THE PRICING OF CONTINGENT CLAIMS AND OPTIMAL POSITIONS IN ASYMPTOTICALLY COMPLETE MARKETS

MICHAIL ANTHROPELOS, SCOTT ROBERTSON, AND KONSTANTINOS SPILIOPOULOS

ABSTRACT. We study utility indifference prices and optimal purchasing quantities for a contingent claim, in an incomplete semi-martingale market, in the presence of vanishing hedging errors and/or risk aversion. Assuming that the average indifference price converges to a well defined limit, we prove that optimally taken positions become large in absolute value at a specific rate. We draw motivation from and make connections to Large Deviations theory, and in particular, the celebrated Gärtner-Ellis theorem. We analyze a series of well studied examples where this limiting behavior occurs, such as fixed markets with vanishing risk aversion, the basis risk model with high correlation, models of large markets with vanishing trading restrictions and the Black-Scholes-Merton model with either vanishing default probabilities or vanishing transaction costs. Lastly, we show that the large claim regime could naturally arise in partial equilibrium models.

1. INTRODUCTION

The goal of this paper is to study the relationship between utility indifference prices and optimal positions for a contingent claim, in a general incomplete semi-martingale market, under the assumption of vanishing hedging errors. In particular, for an exponential utility investor, we wish to verify the heuristic adage that when purchasing optimal quantities one obtains the delicate relationship

$$\text{position size} \times \text{risk aversion} \times \text{incompleteness parameter} \approx \text{constant}.$$

Here, the incompleteness parameter represents the hedging error associated with the claim. From the above we see that as the market becomes complete (or, at least as the given claim in question becomes asymptotically hedgeable), optimal position sizes tend to become large. In fact, optimal position sizes may also become large as risk aversion vanishes in a fixed market, and our analysis is robust enough to cover both cases.

The financial motivation for studying this situation is that large positions are indeed being taken. For example, the over the counter derivatives markets now has more than 700 trillion notional outstanding (see [7]). Other examples include mortgage backed securities, life insurance contracts and mortality derivatives. These products are not completely replicable and a position on them implies unhedgeable risk. Therefore, it is natural to study the situation within the framework of utility based analysis in incomplete markets. Moreover, the observation that position size is connected to hedging error can be understood as follows.

Date: September 21, 2015.

M. Anthropelos is supported in part by the Research Center of the University of Piraeus.

S. Robertson is supported in part by the National Science Foundation under grant number DMS-1312419.

K. Spiliopoulos is supported in part by the National Science Foundation under grant number DMS-1312124.

In a complete market there is only one fair price d for a given claim. Hence, if one is able to purchase claims for price $p \neq d$ then it is optimal to take an infinite position. Of course, in reality one cannot take an infinite position and complete markets are an ideal situation. However, these considerations indicate that large positions may arise endogenously, if the hedging error or risk aversion is small. We also mention that this is the underlying motivation for the indifference price approximations in the basis risk models of [12, 21], which we revisit in the current paper.

Starting at least from [22], utility indifference pricing has attracted a lot of attention, see for example [9] for detailed overview. Recently, indifference pricing for large position sizes has been studied in [8, 33, 34]. In [34] the authors consider a sequence of a particular semi-complete market indexed by n that becomes complete as $n \to \infty$ and, assuming the unhedgeable component of the non-traded asset vanishes in accordance to a Large Deviation Principle (LDP), it is shown that optimal purchase quantities become large at precisely the Large Deviations scaling.

To help motivate our results, let us briefly outline the main idea. Let $n \in \mathbb{N}$ and consider a semi-martingale market with available risky assets for investment S^n, and an investor who owns a non-traded contingent claim B. The investor has exponential utility with risk aversion $a_n > 0$, where, in addition to the assets, we allow the risk aversion to change with n so that $U_{a_n}(x) = -(1/a_n)e^{-a_n x}, x \in \mathbb{R}$. Let $\pi^n \in \mathcal{A}^n$ be an admissible trading strategy and $X^{\pi^n} = (\pi^n \cdot S^n)$ be the resultant wealth process. The optimal utility that the investor can achieve by trading in S^n with initial capital x and q units of B is

$$u^n_{a_n}(x,q) = \sup_{\pi^n \in \mathcal{A}^n} \mathbb{E}\left[U_{a_n}(x + X^{\pi^n}_T + qB)\right]; \quad u^n_{a_n}(x) = u^n_{a_n}(x,0).$$

Then, the average bid utility indifference price $p^n_{a_n}(x,q)$ is defined through the balance equation

$$u^n_{a_n}(x - qp^n_{a_n}(x,q), q) = u^n_{a_n}(x).$$

It is well known that $p^n_{a_n}$ does not depend upon x, and writing $p^n_{a_n}(q)$, takes the form

$$p^n_{a_n}(q) = -\frac{1}{a_n q}\log\left(\mathbb{E}^{\mathbb{Q}^n_0}\left[e^{-a_n q \hat{Y}^n_{a_n}(q)}\right]\right),$$

where $\mathbb{Q}^n_0$ is the minimal entropy measure in the nth market and $\hat{Y}^n_{a_n}(q)$ is related to the normalized residual risk (see [1, 31] amongst others) of owning q units of B. Thus, $p^n_{a_n}$ can be viewed as a "generalized" version of the scaled cummulant generating function $\Lambda_n(q)/q$, where $\Lambda_n(q) := \log\left(\mathbb{E}\left[e^{qY_n}\right]\right)$ for a sequence of random variables $\{Y_n\}$ from Large Deviations theory (see [15] for a classical manuscript). Taking a cue from the celebrated Gärtner-Ellis theorem, which deduces an LDP for the tail probabilities of $\{Y_n\}$ from the assumption that $\lambda \mapsto 1/r_n \Lambda_n(\lambda r_n)$ converges to a sufficiently regular function as $r_n \to \infty$, we naturally ask what conclusions can be deduced from the assumption that $\ell \mapsto p^n_{a_n}(\ell r_n)$ converges to a well defined limit for $\ell \in \mathbb{R}$ and $r_n \to \infty$. Specifically, we assume (see Assumption 3.2) that there exist a sequence $\{r_n\}$ of positive numbers with $r_n \to \infty$ and a $\delta > 0$ such that for all $|\ell| < \delta$ the limit

$$p^\infty(\ell) = \lim_{n\uparrow\infty} p^n_{a_n}(\ell r_n), \tag{1.1}$$

exists, is finite, and is continuous at $\ell = 0$. The price $p^\infty(0)$ is thus the limiting price ignoring position size, and when the market is asymptotically complete, represents the unique arbitrage free price in the limiting complete market: see Section 4.3.

As a first consequence, we prove (see Theorems 4.3, 4.4) that large optimal positions arise endogenously at a rate proportional to r_n. Specifically, for any price $\tilde{p}^n$ which is arbitrage free in the pre-limiting markets, the optimal position size (as defined in [24]) $\hat{q}^n = \hat{q}^n(\tilde{p}^n)$ is such that for n large enough

$$|\hat{q}^n| \approx \ell r_n, \text{ for some } \ell \in (0, \infty),$$

provided that $\tilde{p}^n \to \tilde{p} \neq p^\infty(0)$. Namely, we have $|\hat{q}^n| \to \infty$ at the speed of r_n.

Secondly, in Section 5 we show under which conditions the large claim regime could arise in an equilibrium setting, with a particular focus on justifying the assumption that, asymptotically, one could buy the claim for a price $\tilde{p} \neq p^\infty(0)$. Provided that stock market prices are exogenously given, the equilibrium price of a claim is the one at which the optimal quantities of the investors sum up to zero, meaning that the market of the claim is cleared out. If such a (partial) equilibrium price exists for each $n \in \mathbb{N}$, it is natural to ask where this sequence converges to, and if the prices induce investors to enter the large claim regime. Here, we show that if the investors' random endowments are dominated by r_n, then equilibrium prices converge to $p^\infty(0)$; the unique limiting arbitrage-free price. However, if investors' endowments are growing with rate r_n, equilibrium prices may converge to a limit $\tilde{p} \neq p^\infty(0)$ and hence the large claim regime of Theorems 4.3, 4.4 occurs. This happens when one investor already owns large position in B, and yields a family of examples where the large claim regime is in fact the market's equilibrium. This result helps to explain the large observed volumes in OTC derivative markets and the corresponding extreme prices that often appear (see for instance [2, 7]).

Thirdly, we illustrate through numerous and varied examples that the price convergence in (1.1) holds, and hence is a natural feature of either asymptotically complete markets or vanishing investor's risk aversion in a fixed market. Moreover, in all of these examples we explicitly identify the speed r_n at which optimal positions grow. To be precise, we validate these claims in the following cases: (a) vanishing risk aversion in a fixed market in Section 6.1, (b) basis risk model with high correlation in Section 6.2, (c) large markets with vanishing trading restrictions in Section 6.3, (d) Black-Scholes-Merton model with vanishing default probability in Section 6.4, and (e) vanishing transaction costs in the Black-Scholes-Merton model in Section 7.

The vanishing transaction costs example of Section 7 probably deserves more discussion. The first interesting point is that our theory unifies frictionless markets and markets with frictions, such as transaction costs. In particular, not only do the statements on optimal positions in frictionless markets carry over, but in both cases, the main results turn out to be natural outcomes of the same general statements presented in Appendix C. The second interesting point is that our analysis reveals that the natural relation between risk aversion, a_n, optimal position size, $\hat{q}_n$, and proportion of the transaction costs, λ_n is $a_n \hat{q}_n \lambda_n^2 \approx$ constant. Apart from the conclusion that for fixed risk aversion, this relation indicates that $r_n = \lambda_n^{-2}$, i.e. that

$\hat{q}_n\lambda_n^2 \to \ell \in (0,\infty)$, it also justifies the appropriateness of the limiting asymptotic regimes, which were considered previously without justification; for example, as in [4, 23].

We conclude the introduction mentioning that even though our focus in this paper is on investors with exponential utility, our results are also true within the class of utility functions that decay exponentially for large negative wealths, see Section 4.5. In this case, the optimal position is not necessarily unique. However, we prove that optimizers do exist and that under the assumption of convergence of indifference prices with speed r_n, *for exponential utility*, each optimizer will converge to $\pm\infty$ with speed r_n.

The rest of the paper is organized as follows. In Section 2 we describe in detail the model and the optimal investment problem. In Section 3 we lay down our main assumption on convergence of scaled indifference prices and draw motivations with and connections to Large Deviations theory. In Section 4 we describe the main consequences of the assumption of convergence of scaled indifference prices. Namely, we state the theorems on optimal positions and discuss their consequences. We additionally discuss the limiting behavior for the optimal wealth process, and justify the interpretation that the speed r_n characterizes the speed at which the market approaches completion. Moreover, we prove that the general results on optimal positions are true for all utility functions in the class of utility functions that decay exponentially for large negative wealths. Section 5 contains the results on the partial equilibrium model and on its limiting behavior. Section 6 contains the motivating examples of frictionless markets that satisfy our assumptions. Section 7 contains the example with vanishing transaction costs. Appendices A, B and C contain most of the proofs.

2. The Model, Optimal Investment Problem and Indifference Price

We fix a horizon $T > 0$, probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$, which is assumed to satisfy the usual conditions. Additionally, we assume $\mathcal{F} = \mathcal{F}_T$ and zero interest rates so the risk-free asset is identically equal to 1. For $n \in \mathbb{N}$ we denote by S^n an $\mathbb{R}^{d_n}$-valued, locally bounded semi-martingale which represents the risky assets available for investment. In the sequel, we consider the valuation and the optimal position taking in a contingent claim $B \in \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ assumed to satisfy:

Assumption 2.1. $\mathbb{E}\left[e^{\lambda B}\right] < \infty$ for all $\lambda \in \mathbb{R}$.

Since the assets are changing with n, the class of equivalent local martingale measures are changing with n as well. We denote by $\mathcal{M}^n$ the family of measures $\mathbb{Q}^n \sim \mathbb{P}$ on $\mathcal{F}$ such that S^n is a $\mathbb{Q}^n$ local martingale. Recall for two probability measures $\mu \ll \nu$ the relative entropy of μ with respect to μ is given by $H(\mu \mid \nu) = \mathbb{E}^\nu\left[(d\mu/d\nu)\log(d\mu/d\nu)\right]$. In order to rule out arbitrage in each market, we make the following standard assumption as seen in [14, 18] amongst many others:

Assumption 2.2. For each n, $\tilde{\mathcal{M}}^n := \{\mathbb{Q}^n \in \mathcal{M}^n \,:\, H(\mathbb{Q}^n \mid \mathbb{P}) < \infty\} \neq \emptyset$.

We consider an exponential utility investor with risk aversion $a_n > 0$, where, in addition to the assets, we allow the risk aversion to change with n. Thus, the investor has utility function

$$U_{a_n}(x) = -\frac{1}{a_n}e^{-a_n x}; \qquad x \in \mathbb{R}. \tag{2.1}$$

A trading strategy π^n is admissible if it is predictable, S^n integrable, and if the stochastic integral $X^{\pi^n} := (\pi^n \cdot S^n)$ is a $\mathbb{Q}^n$ supermartingale for all $\mathbb{Q}^n \in \tilde{\mathcal{M}}^n$. The set of admissible trading strategies for the nth market is denoted $\mathcal{A}^n$. For an initial capital x and position $q \in \mathbb{R}$ in the claim B we define

$$u^n_{a_n}(x,q) := \sup_{\pi^n \in \mathcal{A}^n} \mathbb{E}\left[U_{a_n}(x + X^{\pi^n}_T + qB)\right], \tag{2.2}$$

as the optimal utility an investor can achieve by trading in S^n with initial capital x and q units of B. When $q = 0$ so that the investor does not own the claim we denote the value function by

$$u^n_{a_n}(x) := \sup_{\pi^n \in \mathcal{A}^n} \mathbb{E}\left[U_{a_n}(x + X^{\pi^n}_T)\right]. \tag{2.3}$$

The average (bid) utility indifference price $p^n_{a_n}(x,q)$ for initial capital x and q units of B is defined through the balance equation

$$u^n_{a_n}(x - qp^n_{a_n}(x,q), q) = u^n_{a_n}(x). \tag{2.4}$$

We now summarize a number of well known results regarding the utility maximization problem for exponential utility under the current setup and assumptions. For proofs of these facts, see [14, 18, 19, 26, 30, 32].

Since $u^n_{a_n}(x,q) = e^{-a_n x} u^n_{a_n}(0,q)$ we consider without loss of generality that $x = 0$ throughout. The value function without B, $u^n_{a_n}(0)$, is attained by an admissible strategy $\hat{\pi}^n_{a_n}(0)$. Write $\hat{X}^n_{a_n}(0) := X^{\hat{\pi}^n_{a_n}(0)}$ as the optimal wealth process. Additionally, denote by $\mathbb{Q}^n_0 \in \tilde{\mathcal{M}}^n$ the minimal entropy measure, which exists. Then $\mathbb{Q}^n_0$ and $\hat{X}^n_{a_n}(0)$ are related by the formula

$$\left.\frac{d\mathbb{Q}^n_0}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \frac{e^{-a_n \hat{X}^n_{a_n}(0)_T}}{\mathbb{E}\left[e^{-a_n \hat{X}^n_{a_n}(0)_T}\right]}. \tag{2.5}$$

In a similar fashion, the value function for q units of B, $u^n_{a_n}(0,q)$, is also attained for some admissible trading strategy $\hat{\pi}^n_{a_n}(q)$ and we write $\hat{X}^n_{a_n}(q) := X^{\hat{\pi}^n_{a_n}(q)}$ as the resultant wealth process. The indifference price does not depend upon the initial capital and we write $p^n_{a_n}(q)$ instead of $p^n_{a_n}(x,q)$. By its definition, $p^n_{a_n}(q)$ is given by the abstract formula

$$p^n_{a_n}(q) = -\frac{1}{a_n q} \log\left(\frac{u^n_{a_n}(0,q)}{u^n_{a_n}(0)}\right), \tag{2.6}$$

and the total price $qp^n_{a_n}(q)$ admits the variational representation

$$qp^n_{a_n}(q) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(q\mathbb{E}^{\mathbb{Q}^n}[B] + \frac{1}{a_n}\left(H(\mathbb{Q}^n \mid \mathbb{P}) - H(\mathbb{Q}^n_0 \mid \mathbb{P})\right)\right). \tag{2.7}$$

Note that from (2.7) one can easily deduce that for $q \in \mathbb{R}$

$$p^n_{a_n}(q) = p^n_1(a_n q). \tag{2.8}$$

Also, using (2.5) and (2.6) we obtain

$$p^n_{a_n}(q) = -\frac{1}{a_n q} \log\left(\frac{\mathbb{E}\left[e^{-a_n \hat{X}^n_{a_n}(q)_T - a_n qB}\right]}{\mathbb{E}\left[e^{-a_n \hat{X}^n_{a_n}(0)_T}\right]}\right) = -\frac{1}{a_n q} \log\left(\mathbb{E}^{\mathbb{Q}^n_0}\left[e^{-a_n q \hat{Y}^n_{a_n}(q)}\right]\right), \tag{2.9}$$

where

$$\hat{Y}^n_{a_n}(q) := \frac{1}{q}\left(\hat{X}^n_{a_n}(q)_T - \hat{X}^n_{a_n}(0)_T + qB\right). \tag{2.10}$$

$\hat{Y}^n_{a_n}(q)$ is intimately related to the *normalized residual risk process* of [1, 31, 37] amongst others and can be seen as the per unit unhedgeable part of the long position on q units of the claim B.

3. Limiting Prices and Connections to Large Deviations Theory

Equation (2.9) is the starting point for our analysis. To motivate the result we first make connections with the Gärtner-Ellis theorem from Large Deviations stated here for the convenience of the reader.

Theorem 3.1 (Gärtner-Ellis)**.** *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a collection of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of positive reals such that $\lim_{n\uparrow\infty} r_n = \infty$. For each n denote by Λ_n the cummulant generating function for Y_n*

$$\Lambda_n(\lambda) := \log\left(\mathbb{E}\left[e^{\lambda Y_n}\right]\right), \qquad \lambda \in \mathbb{R}. \tag{3.1}$$

Assume the following regarding Λ_n:

1. *For all $\lambda \in \mathbb{R}$ the limit $\Lambda(\lambda) := \lim_{n\uparrow\infty}(1/r_n)\Lambda_n(r_n\lambda)$ exists as an extended real number.*
2. *$\mathcal{D}^0_\Lambda$, the interior of $\mathcal{D}_\Lambda := \{\lambda : \Lambda(\lambda) < \infty\}$, is non-empty with $0 \in \mathcal{D}^0_\Lambda$.*
3. *Λ is differentiable throughout $\mathcal{D}^0_\Lambda$ and steep; i.e. $\lim_{\lambda\to\partial\mathcal{D}_\Lambda} |\nabla\Lambda(\lambda)| = \infty$.*
4. *Λ is lower semi-continuous.*

Then, the random variables $\{Y_n\}_{n\in\mathbb{N}}$ satisfy a Large Deviation Principle (LDP) with speed $\{r_n\}_{n\in\mathbb{N}}$ and good rate function $I(y) = \sup_{\lambda\in\mathbb{R}}(\lambda y - \Lambda(\lambda))$.

To connect Theorem 3.1 with the indifference price in (2.9), assume that the position size q takes the form $q = \ell r_n$ for $\ell \in \mathbb{R}$, where $\{r_n\}_{n\in\mathbb{N}}$ is a sequence of positive reals with $\lim_{n\uparrow\infty} r_n = \infty$. In this case, using (2.9) gives

$$p^n_{a_n}(\ell r_n) = -\frac{1}{a_n \ell r_n}\log\left(\mathbb{E}^{\mathbb{Q}^n_0}\left[e^{-a_n \ell r_n \hat{Y}^n_{a_n}(a_n \ell r_n)}\right]\right) = -\frac{1}{a_n \ell r_n}\Gamma_n(-a_n \ell r_n), \tag{3.2}$$

where, similarly to Λ_n above, we set

$$\Gamma_n(\lambda) := \log\left(\mathbb{E}^{\mathbb{Q}^n_0}\left[e^{\lambda \hat{Y}^n_{a_n}(-\lambda)}\right]\right). \tag{3.3}$$

We thus see that convergence of the indifference prices $p^n_{a_n}(\ell r_n)$ is analogous to the Gärtner-Ellis assumption that the scaled cummulant generating functions $(1/r_n)\Lambda_n(\ell r_n)$ converge. However, besides the dependence of probability measure on n, there is a substantial difference between Γ_n in (3.3) and Λ_n in (3.1): namely, the random variables $\hat{Y}^n_{a_n}(\lambda)$ of (3.3) are changing with λ whereas the random variables Y_n of (2.10) are not. Thus, even though convergence of the scaled indifference prices implies a connection with a LDP for the random variables $\hat{Y}^n_{a_n}(\lambda)$, we do not typically expect a LDP from random variables $\hat{Y}^n_{a_n}(\lambda)$ unless they do not actually depend upon λ. An example where this is the case is presented in Section 6.3 below.

We now make the main assumption in an analogous form to the Gärtner-Ellis theorem.

Assumption 3.2. There exist a sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_{n\uparrow\infty} r_n = \infty$ and a $\delta > 0$ such that for all $|\ell| < \delta$ the limit

$$p^\infty(\ell) := \lim_{n\uparrow\infty} p^n_{a_n}(\ell r_n), \tag{3.4}$$

exists and is finite. In particular, with

$$d_n := p^n_{a_n}(0) = \mathbb{E}^{\mathbb{Q}^n_0}[B], ^* \tag{3.5}$$

the limit $d := p^\infty(0) = \lim_{n\uparrow\infty} d_n$ exists. Furthermore, $p^\infty(\ell)$ is continuous at 0, i.e. $\lim_{\ell\to 0} p^\infty(\ell) = d = p^\infty(0)$.

