LECTURE 2 EXERCISES

1 : Show that if $0 \in D^*_\Lambda$ then $\Lambda^*$ is a good rate function.

2 : Let $\mu$ be a measure on $\mathbb{R}$ such that $\int_\mathbb{R} |x| \mu(dx) < \infty$. Let $\Lambda_\mu, \Lambda^*_\mu$ be the associated cumulant generating function and Legendre transform. Let $p \in \mathbb{R}$.
   i) If $\nu = \delta_p * \mu$ is the convolution of $\mu, \delta_p$ then $\Lambda^*_\nu(x) = \Lambda^*_\mu(x - p)$.
   ii) Prove
   \[
   \int_{\mathbb{R}} \exp \left( \alpha \Lambda^*_\mu(x) \right) \mu(dx) \leq \frac{2}{1 - \alpha} \quad \forall \alpha \in (0, 1)
   \]
   This is from exercises 1.2.9, 1.2.10 in the Deuschel and Stroock book. In ii), let $p = \int_{\mathbb{R}} x \mu(dx)$ and use the facts (and make sure you understand why...)
   \[
   \mu([q, \infty)) \leq e^{-\Lambda^*_\mu(q)} \quad q \geq p
   \]
   \[
   \mu((\infty, q]) \leq e^{-\Lambda^*_\mu(q)} \quad q \leq p
   \]

3 : (This is actually a hold-over from Lecture 1) Prove that if $\mathcal{X}$ is a metric space with distance function $d$ and the $\{\mu_n\}$ are Borel measures then $\{\mu_n\}$ cannot satisfy a LDP with more than one rate function (and hence the rate function must be unique).

4 : Let $I$ be a rate function. Let $\delta > 0$ and set $I^\delta(x) = \min \{I(x) - \delta, \frac{1}{\delta}\}$. Prove that for any set $\gamma$
   \[
   \lim inf_{\delta > 0, x \in \Gamma} I^\delta(x) = \inf_{x \in \Gamma} I(x)
   \]

5 : Do the following exercises from Dembo & Zeitouni.
   i) 2.2.23 (page 35)
   ii) 2.2.36 (page 41)
   iii) 2.2.38 (page 42)