LECTURE 11 EXERCISES

1 : Prove the statements made in class regarding the Perron-Frobenious eigenvalue in the case that the $N \times N$ dimensional matrix B is such that $B(i,j) > 0$ for each $i, j \in 1 \dots N$. Namely, setting

$$\rho = \sup_{\{x \in \mathbb{R}^N:\ x_i \geq 0, x'x = 1\}} \min_{i=1,\dots N} \frac{(Bx)_i}{x_i}$$

show that

1. ρ is an eigenvalue
2. The right eigenvector x is such that $x_i > 0, i = 1 \dots N$.
3. If λ is any other eigevalue then $|\lambda| \leq \rho$.

Hints : Setting $\rho(x) = \min_{i=1,\dots N} \frac{(Bx)_i}{x_i}$ note that $\rho(x)$ is upper semicontinuous. Note that if $x \in \mathbb{R}^N$ is such that $x_i \geq 0, i = 1, \dots N$ then $(Bx)_i > 0, i = 1, \dots N$.

2: Show that for the $N \times N$ stochastic matrix π, that if $\nu \in \mathbb{R}^N, \nu >> 0$ is a left eigenvector, i.e.

$$\sum_{i=1}^{N} \nu_i \pi_{ij} = \nu_j, j = 1, \dots N$$

then

$$I(\nu) = \sup_{u \in \mathbb{R}^N, u >> 0} \sum_{j=1}^{N} \nu_j \log \left(\frac{u_j}{\sum_{i=1}^{N} u_i \pi_{ij}} \right) = 0$$

The meaning of this is that if ν is an invariant measure for π (ν can always be normalized) then $I(\nu) = 0$. This makes sence since the empirical distribution of the Markov chain is converging to the invariant measure.

2 : Do exercises 3.1.8, 3.1.9 on pages 77-78 of the Dembo and Zeitouni book.

3 : Read Section $4.5, 4.5.1$ and $4.5.2$ in the Dembo and Zeitouni book.