3.1. **Discussion.**

3.1.1. *Assumption 3.2 and Vanishing Risk Aversion.* The relation (2.8) allows us to vary risk aversion as well as position size. Specifically, Assumption 3.2 takes the form that for all $|\ell| < \delta$:

$$p^\infty(\ell) = \lim_{n\uparrow\infty} p^n_{a_n}(\ell r_n) = \lim_{n\uparrow\infty} p^n_1(\ell a_n r_n). \tag{3.6}$$

From here, it immediately follows that if the market is fixed: i.e. if $p^n_1(q_n) = p_1(q_n)$ for all n and q_n, then if $a_n \to 0$ we may set $r_n := a_n^{-1} \to \infty$ and Assumption 3.2 holds. Indeed, $p_1(\ell a_n r_n) = p_1(\ell) =: p^\infty(\ell)$ and continuity at 0 follows from [14] which shows that $\lim_{\ell\to 0} p^\infty(\ell) = d = \mathbb{E}^{\mathbb{Q}_0}[B]$. This example is briefly additionally discussed in Section 6.1 below, and Theorems 4.3, 4.3 not withstanding, our focus in the sequel will lie primarily on the case of fixed risk aversion in a sequence of varying markets.

3.1.2. *Assumption 3.2 and Vanishing Hedging Errors.* Though not explicitly stated, for a fixed risk aversion $a_n \equiv a$, Assumption 3.2 implies the hedging errors associated B are vanishing. This follows both from the convergence of scaled indifference prices $p^n_a(\ell r_n)$ and, crucially, from the assumption that p^∞ is continuous at 0. To see this latter point, consider again when the market is fixed so $p^n_a(q_n) = p_a(q_n)$. Here, for a bounded claim B, as shown in [14, 32], we have

$$\lim_{n\uparrow\infty} p_a(\ell r_n) = \begin{cases} \inf\limits_{\mathbb{Q}\in\tilde{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[B], & \ell > 0 \\ \mathbb{E}^{\mathbb{Q}_0}[B], & \ell = 0 \\ \sup\limits_{\mathbb{Q}\in\tilde{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[B], & \ell < 0. \end{cases}$$

Thus, the convergence requirement in Assumption 3.2 holds, but the resultant function p^∞ is not continuous at 0, so Assumption 3.2 cannot hold in a fixed market (or when there is a limiting market but B is not replicable in this market).

Alternatively, consider when all of Assumption 3.2 holds. Firstly, (2.7) implies that $q \mapsto p^n_{a_n}(q)$ is decreasing and $q \mapsto q p^n_{a_n}(q)$ is concave. Thus, $\ell \mapsto \ell p^n_{a_n}(\ell r_n)$ is concave as well and, for $|\ell| < \delta$, so

*See [14] for a proof of this equivalence.

is $\ell \mapsto \ell p^\infty(\ell)$. In particular, $p^\infty(\ell)$ is continuous on $(-\delta, 0)$ and $(0, \delta)$. Thus, additionally assuming continuity of p^∞ at 0 (and hence on all of $(-\delta, \delta)$), we obtain the useful result:

$$\frac{q_n}{r_n} \to \ell \in (-\delta, \delta) \Longrightarrow p^n_{a_n}(q_n) \to p^\infty(\ell). \tag{3.7}$$

Indeed, take $\varepsilon > 0$ so that $(\ell - \varepsilon)r_n \leq q_n \leq (\ell + \varepsilon)r_n$ for all n large enough. Since $p^n_{a_n}(q)$ is decreasing:

$$p^\infty(\ell+\varepsilon) = \lim_{n\uparrow\infty} p^n_{a_n}((\ell+\varepsilon)r_n) \leq \liminf_{n\uparrow\infty} p^n_{a_n}(q_n) \leq \limsup_{n\uparrow\infty} p^n_{a_n}(q_n) \leq \lim_{n\uparrow\infty} p^n_{a_n}((\ell-\varepsilon)r_n) = p^\infty(\ell-\varepsilon).$$

Taking $\varepsilon \downarrow 0$ gives the result. In particular, for all fixed position sizes q and risk aversions a, we have that $\lim_{n\uparrow\infty} p^n_a(q) = d$, and this essentially implies the existence of trading strategies $\pi^n \in \mathcal{A}^n$ which asymptotically hedge B. This argument is expanded upon, in the case of bounded claims and a continuous filtration, in Section 4.3 below.

3.1.3. *On the strict concavity of $\ell \mapsto \ell p^\infty(\ell)$.* Even though $\ell \mapsto \ell p^\infty(\ell)$ is concave under Assumption 3.2, as the example in Section 4.2 below shows, it need not be strictly concave. However, under the assumption of strict concavity a number of nice consequences ensue: for example, see Corollary 4.6 and the equilibrium results in Section 5.

4. Limiting Scaled Indifference Prices and Consequences

We now deduce a number of consequences of Assumption 3.2, the first of which is that the regime where the position size $q = q_n = \ell r_n$ is the appropriate one as $n \uparrow \infty$, if the considered positions are taken optimally. Here, we follow the approach of [24, 33, 34].

4.1. **Optimal Position Taking.** Define

$$\underline{B}_n := \inf_{\mathbb{Q}\in\tilde{\mathcal{M}}^n} \mathbb{E}^{\mathbb{Q}}[B], \qquad \bar{B}_n := \sup_{\mathbb{Q}\in\tilde{\mathcal{M}}^n} \mathbb{E}^{\mathbb{Q}}[B]. \tag{4.1}$$

Assume, for all n, that B cannot be replicated by trading in S^n, and denote by I^n the range of arbitrage free prices for B: i.e.

$$I^n = (\underline{B}_n, \bar{B}_n). \tag{4.2}$$

For $\tilde{p}^n \in I^n$ the optimal position $\hat{q}_n = \hat{q}_n(\tilde{p}^n)$ is defined as the unique (see [24]) solution to the equation

$$\sup_{q\in\mathbb{R}} \left(u^n_{a_n}(-q\tilde{p}^n, q)\right). \tag{4.3}$$

As shown in [24], $\hat{q}_n$ satisfies the first order conditions for optimality

$$\tilde{p}_n = \mathbb{E}^{\mathbb{Q}^{\hat{q}_n(\tilde{p}^n)}}[B], \tag{4.4}$$

where $\mathbb{Q}^{\hat{q}_n(\tilde{p}^n)} \in \tilde{\mathcal{M}}^n$ is the dual optimizer for $\hat{q}_n(\tilde{p}^n)$ units of claim B in that it achieves the infimum in (2.7). To perform the asymptotic analysis we assume consistency (in n) between the markets and non-degeneracy in prices as $n \uparrow \infty$. More precisely:

Assumption 4.1. For $\underline{B}_n, \bar{B}_n$ as in (4.1) we have

$$\underline{B} := \limsup_{n\uparrow\infty} \underline{B}_n < \liminf_{n\uparrow\infty} \bar{B}_n =: \bar{B}. \tag{4.5}$$

Remark 4.2. Let Assumption 3.2 hold. Then, since $\underline{B}_n \leq d_n \leq \bar{B}_n$ for all n it follows that $\underline{B} \leq d \leq \bar{B}$. Assumption 4.1 strengthens this to say that there are $\tilde{p} \neq d$ so that $\tilde{p}$ is arbitrage free for all n large enough. In particular, there are $I^n \ni \tilde{p}^n \to \tilde{p} \neq d$. Now, Assumption 4.1 may fail in two ways. First of all, it may be that I^n is collapsing to the singleton d as $n \uparrow \infty$. In this case, convergence of limiting prices is trivial since $p^n_{a_n}(q_n) \to d$ for all sequences $\{q_n\}$. The second way in which Assumption 4.1 may fail is if there is no consistency between markets in that there is no price $\tilde{p} \neq d$ such that $\tilde{p} \in I^n$ for all n large. Here, we do not have optimizers (along a subsequence) $\hat{q}_n$.

Under Assumption 4.1, we present the first main result, which says that optimal positions are becoming large at a rate which grows at least like ℓr_n for some $\ell \neq 0$.

Theorem 4.3. *Let Assumptions 2.1, 2.2, 3.2 and 4.1 hold. For $I^n \ni \tilde{p}^n \to \tilde{p}$ we have*

- *If $\tilde{p} < d$ then*
$$\liminf_{n\uparrow\infty} \frac{\hat{q}_n(\tilde{p}^n)}{r_n} > 0.$$
- *If $\tilde{p} > d$ then*
$$\liminf_{n\uparrow\infty} \frac{-\hat{q}_n(\tilde{p}^n)}{r_n} > 0.$$

The problem of obtaining upper bounds for $\limsup_{n\uparrow\infty} |\hat{q}_n(\tilde{p}^n)|/r_n$ is more subtle. First of all we need to identify the maximal range where $p^n_{a_n}(\ell r_n)$ converges. To do this, set

$$\delta^+ := \sup\left\{k > 0 \,:\, \lim_{n\uparrow\infty} p^n_{a_n}(\ell r_n) = p^\infty(\ell),\ \forall\, 0 < \ell < k\right\} \in [\delta, \infty]. \tag{4.6}$$

$$\delta_- := \inf\left\{k < 0 \,:\, \lim_{n\uparrow\infty} p^n_{a_n}(\ell r_n) = p^\infty(\ell),\ \forall\, 0 > \ell > k\right\} \in [-\infty, -\delta]. \tag{4.7}$$

As discussed in Section 3.1, $p^n_{a_n}(q)$ is decreasing in q and hence $p^\infty(\ell)$ is decreasing in ℓ. Therefore, the limits

$$p^\infty(\delta^+) := \lim_{\ell\downarrow\delta_-} p^\infty(\ell); \qquad p^\infty(\delta_-) := \lim_{\ell\uparrow\delta^+} p^\infty(\ell), \tag{4.8}$$

exist. Furthermore, since $\underline{B}_n < p^n_{a_n}(\ell r_n) < \bar{B}_n$ for all $\ell \in \mathbb{R}$ we have $\underline{B} \leq p^\infty(\delta^+) \leq p^\infty(\delta_-) \leq \bar{B}$, however, as the example in Section 4.2 below shows, each of these inequalities may be strict. In particular, the range of limiting indifference prices along the rate r_n may deviate from the arbitrage free prices.

With this notation, we now provide the corresponding upper bounds for optimal positions.

Theorem 4.4. *Let Assumptions 2.1, 2.2, 3.2 and 4.1 hold. Define δ^+, δ_- as in* (4.6) *and* (4.7) *respectively. For $I^n \ni \tilde{p}^n \to \tilde{p}$ we have*

- *If* $p^\infty(\delta^+) < \tilde{p} < d$ *then*
$$\limsup_{n\uparrow\infty} \frac{\hat{q}_n(\tilde{p}^n)}{r_n} < \delta^+.$$
- *If* $d < \tilde{p} < p^\infty(\delta_-)$ *then*
$$\limsup_{n\uparrow\infty} \frac{-\hat{q}_n(\tilde{p}^n)}{r_n} < -\delta_-.$$

Note the strict inequality above implies, for example, that when $\delta^+ = \infty$ we have $\limsup_{n\uparrow\infty} \hat{q}_n(\tilde{p}^n)/r_n < \infty$. Lastly, let us discuss when one actually has true convergence. As seen in Section 3.1 the map $\ell \mapsto \ell p^\infty(\ell)$ is concave. Here, we strengthen this by assuming:

Assumption 4.5. The function $\ell \mapsto \ell p^\infty(\ell)$ is strictly concave on (δ_-, δ^+).

Then, we have the following Corollary which ensures the limit $\hat{q}_n/r_n$ actually exists:

Corollary 4.6. *Let Assumptions 2.1, 2.2, 3.2, 4.1 and 4.5 hold. Define* δ^+, δ_- *as in* (4.6) *and* (4.7) *respectively. Let* $I^n \ni \tilde{p}^n \to \tilde{p}$. *If* $p^\infty(\delta^+) < \tilde{p} \neq d < p^\infty(\delta_-)$ *then*
$$\lim_{n\uparrow\infty} \frac{\hat{q}_n(\tilde{p}^n)}{r_n} = \ell \in (\delta_-, \delta^+) \setminus \{0\}.$$

The proofs of Theorems 4.3, 4.4 and of Corollary 4.6 are in Appendix A.

4.2. **Discussion.** Presently, we point out some conclusions and subtleties associated to the above results. First, when we put together Theorems 4.3, 4.4 we see that if the price $\tilde{p}^n \in I^n$ converges to $\tilde{p}$ where $p^\infty(\delta^+) < \tilde{p} < p^\infty(\delta_-), p \neq d$ then up to subsequences we have $\hat{q}_n(\tilde{p}^n)/r_n \to \ell \in (\delta_-, \delta^+) \setminus \{0\}$, which by Corollary 4.6 becomes true convergence if $\ell \mapsto \ell p^\infty(\ell)$ is strictly concave. Note also that by (3.7), under optimal positions we have convergence of indifference prices as well, i.e. $p^n_{a_n}(\hat{q}_n(\tilde{p}^n)) \to p^\infty(\ell)$.

Second, assume for example that $\delta^+ = \infty$. Then, another straightforward calculation shows (recall (4.5))
$$\underline{B} < \tilde{p} < \lim_{\ell\uparrow\infty} p^\infty(\ell) \Longrightarrow \lim_{n\uparrow\infty} \frac{\hat{q}_n(\tilde{p})}{r_n} = \infty,$$
provided of course such a $\tilde{p}$ exists. This offers a converse to Theorem 4.4.

Third, let us briefly discuss the degenerate case where r_n is (chosen) such that $p^\infty(\ell) = d$ for all $\ell \in (\delta_-, \delta^+)$. In this case, a range of different phenomena can occur. For illustration purposes, we consider the following example, taken from [34]. In the nth market, the claim decomposes into a replicable piece D_n (with replicating capital d_n) and a piece Y_n which is independent of S^n. Now, assume $Y_n \sim N(0, \gamma_n)$ under $\mathbb{P}$ and fix the risk aversion $a_n \equiv a$. Here, the indifference price is
$$p^n_a(q) = d_n - \frac{1}{aq} \log\left(\mathbb{E}\left[e^{-aqY_n}\right]\right) = d_n - \frac{1}{2} aq\gamma_n^2.$$
The range of arbitrage free prices maximal: i.e. $\underline{B}_n = -\infty$, $\bar{B}_n = \infty$. For $\tilde{p}^n \in \mathbb{R}$ the optimal purchase quantity found by minimizing $q\tilde{p}^n - qp^n_a(q)$ is
$$\hat{q}_n(\tilde{p}^n) = -\frac{\tilde{p}^n - d_n}{a\gamma_n^2}.$$

Now, assume that $\gamma_n \to 0, d_n \to d$. With $r_n = \gamma_n^{-2} \to \infty$, Assumption 3.2 holds with $p^\infty(\ell) = d - (1/2)a\ell$, $\delta_- = -\infty$ and $\delta^+ = \infty$. Note that $\ell p^\infty(\ell) = \ell d - (1/2)a\ell^2$ is strictly concave. Here, if $\tilde{p}^n \to \tilde{p} \in \mathbb{R}$ we have that

$$\frac{\hat{q}_n(\tilde{p}^n)}{r_n} = -\frac{\tilde{p}^n - d_n}{a} \to -\frac{\tilde{p} - d}{a}.$$

So, both Theorems 4.3, 4.4 hold.

Now, change r_n so that $r_n = \gamma_n^{-1} \to \infty$. Then, Assumption 3.2 still holds with $p^\infty(\ell) = d$, $\delta_- = -\infty$ and $\delta^+ = \infty$. In this instance, however, the map $\ell p^\infty(\ell) = \ell d$ is not strictly concave. Here, if $\tilde{p}^n \to \tilde{p} \in \mathbb{R}$ (which is still arbitrage free since this property does not depend upon r_n) we have

$$\frac{\hat{q}_n(\tilde{p}^n)}{r_n} = -\frac{\tilde{p}^n - d_n}{a\gamma_n}.$$

So, if $\tilde{p} < d$ the ratio goes to ∞, if $\tilde{p} > d$ the ratio goes to $-\infty$ and if $\tilde{p} = d$ then a variety of phenomena can occur depending on the rates at which $\gamma_n \to 0, \tilde{p}^n \to \tilde{p}$ and $d_n \to d$. Even though the behavior is degenerate in this case, it *does not* contradict either Theorem 4.3 or 4.4. In particular, Theorem 4.4 is vacuous in this case since $p^\infty(\ell) = d$ for all ℓ.

The above example is related to the well known fact from Large Deviations that a LDP may hold for the same sequence of random variables with two different rates $\{r_n\}, \{r'_n\}$ with $r_n/r'_n \to 0$. The resulting rate functions however, in an analogous manner to the resultant limiting indifference prices above, may provide drastically different levels of information.

4.3. **On the Normalized Optimal Wealth Process.** For a given n, fixed risk aversion a and position size q_n, recall the optimal wealth process $\hat{X}_a^n(q_n)$ from Section 2. Heuristically, as $|q_n| \to \infty$ one expects $\hat{X}_a^n(q_n)$, as well as the optimal strategy $\hat{\pi}_a^n(q_n)$, to grow on the order of $|q_n|$. However, if we normalize the wealth process by the position size then it is reasonable to ask if some type of convergence takes place. To this end we define the normalized wealth process $\tilde{X}$ via

$$\tilde{X}_a^n(q_n) := \frac{1}{q_n}\hat{X}_a^n(q_n). \tag{4.9}$$

Note that $\tilde{X}_a^n(q_n)$ is in fact a wealth process, obtained from the (acceptable) normalized optimal trading strategy $\tilde{\pi}_a^n(q_n) = (1/q_n)\hat{\pi}_a^n(q_n)$. We wish to stress that convergence of the normalized optimal wealth process is a topic on its own and we do not study this in this paper. However, we mention some interesting and motivating straightforward conclusions.

Let us come back to (2.6), re-written here as $-au_a^n(0)e^{-aq_np_a^n(q_n)} = \mathbb{E}\left[e^{-q_na\left(\tilde{X}_a^n(q_n)+B\right)}\right]$. Since $-au_a^n(0) \le 1$ we immediately see that

$$\mathbb{E}\left[e^{-q_na\left(\tilde{X}_a^n(q_n)+B-p_a^n(q_n)\right)}\right] = -au_a^n(0) \le 1. \tag{4.10}$$

By Markov's inequality we have the elementary estimate:

$$\mathbb{P}\left[\tilde{X}_a^n(q_n) + B - p^n(q_n) \le -\gamma\right] \le e^{-q_na\gamma}; \qquad \gamma \in \mathbb{R}.$$

Thus, we see that for any $q_n \uparrow \infty$ the portfolio obtained by buying one unit of B for $p_a^n(q_n)$ and trading according to the normalized optimal trading strategy provides a super-hedge of 0 in $\mathbb{P}-$probability in that for all $\gamma > 0$

$$\lim_{n\uparrow\infty} \mathbb{P}\left[\tilde{X}_a^n(q_n) + B - p_a^n(q_n) \leq -\gamma\right] = 0, \tag{4.11}$$

and in fact, the convergence to 0 is exponentially fast. This result essentially follows because of risk aversion and is valid under the minimal Assumptions 2.1 and 2.2.

If we consider optimal positions then one can say more and characterize the super-hedge more precisely. We first adapt the set-up of [30] and enforce the following assumptions on the claim B and filtration $\mathbb{F}$:

Assumption 4.7. B is bounded: i.e. $\|B\|_{\mathbb{L}^\infty} < \infty$.

Assumption 4.8. The filtration $\mathbb{F}$ is continuous.

Under Assumptions 4.7, 4.8, Theorem 13 of [30], says that for any q_n

$$q_n B = q_n p_a^n(q_n) + \frac{a}{2}\langle \hat{L}_a^n(q_n)\rangle_T - \hat{L}_a^n(q_n)_T - \hat{X}_a^n(q_n)_T, \tag{4.12}$$

where $\hat{L}_a^n(q_n)$ is a $\mathbb{Q}_0^n$ martingale strongly orthogonal to S^n under $\mathbb{Q}_0^n$. Dividing by q_n and setting $\tilde{L}_a^n(q_n) = (1/q_n)\hat{L}_a^n(q_n)$ as the normalized orthogonal $\mathbb{Q}_0^n$ martingale we obtain

$$\tilde{X}_a^n(q_n)_T + B - p_a^n(q_n) = \frac{aq_n}{2}\langle \tilde{L}_a^n(q_n)\rangle_T - \tilde{L}_a^n(q_n)_T. \tag{4.13}$$

Lastly, as shown in [30, Theorem 19], $\sup_n \left(q_n \mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(q_n)\rangle_T\right]\right) < \infty$, which implies that $\tilde{L}_a^n(q_n)_T$ goes to 0 in $\mathbb{Q}_0^n$-L^2 as $n \uparrow \infty$ if $|q_n| \uparrow \infty$.

