ we have

$$\begin{aligned} -\inf_{x \in \tilde{\Gamma}^\circ} H(x|\mu) &\le \liminf_{n\uparrow\infty} \frac{1}{n}\log\mu_n(\Gamma) \\ &\le \limsup_{n\uparrow\infty} \frac{1}{n}\log\mu_n(\Gamma) \le -\inf_{x\in\bar{\tilde{\Gamma}}} H(x|\mu) \end{aligned}$$

Therefore, deduce that $\{\mu_n\}$ satisfy a LDP with rate function Λ^* by showing for any set Γ

$$\inf_{\nu\in\tilde{\Gamma}} H(\nu|\mu) = \inf_{x\in\Gamma} \inf_{\nu\in M_1(\Sigma): m_\nu = x} H(\nu|\mu) = \inf_{x\in\Gamma} \Lambda^*(x)$$

4 : Do Exercise 2.3.20 on pages 52 of the Dembo & Zeitouni book.