Now, consider when, additionally, Assumptions 3.2 and 4.1 hold and positions are taking optimally: i.e. $q_n = \hat{q}_n = \hat{q}_n(\tilde{p}^n)$ where $I^n \ni \tilde{p}^n \to \tilde{p}$ with $p^\infty(\delta^+) < \tilde{p} < p^\infty(\delta_-), p \neq d$. Then, from Theorems 4.3, 4.4 we have up to subsequences (or, under the Assumptions of Corollary 4.6, for all subsequences) that $\hat{q}_n/r_n \to \ell \in (\delta_-, \delta^+) \setminus \{0\}$ and that $p_a^n(\hat{q}_n) \to p^\infty(\ell)$. Thus, we obtain that in $\mathbb{Q}_0^n$-probability

$$\tilde{X}_a^n(\hat{q}_n)_T + B - p^\infty(\ell) - \frac{a\hat{q}_n}{2}\langle L_a^n(\hat{q}_n)\rangle_T \to 0, \tag{4.14}$$

which implies that the excess hedge is precisely $a\hat{q}_n\langle \tilde{L}_a^n(q_n)\rangle_T/2$ in $\mathbb{Q}_0^n-$probability limit as $n \to \infty$. Even though this result is interesting, one would like to have the same statement under the $\mathbb{P}$ measure. This is true if the measure $\mathbb{P}$ is contiguous with respect to the measure $\mathbb{Q}_0^n$, i.e. that $\mathbb{Q}_0^n(A_n) \to 0$ implies $\mathbb{P}(A_n) \to 0$ for every sequence of measurable sets $\{A_n\}_{n\in\mathbb{N}}$, e.g. Chapter 6 of [39]. The classical Le Cam's first lemma (Lemma 6.4 in [39]) provides sufficient and necessary conditions for contiguity.

Lastly, assume that $q_n = q$ is fixed and come back to (4.13). Taking expectations yields

$$d_n - p_a^n(q) = \frac{aq}{2}\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \hat{L}_a^n(q)\rangle_T\right],$$

where we recall that $d_n = \mathbb{E}^{\mathbb{Q}_0^n}[B]$. As discussed in Section 3.1.2, Assumption 3.2 implies $p_a^n(q) \to d$ and hence $\lim_{n\uparrow\infty} \mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \hat{L}_a^n(q)\rangle_T\right] = 0$ which in turn implies that both $\langle \hat{L}_a^n(q)\rangle_T$, $\hat{L}_a^n(q)_T$ go to zero in $\mathbb{Q}_0^n$ probability as $n \to \infty$. Therefore, for fixed position sizes, we have in view of (4.13), that $\tilde{X}_a^n(q)_T +$

$B - d$ goes to zero in $\mathbb{Q}_0^n$ probability and hence, under the additional contiguity assumption, the claim is asymptotically hedgeable. This makes precise the connection between Assumption 3.2 and vanishing hedging errors mentioned in Section 3.1.2.

4.4. **On a Characterization of r_n.** As in the previous section, we let Assumptions 2.2, 3.2, 4.7 and 4.8 hold. Using the results of [30], we give a characterization for r_n which in a sense justifies the interpretation of r_n as the speed at which the market becomes complete. Recalling (3.5), (4.12) and the normalized orthogonal martingale $\tilde{L}_a^n(q_n)$ we get

$$d_n = p_a^n(q_n) + \frac{aq_n}{2}\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(q_n)\rangle_T\right].$$

Now, let $q_n = \ell r_n$ for some $|\ell| < \delta$ (which, by Corollary 4.6 and (3.7) essentially includes the case of optimal positions). We thus have

$$\lim_{n\uparrow\infty} \frac{r_n}{2}\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(\ell r_n)\rangle_T\right] = \frac{d - p^\infty(\ell)}{a\ell}. \tag{4.15}$$

This conforms to the "asymptotically complete" case. The normalized hedging error under optimal positions $\hat{q}_n \approx \ell r_n$ is approximately (up to a multiplicative constant) $\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(\ell r_n)\rangle_T\right]$. If the market is becoming complete we expect that for $n \to \infty$

$$\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(\ell r_n)\rangle_T\right] \to 0.$$

The speed at which it goes to 0 thus becomes r_n^{-1} and at this scaling we have convergence of prices.

In Sections 6 and 7 we study a number of examples where r_n can be computed explicitly. One would like to have an abstract formula that explicitly characterizes r_n, as (4.15) contains r_n within the normalized hedging error $\langle \tilde{L}_a^n(\ell r_n)\rangle$. Notice that (4.15) holds for all $|\ell| < \delta$. So, one is tempted to take limits as $\ell \to 0$ on both sides, and, if one can interchange the $n \uparrow \infty$ limit with the $\ell \to 0$ limit, pass the latter limit inside the expectation, and if $p^\infty(\ell)$ is both strictly decreasing and differentiable at $\ell = 0$, then for n large enough

$$r_n \approx -\frac{2\dot{p}^\infty(0)}{a} \times \frac{1}{\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(0)\rangle_T\right]}.$$

Here, the interpretation of r_n^{-1} as a market incompleteness factor is much more transparent. Indeed, as shown in [30, Section 6.1] (or formally from (4.12) by dividing by q_n and letting $q_n \to 0$), we have

$$B = \mathbb{E}^{\mathbb{Q}_0^n}[B] - \tilde{L}_a^n(0)_T - \tilde{X}_a^n(0)_T,$$

so that $(\tilde{X}_a^n(0)_T, \tilde{L}_a^n(0)_T)$ provides the Kunita-Watanabe decomposition of $-B$ with respect to the subspace of $L^2(\mathbb{Q}_0^n; \mathcal{F}_T)$ generated by trading in S^n. In other words, $\tilde{L}_a^n(0)$ describes the hedging error associated to B, with size $\mathbb{E}^{\mathbb{Q}_0^n}\left[\langle \tilde{L}_a^n(0)\rangle_T\right] \propto r_n^{-1}$. Thus r_n^{-1} acts as the market incompleteness factor, and, as the market becomes complete, we see that $r_n \to \infty$.

The derivation of this statement is of course heuristic. Rigorous proof of this result seems to be quite hard, but we nevertheless present the argument as it provides more intuition into the problem. We choose to leave the rigorous derivation of this result and further consequences as a future interesting work.

4.5. **Optimal Position Taking for General Utilities.** The optimal position taking results in Theorems 4.3 and 4.4 readily extend to general utility functions on the real line. This essentially follows from [33]. Throughout this section we fix the risk aversion at $a > 0$. Define $\mathcal{U}_a$ as the class of utility functions on $\mathbb{R}$ (i.e. $U \in C^2(\mathbb{R})$, strictly increasing and strictly concave) satisfying

- The absolute risk aversion of U is bounded between two positive constants: i.e. for $0 < \underline{a}_U < \bar{a}_U$:

$$\underline{a}_U \leq \alpha_u(x) := -\frac{U''(x)}{U'(x)} \leq \bar{a}_U; \qquad x \in \mathbb{R}. \tag{4.16}$$

- U decays exponentially with rate a for large negative wealths: i.e.

$$\lim_{x \downarrow -\infty} -\frac{1}{x} \log(-U(x)) = a. \tag{4.17}$$

By (4.16) it follows that U is bounded from above on $\mathbb{R}$ and hence through a normalization we assume $0 = U(\infty) = \lim_{x\uparrow\infty} U(x)$. From [33, Section 2.2] it holds that $U \in \mathcal{U}_a$ satisfies both the Inada conditions $\lim_{x\downarrow-\infty} U'(x) = \infty$, $\lim_{x\uparrow\infty} U'(x) = 0$ and the Reasonable Asymptotic Elasticity conditions $\liminf_{x\downarrow-\infty} xU'(x)/U(x) > 1$, $\limsup_{x\uparrow\infty} xU'(x)/U(x) < 1$. Similarly to (2.2) and (2.3), define the value function in the nth market with initial capital x and q units of the claim as $u_U^n(x,q)$, where if $q = 0$ we write $u_U^n(x)$. Analogously to (2.4), set $p_U^n(x,q)$ as the (average, bid) utility indifference price defined through the equation

$$u_U^n(x - qp_U^n(x,q), q) = u_U^n(x). \tag{4.18}$$

So that $p_U^n(x,q)$ is well defined for $x, q \in \mathbb{R}$ we assume the claim is bounded: i.e. we enforce Assumption 4.7. Under Assumptions 2.2, 4.7 it follows from [32] that for $x, q \in \mathbb{R}$, $p_U^n(x,q)$ is well defined, arbitrage free, decreasing in q with limits (recall (4.2)) $\lim_{q\downarrow-\infty} p^n(x,q) = \bar{B}_n$, $\lim_{q\uparrow\infty} p^n(x,q) = \underline{B}_n$, for each n.

To connect limiting prices for U with those for the exponential utility we impose the following mild asymptotic no arbitrage condition (see [33, pp. 9]):

Assumption 4.9. $\limsup_{n\uparrow\infty} H\left(\mathbb{Q}_0^n \mid \mathbb{P}\right) < \infty$.

Above, $\mathbb{Q}_0^n$ is the minimal entropy measure with density as in (2.5). From [33, Theorem 3.3], it follows from Assumptions 2.2, 3.2, 4.7 and 4.9 that for all $x \in \mathbb{R}$ and $0 < |\ell| < \delta$:

$$\lim_{n\uparrow\infty} p_U^n(x, \ell r_n) = p^\infty(\ell). \tag{4.19}$$

As for $\ell = 0$, since Assumption 3.2 implies p^∞ is continuous at 0, the monotonicity of $p_U^n(x,q)$ yields for $0 < \ell < \delta$ that

$$p^\infty(\ell) = \lim_{n\uparrow\infty} p_U^n(x, \ell r_n) \leq \liminf_{n\uparrow\infty} p_U^n(x,0) \leq \limsup_{n\uparrow\infty} p_U^n(x,0) \leq \lim_{n\uparrow\infty} p_U^n(x, -\ell r_n) = p^\infty(-\ell),$$

so that taking $\ell \downarrow 0$ we obtain that $p_U^n(x,0) \to p^\infty(0)$. Now, for a given arbitrage free price $\tilde{p}^n \in I^n$, we consider the optimal purchase problem

$$\sup_{q\in\mathbb{R}} \left(u_U^n(x - \tilde{p}^n q, q)\right). \tag{4.20}$$

Unlike for the exponential case when the results of [24] yield a unique maximizer, here, to the best our our knowledge, there are no known results on existence/uniqueness of optimizers (see [36] for results with utility functions defined on the positive axis). However, the main results of Theorems 4.3 and 4.4 still hold, as the following theorem shows.

Theorem 4.10. *Let Assumptions 2.2, 3.2, 4.1, 4.7 and 4.9 hold. Assume that $I^n \ni \tilde{p}^n \to \tilde{p}$. Let $x \in \mathbb{R}$ be fixed and recall δ^+, δ_- from* (4.6)*,* (4.7) *respectively. Then*

- *For each n there exists an optimizer $\hat{q}_n = \hat{q}_n(x, \tilde{p}^n)$ to* (4.20)*.*
- *If $p^\infty(\delta^+) < \tilde{p} < d$ then for any sequence of maximizers $\{\hat{q}_n\}$:*

$$0 < \liminf_{n\uparrow\infty} \frac{\hat{q}_n}{r_n} < \limsup_{n\uparrow\infty} \frac{\hat{q}_n}{r_n} < \delta^+. \tag{4.21}$$

- *If $d < \tilde{p} < p^\infty(\delta_-)$ then for any sequence of maximizers $\{\hat{q}_n\}$:*

$$0 < \liminf_{n\uparrow\infty} \frac{-\hat{q}_n}{r_n} < \limsup_{n\uparrow\infty} \frac{-\hat{q}_n}{r_n} < -\delta_-. \tag{4.22}$$

Remark 4.11. As with the exponential case, a sufficient condition for the limits to exist in (4.21) and (4.22) is Assumption 4.5.

5. On Partial Equilibrium Price Quantity and its Limiting Behavior

The concept of indifference pricing has a subjective nature, in the sense that the indifference price of an investor is a way she values unhedgeable positions, and whether or not there is a counter-party to offset a transaction is a different question. In particular, so far we have assumed that a sequence of prices $\tilde{p}^n \in I^n$ converges to $\tilde{p}$, without mentioning whether such prices equilibrate any transactions among different investors. In this section, we address this issue and we justify that such sequence of prices could indeed be the equilibrium prices of the given claim B among (two) investors.

For this, we adapt the notion of the partial equilibrium price quantity (PEPQ). Provided that the stock dynamics are exogenously specified, the equilibrium price of a claim B is the one at which the investors' optimal quantities of the claim sum up to zero, meaning that the market of the claim is cleared out (the word partial refers to the fact the investors specify the equilibrium of the claim and not the stock market). Essentially, the main motivation of this section is to study under Assumption 3.2 when our main optimal position taking results could arise in an equilibrium setting whether all investors act optimally and the price $\tilde{p}^n$ is the equilibrium price in the nth market of a given claim B. In short, the analysis of this section prove that if the investors' risky exposures (random endowments) are dominated by r_n, then $\tilde{p}^n \to d$. However, if investors' endowments are growing like r_n, equilibrium prices $\tilde{p}^n$ could converge to a limit different than d and the results of Theorems 4.3, 4.4 occur. The latter situation, which happens when at least one investor has an already undertaken large position in B, means that there are cases where the large regime is in fact the market's equilibrium, and even more interestingly the equilibrium prices converge to a price different than the unique limiting arbitrage free price.

In the setting of a locally bounded semi-martingale stock market, bounded claims, and exponential utility maximizers, the PEPQ is analyzed in [1]. Specified to the current setup of Section 2, we assume, for each n, there is a group of I investors such that each investor i is endowed with a exogenously given *random endowment*, denoted by $\mathcal{E}_n^i$. For a given bounded claim B, the investors also wish to trade B amongst themselves in such a way that acting optimally (in terms of utility maximization) the market for the claim clears.

For simplicity, we consider the presence of two investors, although we should point out that the results of this section can be generalized for markets with more investors. Recall that I^n from (4.2) denotes the (non-empty) range of arbitrage free prices for B and let $a_n^i > 0$ denote the risk aversion coefficient for investor i. Before we give the exact definition of the PEPQ for a claim B, we need to introduce the notation for the indirect utility and the indifference pricing under the presence of random endowment. Namely, for the random endowment $\mathcal{E}_n^i$ and position size q in B, define, in a similar manner to (2.2), the value function for investor i by

$$u^n_{a_n^i}(x, q|\mathcal{E}_i^n) := \sup_{\pi^n \in \mathcal{A}^n} \mathbb{E}\left[U_{a_n^i}(x + X_T^{\pi^n} + qB + \mathcal{E}_n^i)\right]; \quad i = 1, 2. \tag{5.1}$$

Similarly to (2.4), the average (bid) indifference price of the investor i with random endowment $\mathcal{E}_i^n$ at the nth market is denoted by $p^n_{a_n^i}(q|\mathcal{E}_n^i)$ and is given as the solution of

$$u^n_{a_n^i}(x - qp^n_{a_n^i}(q|\mathcal{E}_n^i), q|\mathcal{E}_n^i) = u^n_{a_n^i}(x|\mathcal{E}_n^i); \quad i = 1, 2. \tag{5.2}$$

Note that the indifference price's independence on the (constant) initial wealth still holds under the presence of the random endowment, which means that we can again assume $x = 0$. Next, for a given $p^n \in I^n$, consider the optimal purchase quantity problem for investor i defined by identifying (compare with (4.3)):

$$\hat{q}_n^i(p^n) = \operatorname*{argmax}_{q \in \mathbb{R}} \left(u^n_{a_n^i}(-qp^n, q|\mathcal{E}_n^i)\right); \quad i = 1, 2. \tag{5.3}$$

A PEPQ is then defined as a pair $(p_*^n, q_*^n) \in I^n \times \mathbb{R}$ such that

$$q_*^n = \hat{q}_n^1(p_*^n) \quad \text{and} \quad -q_*^n = \hat{q}_n^2(p_*^n).$$

In other words, at price p_*^n it is optimal for investor 1 to buy q_*^n and investor 2 to sell q_*^n units of B, thus the market clears out. It is then a matter of simple calculations to get the following condition for the PEPQ for each $n \in \mathbb{N}$ (see also Proposition 5.6 and Corollary 5.7 in [1]):

$$q_*^n = \operatorname*{argmax}_{q \in \mathbb{R}} \left(q\left(p^n_{a_n^1}(q|\mathcal{E}_1^n) + p^n_{a_n^2}(-q|\mathcal{E}_2^n)\right)\right). \tag{5.4}$$

The equilibrium price p_*^n is then given by

$$p_*^n = \mathbb{E}^{\mathbb{Q}_1^n(q_*^n)}[B] = \mathbb{E}^{\mathbb{Q}_2^n(-q_*^n)}[B], \tag{5.5}$$

where $\mathbb{Q}_i^n(q)$ denotes the dual optimizer in $\tilde{\mathcal{M}}^n$ for the position $qB + \mathcal{E}_i^n$ and risk aversion a_n^i (recall the first order condition (4.4) without random endowment)[†]. According to Theorem 5.8 in [1], for a non-replicable

[†]Note that $\mathbb{Q}_i^n(0)$ is not necesarily $\mathbb{Q}_0^n$ due to the presence of $\mathcal{E}_i^n$.

bounded claim B (i.e. satisfying Assumption 4.7) a PEPQ $(p_*^n, q_*^n) \in I^n \times \mathbb{R}$ always exists for each $n \in \mathbb{N}$, and it is unique with $q_*^n \neq 0$ if and only if $a_n^1 \mathcal{E}_1^n - a_n^2 \mathcal{E}_2^n$ is non-replicable.

Now, consider when $n \uparrow \infty$ and Assumption 3.2 holds for each sequence $\{a_n^i\}_{n\in\mathbb{N}}$. The questions that naturally arise are where the sequence of the equilibrium prices converges to and under which conditions the regime of Theorems 4.3, 4.4 occurs. As $n \uparrow \infty$, if one ignores the position size and has non-vanishing risk aversion, the hedging error of positions in B approaches zero and hence it is expected that equilibrium prices converge to price d. It turns out that this is the case provided however that the size of the investors' endowments is dominated by the "market incompleteness" parameter r_n from Assumption 3.2. When at least one of the endowments increases with n sufficiently fast, the equilibrium prices may converge to a limit different than d, which implies a situation similar to the regime of Theorems 4.3, 4.4. In the sequel we provide a family of such examples where the endowment of one of the investor is an increasing position on the claim B.

Before, we present the precise arguments we should clarify how the Assumption 3.2 works in the case of two investors, $i = 1, 2$. The statement that Assumption 3.2 holds for function $p_{a_n^i}^n : \mathbb{R} \mapsto I^n$ (defined in (2.4)), means that there exist a sequence $\left\{r_n^i\right\}_{n\in\mathbb{N}}$ of positive reals with $r_n^i \nearrow \infty$ and a constant $\delta_i > 0$ such that for all $|\ell| < \delta_i$ the limit $p_i^\infty(\ell) := \lim_{n\uparrow\infty} p_{a_n^i}^n(\ell r_n^i)$ exists, is finite and $\lim_{\ell\to 0} p_i^\infty(\ell) = d$. Note that it is readily follows from the relation $p_{a_n^2}^n(q) = p_{a_n^1}^n(q a_n^2/a_n^1)$ (which holds for each n) that if Assumption 3.2 holds for function $p_{a_n^1}^n$, it will also hold for function $p_{a_n^2}^n$ provided that the sequence $\{a_n^2/a_n^1\}_{n\in\mathbb{N}}$ is bounded away from zero and infinity. For this, we could set $r_n^2 := r_n^1 a_n^2/a_n^1$ (possibly going to an increasing subsequence), $p_2^\infty = p_1^\infty$ and $\delta_2 = \delta_1$.

For the proofs of this section we need to introduce the notion of the (bid) indifference price for every arbitrary bounded payoff $C \in \mathbb{L}^\infty$ under risk aversion $a_n > 0$ in the nth market, denoted by $P_{a_n}^n(C)$ and defined as the solution of the following equation

$$\sup_{\pi^n\in\mathcal{A}^n} \mathbb{E}\left[U_{a_n}(x + X_T^{\pi^n} + C - P_{a_n}^n(C))\right] = \sup_{\pi^n\in\mathcal{A}^n} \mathbb{E}\left[U_{a_n}(x + X_T^{\pi^n})\right]; \quad i = 1, 2. \tag{5.6}$$

Note that under this notation $q p_{a_n}^n(q) = P_{a_n}^n(qB)$, for all $q \in \mathbb{R}$ with $p_{a_n}^n$ defined in (2.4). The following Lemma generalizes the findings of Theorems 4.3 and 4.4 under the presence of random endowment provided that the endowment is dominated by the associated r_n.

Lemma 5.1. *Let Assumptions 2.2, 4.1, 4.7 hold and impose Assumption 3.2 for function $p_{a_n^i}^n : \mathbb{R} \mapsto I^n$. If for $i = 1, 2$, $\mathcal{E}_i^n \in \mathbb{L}^\infty$, for each n and $||\mathcal{E}_i^n||_{\mathbb{L}^\infty}/r_n^i \to 0$, then the statements of Theorems 4.3 and 4.4 hold also for the function $p_{a_n^i}^n(\cdot|\mathcal{E}_i^n) : \mathbb{R} \mapsto I^n$.*

Proof. In view of the proof of Theorem 4.3 and under the imposed assumptions, we first have to show that function $p_{a_n^i}^n(\cdot|\mathcal{E}_i^n) : \mathbb{R} \mapsto I^n$ satisfies Assumption C.5. Indeed, the first bullet point follows by a simple change of measure $d\mathbb{P}_i^n/d\mathbb{P} := c_i^n e^{-a_n^i \mathcal{E}_i^n}$, for some constant c_i^n and the corresponding variational representation of the indifference price (2.7) considered under measure $\mathbb{P}_i^n$; while the second bullet point readily follows by the boundedness of claim B. For the third and forth items, it is enough to show that for all $|\ell| < \delta_i$, $\lim_{n\to\infty} p_{a_n^i}^n(\ell r_n^i|\mathcal{E}_i^n) = p_i^\infty(\ell)$. For this, we note that the indifference price of an exponential

utility maximizer under some random endowment can be written as the difference of two indifference prices without endowments (see among others, Appendix of [1] and recall definition (5.6)):

$$qp^n_{a^i_n}(q|\mathcal{E}^n_i) = P^n_{a^i_n}(qB + \mathcal{E}^n_i) - P^n_{a^i_n}(\mathcal{E}^n_i), \quad \forall q \in \mathbb{R}, \tag{5.7}$$

Hence, for any $|\ell| < \delta_i$

$$p^n_{a^i_n}(\ell r^i_n|\mathcal{E}^n_i) = \frac{P^n_{a^i_n}(\ell r^i_n B + \mathcal{E}^n_i) - P^n_{a^i_n}(\mathcal{E}^n_i)}{\ell r^i_n} \leq p^n_{a^i_n}(\ell r^i_n) + 2\frac{||\mathcal{E}^n_i||_{\mathbb{L}^\infty}}{|\ell| r^i_n} \to p^\infty_i(\ell),$$

where the limiting argument follows by the imposed assumptions on function $p^n_{a^i_n}$ and $\mathcal{E}^n_i$. We similarly show that $p^n_{a^i_n}(\ell r^i_n|\mathcal{E}^n_i) \geq p^n_{a^i_n}(\ell r^i_n) - 2\frac{||\mathcal{E}^n_i||_{\mathbb{L}^\infty}}{|\ell| r^i_n} \to p^\infty_i(\ell)$, which finishes the proof that function $q \mapsto p^n_{a^i_n}(q|\mathcal{E}^n_i)$ satisfies Assumption C.5. We then observe that requirements of Proposition C.6 are also met for function $p^n_{a^i_n}(\cdot|\mathcal{E}^n_i) : \mathbb{R} \mapsto I^n$, since by (5.7) it readily follows that $p^n_{a^i_n}(\infty|\mathcal{E}^n_i) = p^n_{a^i_n}(\infty)$. Hence, the rest of the proof follows the same argument lines as the ones in proofs of Theorems 4.3, 4.4. □

Returning to the PEPQ, we exclude trivial cases for each $n \in \mathbb{N}$ by imposing the following assumption.

Assumption 5.2. For each n, $\mathcal{E}^n_i \in \mathbb{L}^\infty$ for both $i = 1, 2$ and $a^1_n\mathcal{E}^n_1 - a^2_n\mathcal{E}^n_2$ is non-replicable.

As mentioned above, this assumption guarantees the existence and the uniqueness of the PEPQ (p^n_*, q^n_*) for each n with $q^n_* \neq 0$. Imposing Assumption 3.2 for indifference prices of both investors, we first address the conditions that give the convergence of the equilibrium prices to d.

Proposition 5.3. *Let Assumptions 2.2, 4.1, 4.7, 5.2 hold, and impose Assumption 3.2 for function $p^n_{a^1_n}(q)$ and Assumption 4.5 for function $qp^\infty_1(q)$. If we further assume that $||\mathcal{E}^n_i||_{\mathbb{L}^\infty}/r^1_n \to 0$, for both $i = 1, 2$ and the sequence $\{a^2_n/a^1_n\}_{n\in\mathbb{N}}$ is bounded away from zero and infinity, the sequence of the partial equilibrium prices p^n_* of claim B converges to d.*

Proof. Let p^n_* denote an arbitrarily chosen convergent subsequence of the equilibrium prices of B with limit $\hat{p}$ (note that $B \in \mathbb{L}^\infty$ guarantees the existence of such subsequence) and assume that $\hat{p} \neq d$, and in particular $\hat{p} < d$.

Under Assumptions 4.7 and 5.2, it follows by Theorem 5.1 of [24] that the map $q \mapsto qp^n_{a^i_n}(q|\mathcal{E}^n_i)$ is strictly concave for each $i = 1, 2$, and also that

$$\mathbb{E}^{\mathbb{Q}^n_i(q)}[B] = \frac{\partial}{\partial q} qp^n_{a^i_n}(q|\mathcal{E}^n_i). \tag{5.8}$$

Now, that $\mathbb{E}^{\mathbb{Q}^n_1(0)}[B] \neq \mathbb{E}^{\mathbb{Q}^n_2(0)}[B]$ holds due to Assumption 5.2. Thus, first assume for some subsequence (still labeled n) that $\mathbb{E}^{\mathbb{Q}^n_1(0)}[B] > \mathbb{E}^{\mathbb{Q}^n_2(0)}[B]$, for sufficiently large n. Then $q^n_* > 0$ and in fact $\mathbb{E}^{\mathbb{Q}^n_1(0)}[B] > p^n_* > \mathbb{E}^{\mathbb{Q}^n_2(0)}[B]$. In view of Theorem 4.3 and Lemma 5.1, we have that the inequality $\hat{p} < d$ implies the existence of a further subsequence of q^n_* (still labeled n) such that $\lim_{n\to\infty} q^n_*/r^1_n = \ell > 0$. We reach then a contradiction if we show that for sufficiently large n, the position $-q^n_*$ is not optimal for investor

2. Since $\hat{p} < d$, we get from Assumption 3.2 that there exists $c > 0$ such that for any sufficiently large n, $p_*^n < \mathbb{E}^{\mathbb{Q}_0^n}[B] - c$. This implies that

$$0 \leq (-q_*^n)\left(p_{a_n^2}^n(-q_*^n|\mathcal{E}_2^n) - p_*^n\right) < (-q_*^n)\left(p_{a_n^2}^n(-q_*^n|\mathcal{E}_2^n) - \mathbb{E}^{\mathbb{Q}_0^n}[B] + c\right),$$

where the first inequality holds because the position $-q_*^n$ is optimal for investor 2 at price p_*^n, for each n. Using the relation (5.7) and the representation (2.7) we get that (recall definition (5.2))

$$\begin{aligned}
0 &< \frac{P_{a_n^2}^n(-q_*^n B + \mathcal{E}_2^n)}{q_*^n} - \frac{P_{a_n^2}^n(\mathcal{E}_2^n)}{q_*^n} + \mathbb{E}^{\mathbb{Q}_0^n}[B] - c \\
&= \inf_{\mathbb{Q}\in\tilde{\mathcal{M}}^n}\left\{\mathbb{E}^{\mathbb{Q}}\left[-B + \frac{\mathcal{E}_2^n}{q_*^n}\right] + \frac{1}{a_n^2 q_*^n}\left(H(\mathbb{Q}\mid\mathbb{P}) - H(\mathbb{Q}_0^n\mid\mathbb{P})\right)\right\} - \frac{P_{a_n^2}^n(\mathcal{E}_2^n)}{q_*^n} + \mathbb{E}^{\mathbb{Q}_0^n}[B] - c \\
&\leq \mathbb{E}^{\mathbb{Q}_0^n}\left[\frac{\mathcal{E}_2^n}{q_*^n}\right] - \frac{P_{a_n^2}^n(\mathcal{E}_2^n)}{q_*^n} - c \leq 2\frac{||\mathcal{E}_2^n||_{\mathbb{L}^\infty}}{q_*^n} - c = 2\frac{||\mathcal{E}_2^n||_{\mathbb{L}^\infty}}{r_n^1}\frac{r_n^1}{q_*^n} - c.
\end{aligned}$$

Since $||\mathcal{E}_2^n||_{\mathbb{L}^\infty}/r_n^1 \to 0$ and $r_n^1/q_*^n \to 1/\ell$ it follows that $c \leq 0$, a contradiction since $c > 0$. Similarly, when $\mathbb{E}^{\mathbb{Q}_1^n(0)}[B] < \mathbb{E}^{\mathbb{Q}_2^n(0)}[B]$, for sufficiently large n, then $q_*^n < 0$ and up to a subsequence $q_*^n/r_n^2 \to -\ell < 0$. In this case, we follow the same arguments to show that the position $-q_*^n$ could not be optimal for the investor 1 for sufficiently large n. Finally, the case where $\hat{p} > d$ is symmetric to the analysis above and hence omitted.

□

Withdrawing however the assumption $||\mathcal{E}_i^n||_{\mathbb{L}^\infty}/r_n \to 0$ could give the interesting cases where the equilibrium prices converge to a price different than the unique arbitrage free price of the limiting market and the regime of Theorems 4.3, 4.4 occurs. A family of such examples are presented in the following Proposition.

Proposition 5.4. *Let Assumptions 2.2, 4.1 and 4.7 hold. Impose also Assumption 3.2 for function $p_1^n(p)$ with constant risk aversion equal to 1 and Assumption 4.5 for the corresponding function $qp^\infty(q)$. If for each $n \in \mathbb{N}$ and $i = 1, 2$, $a_n^i \equiv a_i$ and $\mathcal{E}_i^n \equiv b_i^n B$, for some $a_i > 0$ and $b_i^n \in \mathbb{R}$, the following statements hold:*

i. *For each market $n \in \mathbb{N}$, the unique PEPQ pair (p_*^n, q_*^n) is given by $q_n^* = (a_2 b_2^n - a_1 b_1^n)/(a_1 + a_2)$ and $p_*^n = \mathbb{E}^{\mathbb{Q}^{-ab^n}}[B]$, with $1/a := 1/a_1 + 1/a_2$ and $b^n := b_1^n + b_2^n$.*
ii. *Letting for each $n \in \mathbb{N}$, $b_2^n = \kappa r_n$, for some $\kappa \in (0, \delta_+/a)$ and $b_1^n = b_1 \in \mathbb{R}$, we get that $\lim_{n\to\infty} q_*^n/r_n = \ell > 0$ and $p_*^n \to \hat{p} < d$.*

Proof. The proof of the first item i. is based on standard arguments of the related literature (see for example Theorem 3.2 in [5]). We recall that the equilibrium quantity is the solution of the optimization problem (5.4) and thanks to the strict concavity of the function $q \mapsto qp_{a_i}^n(q|\mathcal{E}_i^n)$ we get that for any $q \in \mathbb{R}$ and every $n \in \mathbb{N}$,

$$q\left(p_{a_1}^n(q|\mathcal{E}_1^n) + p_{a_2}^n(-q|\mathcal{E}_2^n)\right) \leq b^n p_a^n(b^n).$$

We then observe that in fact $b^n p_a^n(b^n) = q_*^n\left(p_{a_1}^n(q_*^n|\mathcal{E}_1^n) + p_{a_2}^n(-q_*^n|\mathcal{E}_2^n)\right)$, which means that q_*^n is indeed the equilibrium quantity. The fact that equilibrium price p_*^n equals to $\mathbb{E}^{\mathbb{Q}^{-ab^n}}[B]$ readily follows by (5.5).

For the second item, we have that $q_n^*/r_n = (a_2\kappa r_n - a_1 b_1)/(a_1+a_2) \to a_2\kappa/(a_1+a_2) > 0$. Since p_*^n is the equilibrium price for each n, we have that $p_*^n < p_1^n(q_*^n|\mathcal{E}_1^n)$, since q_*^n is optimal position for investor 1 at price p_*^n. Then by using the representation (5.7) as in the proof of Lemma 5.1, we get that

$$\lim_{n\to\infty} p_{a_1}^n(q_n^*|\mathcal{E}_1^n) = \lim_{n\to\infty} p_{a_1}^n(a_2\kappa r_n/(a_1+a_2)) = p^\infty(a\kappa).$$

Recall that $p_*^n = \mathbb{E}^{\mathbb{Q}^{-ab^n}}[B]$ and note that strict concavity of the function $q \mapsto qp_{a_1}^n(q|\mathcal{E}_1^n)$ and equation (5.8) give that p_*^n is decreasing in n and hence it has a limiting point $\hat{p}$. Thus, we have that $\lim_{n\to\infty} p_n^* = \hat{p} \le p^\infty(a\kappa) < p^\infty(0) = d$, where the last strict inequality follows by Assumption 4.5. □

Proposition 5.4 indicates that there are cases where the equilibrium quantity increases to infinity at the same time where the equilibrium price is different than the limiting arbitrage free price. It is important to point out here that both investors act optimally at that equilibrium prices even though the limiting price differs than d. The essential element is of course that one of the investor is endowed with a large position on the claim and she is willing to sell portion of her position at a price which induces the other investor acting optimally to enter to a large claim regime too. In other words, Proposition 5.4 justifies the large volume of some OTC derivative markets and the corresponding extreme prices as long as some of the participants in the market are already exposed to a risk that is highly correlated with the payoff of the tradeable derivatives. This situation fits to the observed extreme volumes and prices for example in the Mortgage Backed Securities market in the recent years.

Remark 5.5. The proof of Proposition 5.4 can easily be generalized in the case where the endowments are of the form $\mathcal{E}_i^n = b_i^n B + E_i^n$, with the choices of b_i^n as in the Proposition 5.4 and E_i^n being bounded random endowments such that $||E_i^n||_{\mathbb{L}^\infty}/r_n \to 0$.

6. Examples where the limiting scaled indifference price exist

The power of Assumption 3.2 is its validity in a wide variety of models. In this section we give four well studied market model examples. Then, in the next section we pay particular attention to an example with transactions costs. Remarkably, even though the standard duality results no longer apply, a version of Assumption 3.2 still holds and more importantly, so do the conclusions of Theorems 4.3 and 4.4.

6.1. **Vanishing Risk Aversion in a Fixed Market.**

As shown Section 3.1.1 for a fixed market, if the risk aversion vanishes (i.e. $a_n \to 0$) then Assumption 3.2 holds with $r_n = a_n^{-1}$ and $p^\infty(\ell) = p_1(\ell)$. In addition, as the class of acceptable trading strategies $\mathcal{A}$ is a cone it follows for any q_n that $\hat{\pi}_{a_n}(q_n) = (1/a_n)\hat{\pi}_1(a_n q_n)$. So, for $q_n = \ell r_n = \ell/a_n$, not only do indifference prices trivially converge, but the optimal trading strategy is explicitly known: i.e. it is $(1/a_n)\hat{\pi}_1(\ell) = r_n\hat{\pi}_1(\ell) = (q_n/\ell)\hat{\pi}_1(\ell)$. Note that in this instance the normalized optimal trading strategy trivially converges but does not necessarily provide a super hedge.

6.2. **Basis Risk Model with High Correlation.** This example is considered in detail in [12, 21, 33, 38] amongst others. Here, we have for each n one risky asset S^n which evolves according to

$$\begin{aligned}\frac{dS_t^n}{S_t^n} =&\mu(Y_t)dt + \sigma(Y_t)\left(\rho_n dW_t + \sqrt{1-\rho_n^2}d\tilde{W}_t\right),\\ dY_t =&b(Y_t)dt + a(Y_t)dW_t,\end{aligned}$$

where W and $\tilde{W}$ are two independent Brownian motions. The filtered probability space is the standard two-dimensional augmented Wiener space. The coefficients a, b have appropriate regularity and are such that Y has a unique strong solution taking values in an open subset E of $\mathbb{R}$. Set $\lambda := \mu/\sigma$ as the market price of risk and assume that $\sigma^2(y) > 0, y \in E$ and that λ is bounded on E. $B = B(Y_T)$ for some continuous bounded function B on E. As shown in [33, Section 5.3], $\underline{B}_n = \underline{B} = \inf_{y\in E} B(y)$ and $\bar{B}_n = \bar{B} = \sup_{y\in E} B(y)$ for all n. Set $r_n = (1-\rho_n^2)^{-1}$. As shown in [38] (see also [33]), for a fixed risk aversion $a > 0$ and $\ell \in \mathbb{R}, \ell \neq 0$:

$$p_a^n(\ell r_n) = -\frac{1}{a\ell}\log\left(\frac{\mathbb{E}\left[e^{-\rho_n\int_0^T\lambda(Y_t)dW_t - \frac{1}{2}\int_0^T\lambda^2(Y_t)dt - a\ell B(Y_T)}\right]}{\mathbb{E}\left[e^{-\rho_n\int_0^T\lambda(Y_t)dW_t - \frac{1}{2}\int_0^T\lambda^2(Y_t)dt}\right]}\right).$$

For $\ell = 0$ one has

$$d_n = p_a^n(0) = \mathbb{E}^{\mathbb{Q}_0^n}\left[B(Y_T)\right] = \frac{\mathbb{E}\left[e^{-\rho_n\int_0^T\lambda(Y_t)dW_t - \frac{1}{2}\int_0^T\lambda^2(Y_t)dt}B(Y_T)\right]}{\mathbb{E}\left[e^{-\rho_n\int_0^T\lambda(Y_t)dW_t - \frac{1}{2}\int_0^T\lambda^2(Y_t)dt}\right]}.$$

Thus, if $\rho_n \to 1$ (limit of high correlation) then $r_n \to \infty$ and

$$\begin{aligned}\lim_{n\uparrow\infty} p_a^n(\ell r_n) = p^\infty(\ell) = -\frac{1}{a\ell}\log\left(\mathbb{E}^{\mathbb{Q}}\left[e^{-a\ell B(Y_T)}\right]\right); \qquad \ell \neq 0;\\ \lim_{n\uparrow\infty} p_a^n(0) = p^\infty(0) = \mathbb{E}^{\mathbb{Q}}\left[B(Y_T)\right],\end{aligned}$$

where $\mathbb{Q}$ is the unique martingale measure in the $\rho = 1$ market where the filtration is restricted to $\mathbb{F}^W$. Furthermore, using l'Hopital's rule one obtains $\lim_{\ell\to 0} p^\infty(\ell) = \mathbb{E}^{\mathbb{Q}}\left[B(Y_T)\right] = p^\infty(0)$ so that Assumption 3.2 is satisfied with $\delta = \infty$.

6.3. **Large Markets with Vanishing Trading Restrictions.** The next example is simplified version of the general semi-complete setup considered in [34]. Here, $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to support a sequence of independent Brownian motions $W^1, W^2, ...$ The filtration is the augmented version of $\mathbb{F}^{W^1,W^2,\cdots}$. There is a sequence of (potentially tradeable) assets $S^1, S^2, ...$ with dynamics

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sum_{j=1}^i \sigma^{ij} dW_t^j; \qquad i = 1, 2, 3, ...,$$

where $\mu = (\mu^1, \mu^2, ...)$ satisfies $\sum_{i=1}^\infty (\mu^i)^2 < \infty$ and σ is the lower triangular square root of the symmetric matrix $\Sigma = \left\{\Sigma^{ij}\right\}_{i,j=1,2,...}$, assumed positive definite so that for some $\lambda > 0$ and all $\xi = (\xi^1, \xi^2, ...)$ with $\sum_{i=1}^\infty (\xi^i)^2 < \infty$, we have $\xi'\Sigma\xi \geq \lambda\xi'\xi$.

The claim (as is typical in life insurance markets) is given as the sum of independent, $\mathbb{F}^{W^i}$ adapted claims B^i: $B = \sum_{i=1}^{\infty} B^i$. To make B well defined and amenable to large claim analysis we assume $\mathbb{E}\left[e^{\lambda B^i}\right] < \infty, i = 1, 2, ...$ and $\sum_{i=1}^{\infty} \log\left(\mathbb{E}\left[e^{\lambda B^i}\right]\right) < \infty$ for all $\lambda \in \mathbb{R}$.

For $n = 1, 2, ...$ we construct the nth market by restricting trading to the first n assets. Thus, as $n \uparrow \infty$ the claim is asymptotically hedgeable, though for each n the market is incomplete. As shown in [34], $\underline{B}_n = d^n + \text{ess inf}_{\mathbb{P}}[Y_n]$ and $\bar{B}_n = d^n + \text{ess sup}_{\mathbb{P}}[Y_n]$ where d^n is the unique replicating capital for $\sum_{i=1}^{n} B^i$ and $Y_n := \sum_{i=n+1}^{\infty} B^i$. Under Assumption 3.2, $d^n \to d = \mathbb{E}^{\mathbb{Q}_0}[B]$ where $\mathbb{Q}_0$ is the unique martingale measure in the limiting complete market.

Since $\sum_{i=1}^{\infty} \log\left(\mathbb{E}\left[e^{\lambda B^i}\right]\right) < \infty$ for all $\lambda \in \mathbb{R}$, we know that $\lim_{n\uparrow\infty} \mathbb{E}\left[Y_n^2\right] = 0$. Assume furthermore that Y_n is converging to 0 sufficiently fast so that it satisfies a LDP with scaling $r_n \to \infty$ and good rate function I such that $\{I = 0\} = \{0\}$. Lastly, assume that for some $\delta > 0$, $|\lambda| < \delta$ implies

$$\limsup_{n\uparrow\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} \log\left(\mathbb{E}\left[e^{\lambda r_n B^i}\right]\right) < \infty. \tag{6.1}$$

For example, this will hold if $B^i \sim N(0, \delta_i^2)$, with $\sum_{i=1}^{\infty} \delta_i^2 < \infty$. Fix the risk aversion $a_n = a > 0$. As shown in [34], at $\ell = 0$ we have $\lim_{n\uparrow\infty} p_a^n(0) = d = p^{\infty}(0)$. Furthermore, for $0 < |\ell| < \delta/a$

$$\lim_{n\uparrow\infty} p_a^n(\ell r_n) = p^{\infty}(\ell) = d - \frac{1}{a\ell} \sup_{y\in\mathbb{R}}(-\ell a y - I(y)).$$

Additionally, as can be deduced from $I(y) = 0 \leftrightarrow y = 0$, (6.1) and the lower-semicontinuity of I, it follows that

$$\lim_{\ell\to 0} \frac{1}{a\ell} \sup_{y\in\mathbb{R}}(-\ell a y - I(y)) = 0,$$

so that $p^{\infty}(\ell) \to d = p^{\infty}(0)$ as $\ell \to 0$. Thus, Assumption 3.2 holds. Lastly, it is also shown in [34] that for all $q \in \mathbb{R}$ the normalized residual risk process $\hat{Y}_a^n(q)$ of (2.10) is precisely Y_n and, as such, does not depend upon q.

6.4. **Black-Scholes-Merton Model with Vanishing Default Probability.** This example is taken from [25] and the setup is similar to that considered in [29]. Here, we consider the Black-Scholes-Merton model, except that the stock may default at the first jump time of an independent Poisson process. The claim is a defaultable bond paying 1 if the stock has not defaulted by time T. The owner of the bond wishes to hedge the claim by trading in S^n, but needs to take into account the event of default, since the stock is stuck at 0 after default occurs.

Fix n and let $\lambda_n > 0$. For each n, the probability space is assumed to support a Brownian motion W as well as an independent Poisson process N^n with intensity λ_n. Denote by $\tilde{N}^n$ the compensated Poisson process so that $\tilde{N}_t^n = N_t^n - \lambda_n(\tau_n \wedge t)$, where $\tau_n = \inf\{t \geq 0 : N^n = 1\}$. The filtration is that generated by N^n and W, augmented so that it satisfies the usual conditions. The (single) risky asset S^n evolves

according to

$$\begin{aligned}\frac{dS_t^n}{S_{t-}^n} &= 1_{t\le\tau_n}\left(\mu dt + \sigma dW_t\right) - dN_t^n,\\ &= 1_{t\le\tau_n}\left((\mu+\lambda_n)dt + \sigma dW_t - d\tilde{N}_t^n\right).\end{aligned}$$

The claim is a defaultable bond which pays 1 if S^n defaults before T: i.e. $B = 1_{\tau^n\le T}$[‡]. Here, $\underline{B}_n = 0$ and $\bar{B}_n = 1$, since this is because we can equivalently change the default intensity to take any positive value. Thus, Assumption 4.1 holds even though $d = 1$ and hence $d \notin I^n$ for all n.

As shown in [25], $u_a^n(0,q) = -\frac{1}{a}F^n(0;q)$ where $F^n(\cdot;q)$ solves the ODE

$$\begin{aligned}&\dot{F}^n(t;q) - \lambda F^n(t;q) - \frac{\mu^2}{2\sigma^2}F^n(t;q) + \min_\phi\left(\frac{1}{2}\sigma^2\phi^2F^n(t;q) + \lambda_n e^{\frac{\mu}{\sigma^2}-\phi}\right) = 0; \qquad t\le T,\\ &F^n(T;q) = e^{-aq}.\end{aligned}$$

It is easy to see that the optimal $\hat{\phi}^n$ in the above minimization satisfies $\hat{\phi}^n(t;q)e^{\hat{\phi}^n(t;q)} = \lambda(F^n(t;q))^{-1}e^{\frac{\mu}{\sigma^2}}$, where one can show that $F^n(t;q) > 0$. Now, let $\lambda_n \downarrow 0$ (vanishing default probabilities) and set $r_n = -\log(\lambda_n)$. With $q_n = \ell r_n$, one can show that for $\ell < 1/a$:

$$\lim_{n\uparrow\infty} p_a^n(\ell r_n) = \lim_{n\uparrow\infty} -\frac{1}{\ell a r_n}\log\left(\frac{F^n(0;\ell r_n)}{F^n(0;0)}\right) = p_a^\infty(\ell) = 1.$$

Since

$$\lim_{\ell\to 0} p_a^\infty(\ell) = 1 = \lim_{n\uparrow\infty} p_a^n(0),$$

we see that Assumption 3.2 is satisfied, though the map $\ell \mapsto \ell p^\infty(\ell) = \ell$ is not strictly concave.

7. Vanishing Transaction Costs in the Black-Scholes-Merton Model

In this section we show that the existence of limiting indifference prices and the resultant statements about optimal position taking even extend to models with frictions, where the standard duality results used in Section 2 are not as fully developed (see [11] for a recent treatment of the topic). As such, this example is given its own section.

We consider the Black-Scholes-Merton model with proportional transactions costs, as studied in [4, 6, 10, 13, 20, 23, 27, 28, 35] amongst many others. We take the approach of [10] and especially [4, 23]. Using the notation of [4], the stock S evolves according to a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t; \qquad t\le T. \tag{7.1}$$

Here, the filtered probability space is the standard one-dimensional Wiener space. Now, fix a time $t \le T$ and $s > 0$ and assume $S_t = s$. Denote by X and Y respectively the processes of dollar holdings in the money market and shares of stock owned associated to a trading strategy L, M where $L_t = M_t = 0$ and L represents the cumulative transfers (in shares of stock) from the money market to the stock and M represents

[‡]As the claim depends upon n here it does not fit precisely into the setup of Section 2. However, as inspection of the Propositions in Appendix C shows, the results of Theorems 4.3, 4.4 readily extend to a sequence of claims B_n if they are uniformly bounded.

the cumulative transfers from the stock to the money market. We denote by $\mathcal{A}_t$ the set of (L, M) where L, M are adapted, non-decreasing and left-continuous with $L_t = M_t = 0$.

There is a proportional transaction cost $\lambda \in (0,1)$ by trading. In other words, for a given initial position (x, y) where $x \in \mathbb{R}$ is the initial capital and $y \in \mathbb{R}$ the initial shares held in S the corresponding processes evolve according to

$$\begin{aligned} X_\tau &= X_\tau^{L,M,x,t} = x - \int_t^\tau S_u(1+\lambda)dL_u + \int_t^\tau S_u(1-\lambda)dM_u; \qquad t \le \tau \le T, \\ Y_\tau &= Y_\tau^{L,M,y,t} = y + L_\tau - M_\tau; \qquad t \le \tau \le T. \end{aligned} \tag{7.2}$$

The claim B is a European call option on S: i.e. $B = (S_T - K)^+$, and that the investor is considering selling the call. For an exponential investor with fixed risk aversion $a > 0$ the value function without the claim is given by

$$u_a(x, y; s, t, \lambda) = \sup_{L,M \in \mathcal{A}_t} \mathbb{E}_{s,t}\left[U_a(X_T + Y_T S_T)\right]. \tag{7.3}$$

Here, $\mathbb{E}_{s,t}[\cdot]$ refers to conditioning on time t given $S_t = s$. The value function for q units of the call is

$$u_a(x, y, q; s, t, \lambda) = \sup_{L,M \in \mathcal{A}_t} \mathbb{E}_{s,t}\left[U_a(X_T + Y_T S_T - q(S_T - K)^+)\right]. \tag{7.4}$$

The indifference price $p_a(x, y, q; s, t, \lambda)$ is then defined through the balance equation

$$u_a(x + qp_a(x, y, q; s, t, \lambda), y, q; s, t, \lambda) = u_a(x, y; s, t, \lambda). \tag{7.5}$$

Remark 7.1. $p_a(x, y, q; s, t, \lambda)$ is thus the average *ask* indifference price, as opposed to the average *bid* indifference price defined in Section 2. However, using the notation of Section 2 for a general claim B, the bid and ask prices are related by $p_a^{\text{ask}}(q; B) = -p_a^{\text{bid}}(q; -B)$.

Though the results in [4] are stated in the joint limit of vanishing transactions costs (i.e. $\lambda_n \to 0$) and infinite risk aversion (i.e. $a = a_n \to \infty$), they easily (as the authors therein mention) translate into asymptotics in the joint limit that $\lambda_n \to 0$ and $q = q_n \to \infty$ for a fixed risk aversion a. This translation is made precise in the following proposition.

Proposition 7.2. *Fix $s > 0, 0 \le t \le T$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $\lambda \in (0,1)$ and $a > 0$. The indifference price p_a is independent of x and hence write $p_a = p_a(y, q; s, t, \lambda)$. Now, let $\lambda_n \to 0$ and set $r_n := \lambda_n^{-2}$. For $\ell > 0$ and $q_n = \ell r_n = \ell \lambda_n^{-2}$ we have for all y_n such that $\lim_{n\uparrow\infty} \lambda_n^3 |y_n| = 0$:*

$$\lim_{n\uparrow\infty} p_a(y_n, q_n; s, t, \lambda_n) = p_a^\infty(\ell; s, t) := \Psi(s, t; \sqrt{a\ell}),$$

where for $b > 0$, $\Psi(; b) : (0, \infty) \times [0, T] \mapsto \mathbb{R}$ is the unique continuous viscosity solution to the non-linear Black-Scholes PDE

$$\begin{aligned} \Psi_t + \frac{1}{2}\sigma^2 s^2 \Psi_{ss}\left(1 + S(b^2 s^2 \Psi_{ss})\right) &= 0; \qquad (s,t) \in (0,\infty) \times (0,T); \\ \Psi(s, T) &= (s - K)^+; \qquad s \in (0, \infty); \\ \lim_{s\uparrow\infty} \frac{\Psi(s,t)}{s} &= 1; \qquad t \le T \text{ uniformly in } t. \end{aligned} \tag{7.6}$$

Here, $S : \mathbb{R} \mapsto (-1, \infty)$ *satisfies*

$$\dot{S}(A) = \frac{1 + S(A)}{2\sqrt{AS(A)} - A}; \quad S(0) = 0; \quad \lim_{A \downarrow -\infty} S(A) = -1; \quad \lim_{A \uparrow \infty} S(A)/A = 1.$$

Remark 7.3. The above result allows for y_n to vary since intuitively a position size of q_n in the call would be associated to an initial position of $q_n y$ in the stock for some $y \in \mathbb{R}$. Note that for $y_n = q_n y = \ell y \lambda_n^{-2}$ we have $\lambda_n^3 |y_n| \to 0$.

To obtain the optimal position taking results analogous to Theorems 4.3 and 4.4, it is first necessary to identify the range of limiting prices $p_a^\infty(\ell; s, t)$ in Proposition 7.2 as ℓ varies between 0 and ∞. In other words, we must consider asymptotics for $\Psi(; b)$ for small and large b.

As $b \downarrow 0$, Theorem 7.4 below proves continuity in that $\Psi(s, t; b) \to \Psi(s, t; 0)$. But, for $b = 0$, (7.6) is just the regular Black-Scholes PDE which admits a unique (explicit) classical solution. Thus, as $\ell \downarrow 0$, the limiting indifference price converges to the unique price in complete, $\lambda_n = 0$ market given $S_t = s$.

Theorem 7.4. *Let* $\Psi(; b) : (0, \infty) \times [0, T] \mapsto \mathbb{R}$ *be the unique, continuous, viscosity solution to the non-linear Black-Scholes PDE equation (7.6). Then as* $b \to 0$*, we have locally uniformly that* $\Psi(; b) \to \Psi(; 0)$*, where* $\Psi(; 0)$ *is the unique continuous solution to the linear Black-Scholes PDE.*

Next, we identify the limit of $\Psi(; b)$ as $b \uparrow \infty$. Here, we are guided by the intuition that, thought of as a function of the stock volatility, the Black-Scholes price for a call option converges to the initial price as the volatility becomes large. In fact, a similar phenomenon occurs here as $b \uparrow \infty$, as the following shows:

Theorem 7.5. *For fixed* $s > 0, 0 \le t \le T$ *the map* $b \mapsto \Psi(s, t; b)$ *is increasing with*

$$\lim_{b \uparrow \infty} \Psi(s, t; b) = \begin{cases} (s - K)^+ & t = T \\ s & 0 \le t < T \end{cases}. \tag{7.7}$$

Remark 7.6. An inspection of the proof of Theorem 7.4 below shows that $\Psi(s, t; b)$ is continuously increasing in b. Thus, if $q_n = \ell_n r_n$ where $\ell_n \to \ell \ge 0$ then the indifference prices converge to $\Psi(s, t; \sqrt{a\ell})$.

With the above asymptotics for $p_a^\infty(\ell; s, t)$ in place, we now consider the optimal sale quantity problem in the nth market with transactions cost λ_n. In order to simplify the presentation, we assume that given $S_t = s$ the investor has the opportunity to sell call options at a price $\tilde{p}^n$ in the nth market. To finance this sale, the investor cashes out her initial position in the stock, receiving $ys(1 - \lambda_n)$ for the sale of y shares. Then, with $x + ys(1 - \lambda_n)$ in cash, she identifies the optimal number of options to sell by solving the problem

$$\sup_{q > 0} u_a(x + ys(1 - \lambda_n) + q\tilde{p}^n, 0, q; s, t, \lambda_n). \tag{7.8}$$

In the frictionless case, if $\tilde{p}^n$ is arbitrage free in the nth market, then (see [24]), an optimal $\hat{q}_n$ exists and is unique. When considering transactions costs, rather than identifying the arbitrage free prices in each

market, we use the small and large ℓ asymptotics for $p_a^\infty(\ell;s,t)$ obtained in Theorems 7.4, 7.5 to identify a maximal range of reasonable prices $\tilde{p}^n$ for which one can sell the option. Indeed, from the above theorems

$$\lim_{\ell\downarrow 0} p_a^\infty(\ell;s,t) = \Psi(s,t;0); \qquad \lim_{\ell\uparrow\infty} p_a^\infty(\ell;s,t) = s.$$

It is well known that $\Psi(s,t;0) < s$. Furthermore, if one is going to sell options, the effect of the transactions costs is that the ask price should a) be at least as large as $\Psi(s,t;0)$ and b) be no higher than p since no-one would buy at this price[§]. Thus, the only range of reasonable prices to sell at is $(\Psi(s,t;0),s)$. With this motivation we have:

Theorem 7.7. *Let $\tilde{p}^n \in (\Psi(s,t;0),s)$ for each n with $\tilde{p}^n \to \tilde{p}$ where $\tilde{p} \in (\Psi(s,t;0),s)$. Let $\lambda_n \to 0$. For each n there exists a maximizer $\hat{q}_n > 0$ to (7.8). Additionally, for any sequence $\{\hat{q}_n\}_{n\in\mathbb{N}}$ of maximizers:*

$$\liminf_{n\uparrow\infty} \frac{\hat{q}_n}{r_n} > 0; \qquad \limsup_{n\uparrow\infty} \frac{\hat{q}_n}{r_n} < \infty. \tag{7.9}$$

Thus, up to subsequences, $\hat{q}_n/r_n \to \ell$ and hence for any sequence y_n such that $\lambda_n^3|y_n| \to 0$:

$$\lim_{n\uparrow\infty} p_a(y_n,\hat{q}_n;s,t,\lambda_n) = p_a^\infty(\ell;s,t) = \Psi(s,t;\sqrt{a\ell}).$$

Appendix A. Proofs for Section 4.1

The proofs of Theorems 4.3 and 4.4 are based on a more general result that we prove in Appendix C. Hence, as a precursor to the proofs of Theorem 4.3 and 4.4 we first show that the functions $p^n(q) := p_{a_n}^n(q)$ satisfy Assumption C.5 below.

Lemma A.1. *Let Assumptions 2.1, 2.2, 3.2 and 4.1 hold. Then, $p^n(q) := p_{a_n}^n(q)$ satisfies Assumption C.5.*

Proof of Lemma A.1. As shown in Section 3.1, $p_{a_n}^n(q)$ is decreasing in q and the map $q \mapsto qp_{a_n}^n(q)$ is concave and well defined, finite, for all $q \in \mathbb{R}$. As such, $p_{a_n}^n(q)$ is continuous on $(-\infty,0)$ and $(0,\infty)$ respectively. But, it is well known that continuity at 0 follows as well and in fact $\lim_{q\to 0} p_{a_n}^n(q) = \mathbb{E}^{\mathbb{Q}_0^n}[B] = p_{a_n}^n(0) = d_n$. Thus, bullet point one in Assumption C.5 holds. Regarding bullet point two, let $\gamma > 0$. If $0 < q \le \gamma$ then for any $0 < \ell < \delta^+$ and n sufficiently large so that $r_n \ge \ell/\gamma$:

$$p_{a_n}^n(q) \le p_{a_n}^n(0) = d_n = \mathbb{E}^{\mathbb{Q}_0^n}[B]; \qquad p_{a_n}^n(q) \ge p_{a_n}^n(\ell r_n).$$

If $-\gamma \le q < 0$ then for any $\delta_- < \ell' < 0$ and n so that $r_n \ge -\ell'/\gamma$:

$$p_{a_n}^n(q) \ge p_{a_n}^n(0) = d_n\mathbb{E}^{\mathbb{Q}_0^n}[B]; \qquad p_{a_n}^n(q) \le p_{a_n}^n(\ell' r_n).$$

As such:

$$\limsup_{n\uparrow\infty} \sup_{|q|\le\gamma} q|p_{a_n}^n(q)| \le \gamma \max\left\{|d|, |p^\infty(\ell)|, |p^\infty(\ell')|\right\} = C(\gamma),$$

and bullet point two holds. Bullet points three and four are Assumption 3.2, finishing the result. □

[§]Technically: no one would buy at a price at or above $p(1+\lambda_n)$ because it would then be preferable to buy the stock and not trade. For this to hold as $\lambda_n \downarrow 0$, we require $\tilde{p}^n \le p$. Our results are valid for $\tilde{p}^n < p$.

Proof of Theorem 4.3. For $\tilde{p}^n \in I^n$, the optimal position $\hat{q}_n(\tilde{p}^n)$ is the unique solution of the problem (4.3). Using the explicit formula for U_{a_n} in (2.1) and $p^n_{a_n}$ in (2.6), this optimization problem is equivalent to finding

$$\hat{q}_n(\tilde{p}^n) \in \text{argmin}_{q\in\mathbb{R}} \left(q\tilde{p}^n - qp^n_{a_n}(q)\right). \tag{A.1}$$

The results of the theorem will follow from Proposition C.6 below once the requisite hypotheses are met where $p^n(q) = p^n_{a_n}(q)$. By Lemma A.1, Assumption C.5 holds. Now, let $\tilde{p}^n \in I^n$, $\tilde{p}^n \to \tilde{p}$ where $\tilde{p}$ and $\tilde{p} < d$. Since $p^n(\infty) \leq \tilde{p}^n$ and $d = p^\infty(0)$ we have

$$\limsup_{n\uparrow\infty} p^n(\infty) = \limsup_{n\uparrow\infty} \underline{B}_n \leq \lim_{n\uparrow\infty} \tilde{p}^n = \tilde{p} < d = p^\infty(0).$$

Thus, the conclusions of the theorem follow from Proposition C.6. Similarly let $\tilde{p}^n \in I^n$, $\tilde{p}^n \to \tilde{p}$ where $\tilde{p}$ and $\tilde{p} > d$. Since $p^n(-\infty) \geq \tilde{p}^n$ and $d = p^\infty(0)$ we have

$$\liminf_{n\uparrow\infty} p^n(-\infty) = \liminf_{n\uparrow\infty} \bar{B}_n \geq \lim_{n\uparrow\infty} \tilde{p}^n = \tilde{p} > d = p^\infty(0).$$

Thus, the conclusions of the theorem follow from Proposition C.6 as well, finishing the result.

□

Proof of Theorem 4.4. As in the proof of Theorem 4.3, it is enough to show that requisite hypotheses of Proposition C.6 are met where $p^n(q) = p^n_{a_n}(q)$ and the optimal position $\hat{q}_n(\tilde{p}^n)$ is given in (A.1). Again by Lemma A.1, we have that Assumption C.5 holds. Now, let $\tilde{p}^n \in I^n$, $\tilde{p}^n \to \tilde{p}$ where $\tilde{p}$ and $p^\infty(\delta^+) < \tilde{p} < d$. Since $p^n(\infty) \leq \tilde{p}^n$ and $d = p^\infty(0)$ we have

$$\limsup_{n\uparrow\infty} p^n(\infty) = \limsup_{n\uparrow\infty} \underline{B}_n \leq \lim_{n\uparrow\infty} \tilde{p}^n = \tilde{p} < d = p^\infty(0).$$

Thus, the conclusions of the theorem follow from Proposition C.6. Similarly let $\tilde{p}^n \in I^n$, $\tilde{p}^n \to \tilde{p}$ where $\tilde{p}$ and $p^\infty(\delta_-) > \tilde{p} > d$. Since $p^n(-\infty) \geq \tilde{p}^n$ and $d = p^\infty(0)$ we have

$$\liminf_{n\uparrow\infty} p^n(-\infty) = \liminf_{n\uparrow\infty} \bar{B}_n \geq \lim_{n\uparrow\infty} \tilde{p}^n = \tilde{p} > d = p^\infty(0).$$

Thus, the conclusions of the theorem follow from Proposition C.6 as well, finishing the result. □

Proof of Corollary 4.6. Let, for example, $\tilde{p}^n \to \tilde{p} \in (p^\infty(\delta^+), d)$ so that

$$0 < \underline{\ell} = \liminf_{n\uparrow\infty} \frac{\hat{q}_n(\tilde{p})}{r_n} \leq \limsup_{n\uparrow\infty} \frac{\hat{q}_n(\tilde{p}^n)}{r_n} = \bar{\ell} < \delta^+.$$

Write $\hat{q}_n$ for $\hat{q}_n(\tilde{p}^n)$ and assume for some subsequence (still labeled n) that $\hat{q}_n/r_n \to \ell \in [\underline{\ell}, \bar{\ell}]$. Let $\tau \in [\underline{\ell}, \bar{\ell}]$. By the optimality of $\hat{q}_n$

$$\hat{q}_n\tilde{p}^n - \hat{q}_n p^n_{a_n}(\hat{q}_n) \leq \tau r_n \tilde{p}^n - \tau r_n p^n_{a_n}(\tau r_n).$$

Dividing by r_n, letting $n \uparrow \infty$ and using Assumption 3.2 with (3.7) one obtains

$$\ell\tilde{p} - \ell p^\infty(\ell) \leq \tau\tilde{p} - \tau p^\infty(\tau).$$

Since this works for all $\tau \in [\underline{\ell}, \bar{\ell}]$, we get that

$$\ell\tilde{p} - \ell p^{\infty}(\ell) \leq \inf_{\tau \in [\underline{\ell}, \bar{\ell}]} \left(\tau\tilde{p} - \tau p^{\infty}(\tau)\right).$$

Hence, we see that the only possible limit points for $\hat{q}_n/r_n$ are the minimizers of $\ell\tilde{p} - \ell p^{\infty}(\ell)$ over $[\underline{\ell}, \bar{\ell}]$. But, under the assumption of strict concavity for $\ell p^{\infty}(\ell)$ any minimizer is unique and hence the result follows. □

Proof of Theorem 4.10. To show maximizers exist to the optimal purchase quantity problem in (4.20) we use the following basic result (see [17, Proposition 2.47]): if $U \in \mathcal{U}_a$ then with $\underline{\alpha}_U$, $\bar{\alpha}_U$ of (4.16) it holds for U_a from (2.1) with $a_n \equiv a$ that

$$U(x) = F(U_{\underline{a}_U}(x)); \qquad F(t) = U(U_{\underline{a}_U}^{-1}(t)) = U\left(-\frac{1}{\underline{a}_U}\log\left(-\underline{a}_U t\right)\right);$$
$$U_{\bar{a}_U}(x) = \hat{F}(U(x)); \qquad \hat{F}(t) = U_{\bar{a}_U}(U^{-1}(t)) = -\frac{1}{\bar{a}_U}e^{-\bar{a}_U U^{-1}(t)},$$

and where $F, \hat{F}$ are concave and increasing. Thus, by Jensen's inequality, for any set of random variables $\mathcal{Z}$:

$$\hat{F}^{-1}\left(\sup_{Z\in\mathcal{Z}} \mathbb{E}\left[U_{\bar{a}_U}(Z)\right]\right) \leq \sup_{Z\in\mathcal{Z}} \mathbb{E}\left[U(Z)\right] \leq F\left(\sup_{Z\in\mathcal{Z}} \mathbb{E}\left[U_{\underline{a}_U}(Z)\right]\right),$$

where $\hat{F}^{-1}(s) = U\left(-(1/\bar{a}_U)\log\left(-\bar{a}_U s\right)\right)$ is strictly increasing. Therefore,

$$U\left(-\frac{1}{\bar{a}_U}\log\left(-\bar{a}_U u^n_{\bar{a}_U}(x - q\tilde{p}^n, q)\right)\right) \leq u^n_U(x - q\tilde{p}^n, q) \leq U\left(-\frac{1}{\underline{a}_U}\log\left(-\underline{a}_U u^n_{\underline{a}_U}(x - q\tilde{p}^n, q)\right)\right).$$

Since for any $a > 0$, $u^n_a(x - \tilde{p}^n q, q) = e^{-a(x-\tilde{p}^n q)}u^n_a(0, q)$, we obtain from (2.6) that

$$\begin{aligned} U\left(-\frac{1}{\bar{a}_U}\log(-\bar{a}_U u^n_{\bar{a}_u}(0)) + x - \tilde{p}^n q + q p^n_{\bar{a}_U}(q)\right) &\leq u^n_U(x - \tilde{p}^n q, q) \\ &\leq U\left(-\frac{1}{\underline{a}_U}\log(-\underline{a}_U u^n_{\underline{a}_U}(0)) + x - \tilde{p}^n q + q p^n_{\underline{a}_U}(q)\right). \end{aligned} \tag{A.2}$$

Now, let $\tilde{p}^n \in I^n = (\underline{B}_n, \bar{B}_n)$. As $\lim_{q\uparrow\infty} p^n_{\underline{a}_U}(q) = \underline{B}_n$, $\lim_{q\downarrow-\infty} p^n_{\underline{a}_U}(q) = \bar{B}_n$ we have

$$\lim_{|q|\uparrow\infty} q(p^n_{\underline{a}_U}(q) - \tilde{p}^n) = -\infty,$$

and hence from the second inequality in (A.2) and $\lim_{x\downarrow-\infty} U(x) = -\infty$ (which follows from (4.17)) we obtain

$$\lim_{q\uparrow\infty} u^n_U(x - \tilde{p}^n q, q) = -\infty, \lim_{q\downarrow-\infty} u^n_U(x - \tilde{p}^n q, q) = -\infty.$$

As $U(x - \tilde{p}^n q - |q|\|B\|_{\mathbb{L}^\infty}) \leq u^n_U(x - \tilde{p}^n q, q) \leq 0$, any maximizing sequence $\{q^n_m\}_{m\in\mathbb{N}}$ must be bounded and has an accumulation point $\hat{q}_n$. Now, $u^n_U(x - \tilde{p}^n q, q)$ admits the variational representation (see [32])

$$u^n_U(x - \tilde{p}^n q, q) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n, y>0}\left(y(x - \tilde{p}^n q) + yq\mathbb{E}^{\mathbb{Q}^n}[B] + \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}}\Big|_{\mathcal{F}_T}\right)\right]\right), \tag{A.3}$$

where

$$V(y) := \sup_{x\in\mathbb{R}} \left(U(x) - xy\right). \tag{A.4}$$

Thus, we see that $q \mapsto u_U^n(x - \tilde{p}^n q, q)$ is concave, hence continuous on $\mathbb{R}$ and $\hat{q}_n$ is indeed a maximizer.

We next show for $p^\infty(\delta^+) < \tilde{p} < d$ and $I^n \ni \tilde{p}^n \to \tilde{p}$ that (4.21) holds (the corresponding proof for negative positions in (4.22) is omitted as it is the exact same). We first claim that for n large enough, any maximizer $\hat{q}_n$ is positive. Indeed, since $d_n \to d$ where $d_n = \mathbb{E}^{\mathbb{Q}_0^n}[B] = p_a^n(0)$ (for any $a > 0$) and $\tilde{p} < d$, $\tilde{p}^n \to \tilde{p}$ we can find n large enough so that $\tilde{p}^n < d_n$. Thus, for $q < 0$ we have (since $p_a^n(q)$ is decreasing in q for any $a > 0$) that

$$qp_{\underline{a}_U}^n(q) - q\tilde{p}^n \le q\left(d_n - \tilde{p}^n\right) \le 0.$$

In view of (A.2) this implies for $q \le 0$ that

$$u_U^n(x - \tilde{p}^n q, q) \le U\left(-\frac{1}{\underline{a}_U}\log\left(-\underline{a}_U u_{\underline{a}_U}^n(0)\right) + x\right). \tag{A.5}$$

Now, let $\ell > 0$ be so that $\ell\bar{a}_U/a < \delta^+$. At $q = \ell r_n$ we have

$$p_{\bar{a}_U}^n(\ell r_n) - \tilde{p}^n = p_a^n(\bar{a}_U\ell/ar_n) - \tilde{p}^n \to p^\infty(\bar{a}_U\ell/a) - \tilde{p}.$$

Since $\tilde{p} < p^\infty(0)$ and p^∞ is continuous at 0 we can find an ℓ small enough so the above quantity is strictly positive for n large. Thus, from (A.2) we see that

$$u_U^n(x - \tilde{p}^n\ell r_n, \ell r_n) \ge U\left(-\frac{1}{\bar{a}_U}\log(-\bar{a}_U u_{\bar{a}_u}^n(0)) + x - \tilde{p}^n\ell r_n + \ell r_n p_{\bar{a}_U}^n(\ell r_n)\right).$$

As $n \uparrow \infty$ the right hand side above converges to 0 whereas the right hand side of (A.5), in view of Assumption 4.9 is bounded above by $U(C + x) < 0$ for some constant C. Thus, for large enough n, no maximizer can be non-positive.

Now, let $\{\hat{q}_n\}_{n\in\mathbb{N}}$ be a sequence of (positive) maximizers. We prove the lower bound in (4.21) by contradiction; i.e. assume $\liminf_{n\uparrow\infty} \hat{q}_n/r_n = 0$ and take a sequence (still labeled n) where $\hat{q}_n/r_n \to 0$. Let $0 < \ell < \delta^+\bar{a}_U/a$ and assume $\hat{q}_n/r_n \le \ell$. Since $\hat{q}_n$ was an optimizer, we obtain from (A.2) that

$$-\frac{1}{\bar{a}_U}\log(-\bar{a}_U u_{\bar{a}_U}^n(0)) + x - \tilde{p}^n\ell r_n + \ell r_n p_{\bar{a}_U}^n(\ell r_n) \le -\frac{1}{\underline{a}_U}\log(-\underline{a}_U u_{\underline{a}_U}^n(0)) + x - \tilde{p}^n\hat{q}_n + \hat{q}_n p_{\underline{a}_U}^n(\hat{q}_n).$$

Since $\ell r_n > 0$

$$-\frac{1}{\ell r_n\bar{a}_U}\log(-\bar{a}_U u_{\bar{a}_U}^n(0)) + \frac{x}{\ell r_n} - \tilde{p}^n + p_{\bar{a}_U}^n(\ell r_n) \le -\frac{1}{\ell r_n\underline{a}_U}\log(-\underline{a}_U u_{\underline{a}_U}^n(0)) + \frac{x}{\ell r_n} + \frac{\hat{q}_n}{\ell r_n}\left(p_{\underline{a}_U}^n(\hat{q}_n) - \tilde{p}^n\right).$$

For any $a > 0$, $-(1/a) \le u_a^n(0) = -(1/a)e^{-H\left(\mathbb{Q}_0^n \mid \mathbb{P}\right)}$. Additionally, from (2.7) it holds for any $a, b > 0$ that $p_a^n(q) = p_b^n(aq/b)$. Thus by Assumptions 3.2 and 4.9

$$p^\infty\left(\frac{\bar{a}_U\ell}{a}\right) - \tilde{p} \le \liminf_{n\uparrow\infty} \frac{\hat{q}_n}{\ell r_n}\left(p_{\underline{a}_u}^n(\hat{q}_n) - \tilde{p}^n\right) = 0,$$

where the last equality follows since $\hat{q}_n/r_n \to 0$, $\tilde{p}^n \to \tilde{p}$ and $|p_{\underline{a}_U}^n(q)| \le \|B\|_{\mathbb{L}^\infty}$. Taking $\ell \downarrow 0$ gives $\tilde{p} \ge p^\infty(0)$ a contradiction. Therefore, $\liminf_{n\uparrow\infty} \hat{q}_n/r_n > 0$.

To obtain the upper bound in (4.21), we first claim that

$$p_U^n(x, \hat{q}_n) \geq \tilde{p}^n. \tag{A.6}$$

Assuming (A.6) the upper bound in (4.21) readily follows: indeed, assume $\limsup_{n\uparrow\infty} \hat{q}_n/r_n = k \geq \delta^+$ and take a subsequence (still labeled n) so that $\hat{q}_n/r_n \to k$. Let $0 < \ell < \delta^+$ so that $\hat{q}_n/r_n \geq \ell$ for n large enough. Since $p_U^n(x,q)$ is decreasing in q, (A.6) implies $\tilde{p}^n \leq p_U^n(x, \ell r_n)$. Taking $n \uparrow \infty$ gives $\tilde{p} \leq p^\infty(\ell)$ and then taking $\ell \uparrow \delta^+$ gives $\tilde{p} \leq p^\infty(\delta^+)$. But, this is a contradiction and hence (4.21) holds.

To prove (A.6), come back to (A.3). Write $Z^{\mathbb{Q},n} := d\mathbb{Q}^n/d\mathbb{P}|_{\mathcal{F}_T}$. From (A.3) it follows for any $y > 0$ that

$$\frac{u_U^n(x - \tilde{p}^n q, q) - u_U^n(x)}{y} + \tilde{p}^n q \leq q\mathbb{E}^{\mathbb{Q}^n}[B] + \frac{1}{y}\left(\mathbb{E}\left[V(yZ^{\mathbb{Q},n})\right] + xy - u_U^n(x)\right). \tag{A.7}$$

Consider the problem

$$\inf_{y>0} \frac{1}{y}\left(\mathbb{E}\left[V(yZ^{\mathbb{Q},n})\right] + xy - u_U^n(x)\right). \tag{A.8}$$

According to [33, Lemma A.4] the map $y \mapsto \mathbb{E}\left[V(yZ^{\mathbb{Q},n})\right]$ is differentiable with derivative $\mathbb{E}\left[Z^{\mathbb{Q},n}V'(yZ^{\mathbb{Q},n})\right]$. Thus, we see the derivative of the above map is

$$\frac{1}{y^2}\left(\mathbb{E}\left[yZ^{\mathbb{Q},n}V'(yZ^{\mathbb{Q},n}) - V(yZ^{\mathbb{Q},n})\right] + u_U^n(x)\right) = \frac{1}{y^2}\left(\mathbb{E}\left[\int_0^{yZ^{\mathbb{Q},n}} \tau V''(\tau)d\tau\right] + u_U^n(x)\right),$$

where the last equality follows since $(d/d\tau)(\tau V'(\tau) - V(\tau)) = \tau V''(\tau)$ and since $U \in \mathcal{U}_a$ implies $\lim_{\tau\downarrow 0} \tau V'(\tau) = \lim_{\tau\downarrow 0} V(\tau) = 0$. Since $U \in \mathcal{U}_a$ and Assumption 4.9 imply $u_U^n(x) < 0$, the strict convexity of V yields a unique $y^{\mathbb{Q},n}$ solving (A.8) and this y satisfies the first order conditions

$$-u_U^n(x) = \mathbb{E}\left[\int_0^{y^{\mathbb{Q},n}Z^{\mathbb{Q},n}} \tau V''(\tau)d\tau\right].$$

A straightforward calculation shows $\tau V''(\tau) = 1/\alpha_U(I(\tau))$ where $I(\tau) = (U')^{-1}(\tau)$. Since $U \in \mathcal{U}_a$ implies $0 < \underline{a}_U < \alpha_U(x) < \bar{a}_U$ on $\mathbb{R}$ we see that $\mathbb{E}\left[Z^{\mathbb{Q},n}\right] = 1$ gives

$$\frac{1}{\bar{a}_U} y^{\mathbb{Q},n} \leq -u_U^n(x) \leq \frac{1}{\underline{a}_U} y^{\mathbb{Q},n},$$

or equivalently, that $-\underline{a}_U u_U^n(x) \leq y^{\mathbb{Q},n} \leq -\bar{a}_U u_U^n(x)$. Using this $y^{\mathbb{Q},n}$ in (A.7) gives

$$\begin{aligned}\frac{u_U^n(x - \tilde{p}^n q, q) - u_U^n(x)}{y^{\mathbb{Q},n}} + \tilde{p}^n q &\leq q\mathbb{E}^{\mathbb{Q}^n}[B] + \frac{1}{y^{\mathbb{Q},n}}\left(\mathbb{E}\left[V(y^{\mathbb{Q},n}Z^{\mathbb{Q},n})\right] + xy - u_U^n(x)\right) \\ &= q\mathbb{E}^{\mathbb{Q}^n}[B] + \inf_{y>0}\frac{1}{y}\left(\mathbb{E}\left[V(yZ^{\mathbb{Q},n})\right] + xy - u_U^n(x)\right).\end{aligned}$$

We have already shown the existence of a $\hat{q}_n > 0$ which maximizes $u_U^n(x - \tilde{p}^n q, q)$ and shown that for n large enough $u^n(x - \tilde{p}\hat{q}_n, \hat{q}_n) > u_U^n(x)$. Thus, for this $\hat{q}_n$ we have, using the inequalities for $y^{\mathbb{Q},n}$ that

$$-\frac{1}{\bar{a}_U u_U^n(x)}\left(u_U^n(x - \tilde{p}^n\hat{q}_n, \hat{q}_n) - u_U^n(x)\right) + \tilde{p}^n\hat{q}_n \leq \hat{q}_n\mathbb{E}^{\mathbb{Q}^n}[B] + \inf_{y>0}\frac{1}{y}\left(\mathbb{E}\left[V(yZ^{\mathbb{Q},n})\right] + xy - u_U^n(x)\right),$$

or, since this inequality is valid for any $\mathbb{Q}^n \in \tilde{\mathcal{M}}^n$ that

$$\begin{aligned} u_U^n(x - \tilde{p}^n \hat{q}_n, \hat{q}_n) - u_U^n(x) - \bar{a}_U u_U^n(x) \tilde{p}^n \hat{q}_n \\ \leq -\bar{a}_u u_U^n(x) \left(\inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\hat{q}_n \mathbb{E}^{\mathbb{Q}^n}[B] + \inf_{y>0} \frac{1}{y} \left(\mathbb{E}\left[V(yZ^{\mathbb{Q},n}) \right] + xy - u_U^n(x) \right) \right) \right) \\ = -\bar{a}_U u_U^n(x) \hat{q}_n p_U^n(x, \hat{q}_n), \end{aligned}$$

where the last equality follows from [32, Proposition 7.1]. We thus obtain the bounds

$$u_U^n(x) \leq u_U^n(x - \tilde{p}^n \hat{q}_n, \hat{q}_n) \leq u_U^n(x) - \bar{a}_U u_U^n(x) \hat{q}_n \left(p_U^n(x, \hat{q}_n) - \tilde{p}^n \right). \tag{A.9}$$

which, since $u_U^n(x) < 0, \hat{q}_n > 0$ implies (A.6), finishing the result. □

APPENDIX B. PROOFS FROM SECTION 7

We begin with a lemma[¶] showing how the indifference price scales with the initial position and risk aversion. This is an easy consequence of the fact that $\mathcal{A}_t$ is a cone:i.e. for each $c > 0$, $(L, M) \in \mathcal{A}_t \Leftrightarrow (cL, cM) \in A_t$. Throughout, we assume that $x, y \in \mathbb{R}$, $0 \leq t \leq T$, $s > 0$, $a > 0$ and $\lambda \in (0, 1)$ (resp. $\lambda_n \in (0, 1)$).

Lemma B.1. *For p_a as in* (7.5) *and $q > 0$:*

$$p_a(qx, qy, q; s, t, \lambda) = p_{qa}(x, y, 1; s, t, \lambda). \tag{B.1}$$

Proof of Lemma B.1. For $(L, M) \in \mathcal{A}_t$ and X, Y as in (7.2) note that

$$-a\left(X_T^{L,M,qx,t} + Y_T^{L,M,qy,t} - q(S_T - K)^+ \right) = -qa\left(X_T^{L/q,M/q,x,t} + Y_T^{L/q,M/q,x,t} - (S_T - K)^+ \right). \tag{B.2}$$

As $\mathcal{A}_t$ is a cone:

$$\inf_{(L,M)\in\mathcal{A}_t} \mathbb{E}_{s,t}\left[e^{-a\left(X_T^{L,M,qx,t} + Y_T^{L,M,qy,t} - q(S_T-K)^+\right)} \right] = \inf_{(L,M)\in\mathcal{A}_t} \mathbb{E}_{s,t}\left[e^{-qa\left(X_T^{L,M,x,t} + Y_T^{L,M,y,t} - (S_T-K)^+\right)} \right].$$

By removing $(S_T - K)^+$ from the above calculations we obtain from (7.3) and (7.4):

$$u_a(qx, qy, q; s, t, \lambda) = qu_{qa}(x, y, 1; s, t, \lambda); \qquad u_a(qx, qy; s, t, \lambda) = qu_{qa}(x, y; s, t, \lambda). \tag{B.3}$$

It is clear for $x' \in \mathbb{R}$ that $u_{qa}(x + x', y, 1; s, t, \lambda) = e^{-qax'} u_{qa}(x, y, 1; s, t, \lambda)$. To make the notation cleaner set $p = p_a(qx, qy, q; s, t, \lambda)$ and $p' = p_{qa}(x, y, 1; s, t, \lambda)$ so that (B.1) becomes $p = p'$. Using the above

[¶]See the comment in [4, Section 2.1].

facts:

$$\begin{aligned} u_{qa}(x,y;s,t,\lambda) &= \frac{1}{q}u_a(qx,qy;s,t,\lambda) = \frac{1}{q}u_a(qx+qp,qy,q;s,t,\lambda); \\ &= \frac{1}{q}u_a(qx+qp'+q(p-p'),qy,q;s,t,\lambda); \\ &= u_{qa}(x+p'+(p-p'),y,1;s,t,\lambda) \\ &= e^{-qa(p-p')}u_{qa}(x+p',y,1;s,t,\lambda); \\ &= e^{-qa(p-p')}u_{qa}(x,y;s,t,\lambda). \end{aligned}$$

Thus, $p = p'$.

□

As in [4, pp. 374-375], for $\varepsilon > 0$ define

$$v^{\varepsilon}(x,y,s,t;\lambda) := 1 + \frac{1}{\varepsilon}u_{1/\varepsilon}(x,y,1;s,t,\lambda); \qquad v^{\varepsilon,f}(x,y,s,t;\lambda) := 1 + \frac{1}{\varepsilon}u_{1/\varepsilon}(x,y;s,t,\lambda). \tag{B.4}$$

Next, define

$$\begin{aligned} z^{\varepsilon}(x,y,s,t;\lambda) &:= x + sy + \varepsilon\log\left(1 - v^{\varepsilon}(x,y,s,t;\lambda)\right); \\ &= x + sy + \varepsilon\log\left(-\frac{1}{\varepsilon}u_{1/\varepsilon}(x,y,1;s,t,\lambda)\right), \\ z^{\varepsilon,f}(x,y,s,t;\lambda) &:= x + sy + \varepsilon\log\left(1 - v^{\varepsilon,f}(x,y,s,t;\lambda)\right); \\ &= x + sy + \varepsilon\log\left(-\frac{1}{\varepsilon}u_{1/\varepsilon}(x,y;s,t,\lambda)\right). \end{aligned} \tag{B.5}$$

Note that by definition $x + py - z^{\varepsilon}$ and $x + py - z^{\varepsilon,f}$ are the respective certainty equivalents in the λ transactions costs market with and without the claim. Furthermore:

Lemma B.2. *$z^{\varepsilon}, z^{\varepsilon,f}$ from* (B.5) *are independent of x and hence write $z^{\varepsilon}(y,s,t;\lambda), z^{\varepsilon,f}(y,s,t;\lambda)$. Furthermore:*

$$\begin{aligned} \Psi(s,t;0) - \frac{\varepsilon\mu^2}{2\sigma^2}(T-t) \le z^{\varepsilon}(y,s,t;\lambda) &\le s(1+\lambda|y-1|); \\ -\frac{\varepsilon\mu^2}{2\sigma^2}(T-t) \le z^{\varepsilon,f}(y,s,t;\lambda) &\le \lambda s|y|, \end{aligned} \tag{B.6}$$

where μ is the drift of S as in (7.1) *and $\Psi(s,t;0)$ is the Black-Scholes price in the frictionless model. Next, for a fixed (y,s,t) and ε, both $z^{\varepsilon}, z^{\varepsilon,f}$ are increasing in λ. Lastly, for a fixed (y,s,t) and λ, both z^{ε} and $z^{\varepsilon,f}$ are continuous and decreasing in ε on $(0,\infty)$.*

Proof of Lemma B.2. That $z^\varepsilon, z^{\varepsilon,f}$ are independent of x and (B.6) holds both follow from [4, Proposition 2.1]. Next, using the definition of v^ε in (B.4) and (7.2) we have

(B.7)

$$\begin{aligned} &z^\varepsilon(y,s,t;\lambda) - sy \\ &\quad = \inf_{(L,M)\in\mathcal{A}_t} \varepsilon \log\left(\mathbb{E}_{s,t}\left[e^{-\frac{1}{\varepsilon}\left(-\int_t^T S_\tau(1+\lambda)dL_\tau + \int_t^T S_\tau(1+\lambda)dM_\tau + yS_T + S_T(L_T-M_T) - (S_T-K)^+\right)}\right]\right) \\ &\quad = \inf_{(L,M)\in\mathcal{A}_t} \varepsilon \log\left(\mathbb{E}_{s,t}\left[e^{-\frac{1}{\varepsilon}\left(-\int_t^T S_\tau dL_\tau + \int_t^T S_\tau dM_\tau + yS_T + S_T(L_T-M_T) - (S_T-K)^+\right)} e^{\frac{\lambda}{\varepsilon}\int_t^T S_\tau(dL_\tau + dM_\tau)}\right]\right). \end{aligned}$$

It is thus evident that $z^\varepsilon(y,s,t;\lambda)$ is increasing in λ. Since the same formula holds for $z^{\varepsilon,f}$, just absent the $(S_T-K)^+$ term, $z^{\varepsilon,f}(y,s,t;\lambda)$ is also increasing in λ. Also, that $z^\varepsilon(y,s,t;\lambda), z^{\varepsilon,f}(y,s,t;\lambda)$ are decreasing in ε follows from Holder's inequality. Lastly, note that the map

$$\gamma \mapsto \inf_{(L,M)\in\mathcal{A}_t} \mathbb{E}_{s,t}\left[e^{-\gamma\left(-\int_t^T S_\tau(1+\lambda)dL_\tau + \int_t^T S_\tau(1+\lambda)dM_\tau + yS_T + S_T(L_T-M_T) - (S_T-K)^+\right)}\right],$$

is convex on $(0,\infty)$ (and again, also when the $(S_T-K)^+$ term is absent). Indeed, take $0<\gamma_1<\gamma_2$ and $0<\lambda<1$. Set $\gamma_\lambda = \lambda\gamma_1 + (1-\lambda)\gamma_2$ and let $(L_1,M_2),(L_2,M_2)\in\mathcal{A}_t$. Since $z\mapsto e^{-z}$ is convex and

$$(L,M) = \frac{\lambda\gamma_1}{\gamma_\lambda}(L_1,M_1) + \frac{(1-\lambda)\gamma_2}{\gamma_\lambda}(L_2,M_2) \in \mathcal{A}_t$$

the convexity follows by first minimizing over (L_1,M_1) then over (L_2,M_2). Since convex functions are continuous on the interior of their effective domain and since $z^\varepsilon, z^{\varepsilon,f}$ are finite by (B.6) we see that $z^\varepsilon(y,s,t;\lambda), z^{\varepsilon,f}(y,s,t;\lambda)$ are continuous in ε on $(0,\infty)$.

□

Proof of Proposition 7.2. Using Lemma B.1 at $q=(\varepsilon a)^{-1}$ gives

$$p_a\left(\frac{x}{\varepsilon a},\frac{y}{\varepsilon a},\frac{1}{\varepsilon a};s,t;\lambda\right) = p_{1/\varepsilon}\left(x,y,1;s,t,\lambda\right),$$

so that

$$v^\varepsilon\left(x + p_a\left(\frac{x}{\varepsilon a},\frac{y}{\varepsilon a},\frac{1}{\varepsilon a};s,t;\lambda\right),y,p,t;\mu\right) = v^{\varepsilon,f}(x,y,s,t;\lambda).$$

Thus, using (B.4), (B.5) one obtains, since Lemma B.2 shows $z^\varepsilon, z^{\varepsilon,f}$ are independent of the capital x, that

$$\begin{aligned} p_a\left(\frac{x}{\varepsilon a},\frac{y}{\varepsilon a},\frac{1}{\varepsilon a};s,t,\lambda\right) &= z^\varepsilon\left(x + p_a\left(\frac{x}{\varepsilon\gamma},\frac{y}{\varepsilon\gamma},\frac{1}{\varepsilon a};s,t;\lambda\right),y,s,t;\lambda\right) - z^{\varepsilon,f}(x,y,s,t;\lambda) \\ &= z^\varepsilon\left(y,s,t;\lambda\right) - z^{\varepsilon,f}(y,s,t;\lambda). \end{aligned}$$

Thus, p_a is independent of x. The conclusions of the theorem now readily follow: namely let $r_n = \lambda_n^{-2}$ and set $q_n = \ell r_n$. Let $y_n\in\mathbb{R}$. Take $\varepsilon_n = \lambda_n^2/(a\ell) = (q_n a)^{-1}$ so that $q_n = (\varepsilon_n a)^{-1}$ and $\lambda_n = \sqrt{\varepsilon_n}\sqrt{a\ell}$. We

then have

$$p_a(y_n, q_n; s, t; \lambda_n) = p_a\left(\frac{y_n\lambda_n^2/\ell}{\varepsilon_n a}, \frac{1}{\varepsilon_n a}; s, t, \sqrt{\varepsilon_n}\sqrt{a\ell}\right)$$
$$= z^{\varepsilon_n}\left(\frac{y_n\lambda_n^2}{\ell}, s, t; \sqrt{\varepsilon_n}\sqrt{a\ell}\right) - z^{\varepsilon_n, f}\left(\frac{y_n\lambda_n^2}{\ell}, s, t; \sqrt{\varepsilon_n}\sqrt{a\ell}\right).$$

Now, by [4, Theorem 3.1] we have for any $y_0 \in \mathbb{R}$ that

$$\lim_{n\uparrow\infty} z^{\varepsilon_n}\left(y_0, s, t; \sqrt{\varepsilon_n}\sqrt{a\ell}\right) = \Psi(s, t; \sqrt{a\ell}); \qquad \lim_{n\uparrow\infty} z^{\varepsilon_n, f}\left(y_0, s, t; \sqrt{\varepsilon_n}\sqrt{a\ell}\right) = 0. \tag{B.8}$$

Furthermore, as shown on [4, pp. 389]

$$\left|z^{\varepsilon_n}\left(\frac{y_n\lambda_n^2}{\ell}, s, t; \sqrt{\varepsilon_n}\sqrt{a\ell}\right) - z^{\varepsilon_n}(0, s, t; \sqrt{\varepsilon_n}\sqrt{a\ell})\right| \leq \lambda_n s\frac{\lambda_n^2|y_n|}{\ell},$$

with the same inequality also holding for $z^{\varepsilon_n, f}$. Thus, if $\lim_{n\uparrow\infty}\lambda_n^3|y_n| = 0$ we see that

$$\lim_{n\uparrow\infty} p_a(y_n, q_n; s, t; \lambda_n) = \Psi(p, t; \sqrt{a\ell}),$$

which is the desired result.

□

Proof of Theorem 7.4 . The proof of convergence follows the weak viscosity limits of [3], see also Chapter VII of [16]. Let us define

$$\Psi^*(s, t) = \limsup_{\rho\downarrow 0}\limsup_{b\downarrow 0}\sup\left\{\Psi(\hat{s}, \hat{t}; b) : |s - \hat{s}| + |t - \hat{t}| < \rho\right\},$$

and

$$\Psi_*(s, t) = \liminf_{\rho\downarrow 0}\liminf_{b\downarrow 0}\inf\left\{\Psi(\hat{s}, \hat{t}; b) : |s - \hat{s}| + |t - \hat{t}| < \rho\right\}.$$

Step 1: $\Psi^*(s, t)$ is a viscosity subsolution to the linear Black-Scholes equation.

Let $w(s, t)$ be a smooth test function and assume that $(s_0, t_0) \in (0, \infty) \times [0, T]$ is a strict local maximizer of the difference $\Psi^*(s, t) - w(s, t)$ on $[0, \infty) \times [0, T]$ such that $\Psi^*(s_0, t_0) = w(s_0, t_0)$. We may, and will do so, assume that $w_{ss}(s_0, t_0) \neq 0$. We verify that Ψ^* is a viscosity subsolution, by proving that if $t_0 < T$, then

$$-w_t(s_0, t_0) - \frac{1}{2}s_0^2\sigma^2 w_{ss}(s_0, t_0) \leq 0,$$

whereas if $t_0 = T$, then either the previous inequality holds or $\Psi^*(s_0, T) \leq (s_0 - K)^+$.

Let us assume that either $t_0 < T$ or that $t_0 = T$ and $\Psi^*(s_0, T) > (s_0 - K)^+$. Consider a sequence $b_n \downarrow 0$ and local maximizers $(s_n, t_n) \in (0, \infty) \times [0, T)$ of the function

$$(s, t) \mapsto \Psi(s, t; b_n) - w(s, t),$$

such that

$$(s_n, t_n) \to (s_0, t_0), \Psi(s_n, t_n; b_n) \to \Psi^*(s_0, t_0), \text{ and } \Psi(s_n, t_n; b_n) - w(s_n, t_n) \to 0.$$

The existence of such a sequence and maximizers is shown in [3]. Notice that for n large enough we have $t_n < T$. Indeed, if $t_0 < T$, then $t_n < T$ for large enough n follows by the convergence $t_n \to t_0$. Let's now assume that $t_0 = T$ and $\Psi^*(s_0, T) > (s_0 - K)^+$ and let $t_n = T$. We calculate

$$\Psi^*(s_0, t_0) = \lim_{n\to\infty} \Psi(s_n, T; b_n) = (s_0 - K)^+.$$

But, since we have assumed that $\Psi^*(s_0, T) > (s_0 - K)^+$ we get a contradiction, which implies that $t_n < T$ for all n large enough.

Let us set now $k_n = \Psi(s_n, t_n; b_n) - w(s_n, t_n)$ and define the operator

$$\mathcal{G}_b[\Psi] = \frac{1}{2}\sigma^2 s^2 \Psi_{ss}(s,t)\left(1 + S\left(bs^2\Psi_{ss}(s,t)\right)\right).$$

By the fact that $\Psi(; b_n)$ is a continuous viscosity solution of (7.6) and that the function $A \mapsto A(1+S(A))$ is increasing function, we get the following

$$0 \geq -w_t(s_n, t_n) - \mathcal{G}_{b_n}[w(s_n, t_n) + k_n].$$

Taking now $n \to \infty$ and using the fact that $\ell_n \to 0$, $(s_n, t_n) \to (s_0, t_0)$, $k_n \to 0$ and $S(0) = 0$, we get

$$-w_t(s_0, t_0) - \frac{1}{2}\sigma^2 s_0^2 w_{ss}(s_0, t_0) \leq 0,$$

completing the proof of the viscosity subsolution property of Ψ^*.

Step 2: $\Psi_*(s,t)$ is a viscosity supersolution to the linear Black-Scholes equation.

The proof if this step is almost identical to the proof of the previous step. Let $w(s,t)$ be a smooth test function and assume that $(s_0, t_0) \in (0,\infty) \times [0,T]$ is a strict global minimizer of the difference $\Psi_*(s,t) - w(s,t)$ on $[0,\infty) \times [0,T]$ such that $\Psi_*(s_0, t_0) = w(s_0, t_0)$. We may, and will do so, assume that $w_{ss}(s_0, t_0) \neq 0$. We verify that Ψ_* is a viscosity supersolution, by proving that if $t_0 < T$, then

$$-w_t(s_0, t_0) - \frac{1}{2}s^2\sigma^2 w_{ss}(s_0, t_0) \geq 0.$$

If $t_0 = T$, then by construction we have the supersolution property $\Psi_*(s, T) \geq (s - K)^+$. We need to show the viscosity property.

Consider a sequence $b_n \downarrow 0$ and local minimizers $(s_n, t_n) \in (0,\infty) \times [0,T)$ of the function

$$(s,t) \mapsto \Psi(s,t; b_n) - w(s,t),$$

such that

$$(s_n, t_n) \to (s_0, t_0), \Psi(s_n, t_n; b_n) \to \Psi_*(s_0, t_0), \text{ and } \Psi(s_n, t_n; b_n) - w(s_n, t_n) \to 0.$$

The existence of such a sequence and minimizers is shown in [3]. Notice that, as in the viscosity subsolution case, for n large enough, we have that $t_n < T$.

By the fact that $\Psi(; b_n)$ is a viscosity solution of (7.6) and that the function $A \mapsto A(1 + S(A))$ is increasing function, we get the following

$$0 \leq -w_t(s_n, t_n) - \mathcal{G}_{b_n}[w(s_n, t_n) + k_n].$$

Taking now $n \to \infty$ and using the fact that $\ell_n \to 0$, $(s_n, t_n) \to (s_0, t_0)$, $k_n \to 0$ and $S(0) = 0$, we get

$$-w_t(s_0, t_0) - \frac{1}{2}\sigma^2 s_0^2 w_{ss}(s_0, t_0) \geq 0,$$

completing the proof of the viscosity supersolution property of Ψ_*.

Step 3: Putting the estimates together

By construction we have that $\Psi_* \leq \Psi^*$. Then a comparison argument as in proof of Theorem 3.1 of [4], or equivalently see Section VII.8 of [16], gives the opposite inequality, i.e., $\Psi_* \geq \Psi^*$. Thus we have that $\Psi_* = \Psi^*$ and the function $\Psi^0 = \Psi_* = \Psi^*$ is solution to the equation

$$\Psi_t + \frac{1}{2}\sigma^2 s^2 \Psi_{ss} = 0; \qquad \Psi(T, s) = (s - K)^+.$$

Classical arguments, e.g. Theorem 7.1 of [16], then imply that the equality $\Psi_* = \Psi^*$ implies the local uniform convergence $\Psi^\ell \to \Psi^0$ as $\ell \to 0$. This completes the proof of the theorem. □

Proof of Theorem 7.5. From Lemma B.2 at $\lambda = b\sqrt{\varepsilon}$ it follows that $z^\varepsilon(y, s, t; b\sqrt{\varepsilon})$ is increasing in b. Since [4, Theorem 3.1] implies $\lim_{\varepsilon \to 0} z^\varepsilon(y, s, t; b\sqrt{\varepsilon}) = \Psi(s, t; b)$, it follows that $\Psi(s, t; b)$ is increasing in b. As for the asymptotics in (7.7) by construction $\Psi(s, T; b) = (s - K)^+$ for $p > 0, b > 0$. Thus, we only consider when $t < T$. Here, we recall from Proposition 7.2 that $\lim_{A \uparrow \infty} S(A)/A = 1$. Furthermore, as shown in [4], $S(A) > 0$ for $A > 0$. Thus, let $\gamma > 0$ and pick A_γ so that $S(A) \geq (1 - \gamma)A$ for $A \geq A_\gamma$.

Now, let $\psi : (0, \infty) \times [0, T]$ be a smooth function with $\psi_{ss} \geq 0$. Write

$$H[\psi] := \psi_t + \frac{1}{2}\sigma^2 s^2 \psi_{ss} \left(1 + S(b^2 s^2 \psi_{ss})\right).$$

We have the following basic estimate, since $\psi_{ss} \geq 0$ and $A \mapsto A(1 + S(A))$ is increasing:

$$\begin{aligned}
H[\psi] &\geq \psi_t + 1_{s^2\psi_{ss} \geq \frac{A_\gamma}{b^2}} \left(\frac{1}{2}\sigma^2 s^2 \psi_{ss} \left(1 + (1 - \gamma) b^2 s^2 \psi_{ss}\right)\right), \\
&= \psi_t + 1_{s^2\psi_{ss} \geq \frac{A_\gamma}{b^2}} \left(\frac{1 - \gamma}{2}\sigma^2 \left(b s^2 \psi_{ss} + \frac{1}{2(1 - \gamma)b}\right)^2 - \frac{\sigma^2}{8b^2(1 - \gamma)}\right), \\
&\geq \psi_t - \frac{\sigma^2}{8b^2(1 - \gamma)} + \frac{1 - \gamma}{2}\sigma^2 \left(b s^2 \psi_{ss} + \frac{1}{2(1 - \gamma)b}\right)^2 - 1_{s^2\psi_{ss} < \frac{A_\gamma}{b^2}} \frac{1 - \gamma}{2}\sigma^2 \left(b s^2 \psi_{ss} + \frac{1}{2(1 - \gamma)b}\right)^2, \\
&\geq \psi_t - \frac{\sigma^2}{8b^2(1 - \gamma)} + \frac{1 - \gamma}{2}\sigma^2 \left(b s^2 \psi_{ss} + \frac{1}{2(1 - \gamma)b}\right)^2 - \frac{1 - \gamma}{2}\sigma^2 \left(\frac{A_\gamma}{b} + \frac{1}{2(1 - \gamma)b}\right)^2, \\
&= \psi_t - \frac{\sigma^2 K_\gamma}{2b^2} + \frac{1 - \gamma}{2}\sigma^2 \left(b s^2 \psi_{ss} + \frac{1}{2(1 - \gamma)b}\right)^2,
\end{aligned}$$

where

$$K_\gamma := \frac{1}{4(1 - \gamma)} + (1 - \gamma)\left(A_\gamma + \frac{1}{2(1 - \gamma)}\right).$$

To recap, we have for ψ smooth with $\psi_{ss} \geq 0$ that

$$H[\psi] \geq \psi_t - \frac{\sigma^2 K_\gamma}{2b^2} + \frac{1 - \gamma}{2}\sigma^2 \left(b s^2 \psi_{ss} + \frac{1}{2(1 - \gamma)b}\right)^2. \tag{B.9}$$

Now, let $C > 0$ and denote by $\phi(s,t;C)$ the Black-Scholes price at (s,t) for a call option with strike K, maturity T when the interest rate is 0 and the asset volatility is C. Let $M \in \mathbb{R}$ and consider the function

$$\psi(s,t) = \phi(s,t;C) - M(T-t).$$

Clearly, ψ is smooth and from the explicit formula for $\phi(s,t;C)$ it follows that $\psi_{ss} \geq 0$. We then have from (B.9) (writing ϕ^C to denote the dependence upon C) that

$$\begin{aligned} H[\psi] &\geq \phi_t^C + M - \frac{\sigma^2 K_\gamma}{2a^2} + \frac{1}{2}(1-\gamma)\sigma^2 \left(bs^2\phi_{ss}^C + \frac{1}{2(1-\gamma)b} \right)^2; \\ &= -\frac{1}{2}C^2 s^2 \phi_{ss}^C + M - \frac{\sigma^2 K_\gamma}{2b^2} + \frac{1}{2}(1-\gamma)\sigma^2 \left(bs^2\phi_{ss}^C + \frac{1}{2(1-\gamma)b} \right)^2. \end{aligned}$$

The quadratic form $(1/2)(1-\gamma)\sigma^2 b^2 x^2 + (1/2)(\sigma^2 - C^2)x$ is bounded below by

$$-\frac{1}{8}\frac{(\sigma^2 - C^2)^2}{(1-\gamma)\sigma^2 b^2}.$$

Plugging this into the above (with $s^2\phi_{ss}^C$ playing the role of x) yields

$$H[\psi] \geq -\frac{(\sigma^2 - C^2)^2}{8(1-\gamma)\sigma^2 b^2} + M - \frac{\sigma^2 K_\gamma}{2b^2} + \frac{\sigma^2}{8(1-\gamma)b^2}.$$

Clearly, setting

$$M = \frac{(\sigma^2 - C^2)^2}{8(1-\gamma)\sigma^2 b^2} + \frac{\sigma^2 K_\gamma}{2b^2} - \frac{\sigma^2}{8(1-\gamma)b^2} = \frac{C^4}{8(1-\gamma)\sigma^2 b^2} - \frac{C^2}{4(1-\gamma)b^2} + \frac{\sigma^2 K_\gamma}{2b^2}, \tag{B.10}$$

yields that $H[\psi] \geq 0$ and hence by the comparison argument shown in [4, Theorem 3.1, pp. 395-396] it follows that $\Psi(s,t;b) \geq \psi(s,t)$. To connect with the results therein, set

$$z^*(s,t) = \psi(s,t) = \phi(s,t;C) - M(T-t); \qquad z_*(s,t) = \Psi(s,t;b),$$

and note that z^* is a (classical) sub-solution; z_* is a continuous viscosity super-solution; $\lim_{s\uparrow\infty} z^*(s,t)/s = 1$, $\lim_{s\uparrow\infty} z_*(s,t)/s = 1$ uniformly in $0 \leq t \leq T$; and that $z^*(0,t) = -M(a)(T-t) \leq z_*(0,t) = 0$ for any $t \leq T$ if $C > \sqrt{2\sigma}$. Thus, the argument in [4, pp. 395-396] goes through.

Now, so far the choice of $C > 0$ was arbitrary. Consider then when $C = b^{1/4}$. Here we have as $b \to \infty$ that

$$\begin{aligned} C &= C(b) \to \infty, \\ M &= M(b) = \frac{1}{8(1-\gamma)\sigma^2 b} - \frac{1}{4(1-\gamma)b^{3/2}} + \frac{\sigma^2 K_\gamma}{2b^2} \to 0. \end{aligned}$$

Thus, we have from the comparison principle that

$$\liminf_{b\uparrow\infty} \Psi(s,t;b) \geq \liminf_{b\uparrow\infty} \phi(s,t;C(b)) - M(b)(T-t) = s,$$

where the last equality follows from the well known fact that the price of a call in the Black-Scholes model converges to the initial stock price as the volatility approaches infinity. This completes the proof since it was shown in [4, Proposition 2.1, Theorem 3.1] that $\Psi(s,t;b) \leq s$ for all $b > 0$. □

Proof of Theorem 7.7. For $\tilde{p}^n \in (\Psi(s,t;0), s)$ recall the optimal sale quantity problem in (7.8):

$$\max_{q>0} u_a(x + ys(1-\lambda_n) + q\tilde{p}^n, 0, q; s, t; \lambda_n).$$

With $\tilde{x} = x + ys(1-\lambda_n)$ we have, in view of (B.3) and (B.4), (B.5), that for $q > 0$:

(B.11)

$$u_a(\tilde{x} + q\tilde{p}^n, 0, q; s, t; \lambda_n) = \frac{1}{a}\left(v^{\frac{1}{qa}}\left(\frac{\tilde{x}}{q} + \tilde{p}^n, 0, s, t; \lambda_n\right) - 1\right) = -\frac{1}{a}e^{-a\left(\tilde{x}+q\tilde{p}^n - qz^{\frac{1}{qa}}(0,s,t;\lambda_n)\right)},$$

and hence it suffices to consider the optimization problem

$$\sup_{q>0}\left(q\tilde{p}^n - qz^{\frac{1}{qa}}(0,s,t;\lambda_n)\right) = -\inf_{q>0}\left(q(-\tilde{p}^n) - q\left(-z^{\frac{1}{qa}}(0,s,t;\lambda_n)\right)\right). \tag{B.12}$$

The existence of a maximizer $\hat{q}_n > 0$, as well as the asymptotic behavior of $\hat{q}_n/r_n$ in (7.9) as $\lambda_n \to 0$ will follow from Proposition C.2 below once the requisite hypotheses are shown to hold. Here, p^n is the map

$$q \mapsto p^n(q) = -z^{\frac{1}{qa}}(0,s,t;\lambda_n).$$

We first consider Assumption C.1. As for bullet point one, note that by Lemma B.2, p^n is continuous and non-increasing on $(0,\infty)$. Regarding bullet point two, (B.6) gives

$$-qs(1+\lambda_n) \le qp^n(q) \le -q\Psi(s,t;0) + \frac{\mu^2}{2a\sigma^2}(T-t),$$

so that for any $\gamma > 0$

$$\limsup_{n\uparrow\infty}\sup_{q\le\gamma} q|p^n(q)| \le \gamma \max\left\{\Psi(s,t;0) + \frac{\mu^2}{2a\sigma^2}(T-t), \gamma s\right\} := C(\gamma) < \infty,$$

verifying bullet point two. Regarding bullet point three, from (B.8) where $\varepsilon_n = \lambda_n^2/(a\ell)$, $q_n = \ell r_n$ and $r_n = \lambda_n^{-2}$ it holds for all $\ell > 0$ that $p^n(\ell r_n) \to -\Psi(s,t;\sqrt{a\ell}) = p^\infty(\ell)$. Thus, bullet point three holds with $\delta = \delta^+ = \infty$. Lastly, regarding bullet point four, since Theorem 7.5 shows that $\lim_{\ell\uparrow\infty}\Psi(s,t;\sqrt{a\ell}) = -\lim_{\ell\uparrow\infty} p^\infty(\ell) = -s$ and $s > \Psi(s,t;0) = -p_+^\infty(0)$, bullet point four holds (see the sufficient condition below Assumption C.1). Therefore, Assumption C.1 holds. Lastly, as stated above for $\tilde{p} \in (\Psi(s,t;0), s)$ we have

$$-s = \lim_{\ell\uparrow\infty} p^\infty(\ell) < -\tilde{p} < p_+^\infty(0) = \lim_{\ell\downarrow 0}(-\Psi(s,t;\sqrt{a\ell})) = -\Psi(s,t;0).$$

Therefore, the results of Proposition C.2 go through, finishing the proof.

□

APPENDIX C. TECHNICAL SUPPORTING RESULTS

The following propositions provide the main technical tools to prove the optimal position taking results in both the frictionless and transactions cost cases. To seamlessly integrate with the transaction costs case, results are separated into long and short positions.

C.1. **Long Positions.** Assume:

Assumption C.1. $\{p^n\}$ is a family of functions defined on $(0,\infty)$ such that

- For each n, p^n is non-increasing and continuous.
- There exists a $\gamma > 0$ such that $\limsup_{n\uparrow\infty} \sup_{q\leq\gamma} q|p^n(q)| = C(\gamma) < \infty$.
- There exists $r_n \to \infty$ and $\delta > 0$ such that for $0 < \ell < \delta$ we have $\lim_{n\uparrow\infty} p^n(\ell r_n) = p^\infty(\ell)$.
- With $p^\infty_+(0) := \lim_{\ell\downarrow 0} p^\infty(\ell)$ and $p^n(\infty) := \lim_{q\uparrow\infty} p^n(q)$ we have $\limsup_{n\uparrow\infty} p^n(\infty) < p^\infty_+(0)$.

To find the maximal upper bound of convergence, set

$$\delta^+ := \sup\left\{k > 0 \mid \lim_{n\uparrow\infty} p^n(\ell r_n) = p^\infty(\ell),\ \forall\, 0 \leq \ell < k\right\} \in [\delta,\infty]. \tag{C.1}$$

Note that for $0 < \ell < \delta^+$ we have $p^n(\infty) \leq p^n(\ell r_n)$ so that $\limsup_{n\uparrow\infty} p^n(\infty) \leq p^\infty(\ell) \leq p^\infty_+(0)$. As such, a sufficient condition for bullet point four in Assumption C.1 to hold is that $p^\infty(\ell) < p^\infty_+(0)$ for some $0 < \ell < \delta^+$.

Under Assumption C.1 we have the following result for positive position sizes:

Proposition C.2. *Let Assumption C.1 hold. Let $\tilde{p}^n \to \tilde{p}$.*

- *If $\limsup_{n\uparrow\infty} p^n(\infty) < \tilde{p} < p^\infty_+(0)$ then for n large enough the optimization problem*

$$\inf_{q>0} \left(q\tilde{p}^n - qp^n(q)\right), \tag{C.2}$$

  admits a minimizer $\hat{q}_n > 0$.
- *If $\limsup_{n\uparrow\infty} p^n(\infty) < \tilde{p} < p^\infty_+(0)$ then for any sequence of minimizers $\{\hat{q}_n\}$:*

$$0 < \liminf_{n\uparrow\infty} \frac{\hat{q}_n}{r_n}. \tag{C.3}$$

- *If additionally $\lim_{\ell\uparrow\delta^+} p^\infty(\ell) < \tilde{p} < p^\infty_+(0)$ then for any sequence $\{\hat{q}_n\}$ of minimizers:*

$$\limsup_{n\uparrow\infty} \frac{\hat{q}_n}{r_n} < \delta^+. \tag{C.4}$$

Proof of Proposition C.2. First consider the minimization problem in (C.2). Since $\tilde{p}^n \to \tilde{p}$ there is some $\varepsilon > 0$ and N_ε so that $n \geq N_\varepsilon$ implies $\limsup_{n\uparrow\infty} p^n(\infty) + \varepsilon < \tilde{p}^n < p^\infty_+(0) - \varepsilon$. Next, choose $\ell > 0$ small enough so that $\tilde{p}^n < p^\infty(\ell) - \varepsilon/2$. By enlarging N_ε we know for $n \geq N_\varepsilon$ that $p^n(\infty) \leq \limsup_{n\uparrow\infty} p^n(\infty) + \varepsilon/2$ and $p^\infty(\ell) < p^n(\ell r_n) + \varepsilon/4$ and hence

$$p^n(\infty) + \varepsilon/2 \leq \tilde{p}^n \leq p^n(\ell r_n) - \varepsilon/4. \tag{C.5}$$

For a fixed n, note that $\lim_{q\uparrow\infty}(\tilde{p}^n - p^n(q)) = \tilde{p}^n - p^n(\infty) \geq \varepsilon/2$. Thus, if $\{\hat{q}^m_n\}_{m\in\mathbb{N}}$ is a minimizing sequence for (C.2), then $\{\hat{q}^m_n\}$ is bounded and hence has an accumulation point $\hat{q}_n$. We now show that $\hat{q}_n \neq 0$, which combined with the continuity of $qp^n(q)$ proves $\hat{q}_n > 0$ is a minimizer. To see that $\hat{q}_n \neq 0$ note that with the γ from Assumption C.1:

$$\liminf_{q\downarrow 0}(q\tilde{p}^n - qp^n(q)) = -\limsup_{q\downarrow 0} qp^n(q) \geq -\sup_{q\leq\gamma} q|p^n(q)|.$$

For the given ε, by enlarging N_ε we may assume that for $n \geq N_\varepsilon$

$$\liminf_{q\downarrow 0}(q\tilde{p}^n - qp^n(q)) \geq -\limsup_{n\uparrow\infty}\sup_{q\leq\gamma} q|p^n(q)| - \varepsilon = -C(\gamma) - \varepsilon.$$

But, for the ℓ from (C.5):

$$\ell r_n \tilde{p}^n - \ell r_n p^n(\ell r_n) \leq -\ell r_n \varepsilon/4. \tag{C.6}$$

Combining the last two displays we get that for the chosen n, we have

$$-\ell r_n \varepsilon/4 \geq -C(\gamma) - \varepsilon.$$

However, by potentially enlarging N_ε, and since $r_n \to \infty$, we can always arrange things so that $-\ell r_n\varepsilon/4 < -C(\gamma) - \varepsilon$. This leads to a contradiction, proving that $\hat{q}_n \neq 0$.

Now, let $\{\hat{q}_n\}$ be a sequence of minimizers. We first claim that $\liminf_{n\uparrow\infty}\hat{q}_n > 0$. Indeed, assume there is a subsequence (still labeled n) so that $\lim_{n\uparrow\infty}\hat{q}_n = 0$. We then have, using the γ of Assumption C.1 that

$$\liminf_{n\uparrow\infty}(\hat{q}_n\tilde{p}^n - \hat{q}_n p^n(\hat{q}_n)) = -\limsup_{n\uparrow\infty}\hat{q}_n p^n(\hat{q}_n) \geq -\limsup_{n\uparrow\infty}\sup_{q\leq\gamma} q|p^n(q)| = -C(\gamma).$$

But, this directly violates the minimality of $\hat{q}_n$ in view of (C.6). As such, there is some $K > 0$ so that $\hat{q}_n \geq K$ for n large enough.

Now, assume that $\liminf_{n\uparrow\infty}\hat{q}_n/r_n = 0$ and take a subsequence such that $\lim_{n\uparrow\infty}\hat{q}_n/r_n = 0$. For all $0 < c < \delta^+$ we see

$$\tilde{p}^n - p^n(cr_n) \geq \frac{\hat{q}_n}{cr_n}\left(\tilde{p}^n - p^n(\hat{q}_n)\right). \tag{C.7}$$

As $n \uparrow \infty$ we know that $\tilde{p}^n - p^n(cr_n) \to \tilde{p} - p^\infty(c)$, $\hat{q}_n/(cr_n) \to 0$ and $\tilde{p}^n \to \tilde{p}$. Recall that $\liminf_{n\uparrow\infty}\hat{q}_n \geq K$ and the γ from Assumption C.1. Note that if $K > \gamma$ then

$$p^n(K) \leq p^n(\gamma) = \frac{1}{\gamma}\gamma p^n(\gamma) \leq \frac{1}{\gamma}\sup_{q\leq\gamma} q|p^n(q)|,$$

whereas if $K \leq \gamma$ then

$$p^n(K) = \frac{1}{K}Kp^n(K) \leq \frac{1}{K}\sup_{q\leq\gamma} q|p^n(q)|.$$

Putting these together gives

$$\limsup_{n\uparrow\infty} p^n(\hat{q}_n) \leq \frac{1}{\gamma\wedge K}\limsup_{n\uparrow\infty}\sup_{q\leq\gamma} q|p^n(q)| = \frac{C(\gamma)}{\gamma\wedge K}.$$

Thus, taking $n \uparrow \infty$ in (C.7) gives $\tilde{p} \geq p^\infty(c)$. Taking $c \downarrow 0$ gives $\tilde{p} \geq p^\infty_+(0)$ a contradiction. Therefore, (C.3) holds.

Next, assume that $\limsup_{n\uparrow\infty}\hat{q}_n/r_n \geq \delta+$ and take a subsequence so that $\lim_{n\uparrow\infty}\hat{q}_n/r_n = k \geq \delta^+$. For each $c < \delta^+$ we have $\hat{q}_n/r_n \geq c$ and hence for any $K > 0$, $\hat{q}_n \geq K$ for n large enough. Thus, we have

$$K\tilde{p}^n - Kp^n(K) \geq \hat{q}_n\left(\tilde{p}^n - p^n(\hat{q}_n)\right) \geq \hat{q}_n\left(\tilde{p}^n - p^n(cr_n)\right). \tag{C.8}$$

Clearly, $K\tilde{p}^n/\hat{q}_n \to 0$. Additionally, for any $0 < c' < \delta^+$:

$$\liminf_{n\uparrow\infty} \frac{p^n(K)}{\hat{q}_n} \geq \liminf_{n\uparrow\infty} \frac{p^n(c'r_n)}{\hat{q}_n} = 0.$$

Thus, dividing by $\hat{q}_n$ in (C.8) and taking $n \uparrow \infty$ yields $0 \geq \tilde{p} - p^\infty(c)$. Taking $c \uparrow \delta^+$ gives that $\tilde{p} \leq \lim_{c\uparrow\delta^+} p^\infty(c)$, which is a contradiction. Therefore, (C.4) holds.

□

C.2. **Short Positions.** We just state the result for $q < 0$ as the proof is the exact same. First, we assume:

Assumption C.3. $\{p^n\}$ is a family of functions defined on $(-\infty, 0)$ such that

- For each n, p^n is non-increasing and continuous.
- There exists a $\gamma < 0$ such that $\limsup_{n\uparrow\infty} \sup_{q\geq\gamma} q|p^n(q)| = C(\gamma) < \infty$.
- There exists $r_n \to \infty$ and $\delta > 0$ such that for $-\delta < \ell < 0$ we have $\lim_{n\uparrow\infty} p^n(\ell r_n) = p^\infty(\ell)$.
- With $p^\infty_-(0) := \lim_{\ell\uparrow 0} p^\infty(\ell)$ and $p^n(-\infty) := \lim_{q\downarrow-\infty} p^n(q)$ we have $p^\infty_-(0) < \liminf_{n\uparrow\infty} p^n(-\infty)$.

To find the minimal lower bound of convergence, set

$$\delta_- := \inf\left\{k < 0 \mid \lim_{n\uparrow\infty} p^n(\ell r_n) = p^\infty(\ell),\ \forall 0 \geq \ell > k\right\} \leq \in [-\infty, \delta_-]. \tag{C.9}$$

As before, we have for any $\delta_- < \ell < 0$ that $p^\infty_-(0) \leq p^\infty(\ell) \leq \liminf_{n\uparrow\infty} p^n(-\infty)$ so that a sufficient condition for bullet point four above to hold is that $p^\infty_-(0) < p^\infty(\ell)$ for some $\delta_- < \ell < 0$. The main result now reads:

Proposition C.4. *Let Assumption C.3 hold. Let $\tilde{p}^n \to \tilde{p}$.*

- *If $p^\infty_-(0) < \tilde{p} < \liminf_{n\uparrow\infty} p^n(-\infty)$ then for n large enough the optimization problem*

$$\inf_{q<0} \left(q\tilde{p}^n - qp^n(q)\right), \tag{C.10}$$

  admits a minimizer $\hat{q}_n < 0$.
- *If $p^\infty_-(0) < \tilde{p} < \liminf_{n\uparrow\infty} p^n(-\infty)$ then for any sequence of minimizers $\{\hat{q}_n\}$:*

$$0 < \liminf_{n\uparrow\infty} \frac{-\hat{q}_n}{r_n}. \tag{C.11}$$

- *If additionally $p^\infty_-(0) < \tilde{p} < \lim_{\ell\downarrow\delta_-} p^\infty(\ell)$ then for any sequence $\{\hat{q}_n\}$ of minimizers:*

$$\limsup_{n\uparrow\infty} \frac{-\hat{q}_n}{r_n} < -\delta_-. \tag{C.12}$$

C.3. **Long and Short Positions.** We now combine the long and short results of the previous section into one result which will be used to prove the frictionless results of Section 4. Here, we assume

Assumption C.5. $\{p^n\}_{n\in\mathbb{N}}$ is a sequence of functions on $\mathbb{R}$ such that

- For each n, p^n is non-increasing and continuous.
- There exists a $\gamma > 0$ such that $\limsup_{n\uparrow\infty} \sup_{|q|\leq\gamma} q|p^n(q)| = C(\gamma) < \infty$.
- There exists $r_n \to \infty$ and $\delta > 0$ such that for $|\ell| < \delta$ we have $p^n(\ell r_n) \to p^\infty(\ell)$.

- $\lim_{\ell\to 0} p^\infty(\ell) = p^\infty(0)$.

Proposition C.6. *Let Assumption C.5 hold and define δ^+, δ_- as in* (C.1) *and* (C.9)*. Let $\tilde{p}^n \to \tilde{p}$.*

- *Assume that $\limsup_{n\uparrow\infty} p^n(\infty) < p^\infty(0)$. If $\limsup_{n\uparrow\infty} p^n(\infty) < \tilde{p} < p^\infty(0)$ then for n large enough any minimizer to the optimization problem $\inf_{q\in\mathbb{R}} (q\tilde{p} - qp^n(q))$ is positive. Furthermore, for any sequence of minimizers $\{\hat{q}_n\}_{n\in\mathbb{N}}$ we have that $0 < \liminf_{n\uparrow\infty} \hat{q}_n/r_n$. If additionally $\lim_{\ell\uparrow\delta^+} p^\infty(\ell) < \tilde{p} < p^\infty(0)$ then for any sequence of minimizers $\{\hat{q}_n\}_{n\in\mathbb{N}}$ we have that $\limsup_{n\uparrow\infty} \hat{q}_n/r_n < \delta^+$.*
- *Assume that $p^\infty(0) < \liminf_{n\uparrow\infty} p^n(-\infty)$. If $p^\infty(0) < \tilde{p} < \liminf_{n\uparrow\infty} p^n(-\infty)$ then for n large enough, any minimizer to the optimization problem $\inf_{q\in\mathbb{R}} (q\tilde{p}^n - qp^n(q))$ is negative. Furthermore, for any sequence of minimizers $\{\hat{q}_n\}_{n\in\mathbb{N}}$ we have that $0 < \liminf_{n\uparrow\infty} -\hat{q}_n/r_n$. If additionally $p^\infty(0) < \tilde{p} < \lim_{\ell\downarrow\delta_-} p^\infty(\ell)$ then for any sequence of minimizers $\{\hat{q}_n\}$ we have that $\limsup_{n\uparrow\infty} -\hat{q}_n/r_n < -\delta_-$.*

Proof of Proposition C.6. We will prove the results for $\limsup_{n\uparrow\infty} p^n(\infty) < \tilde{p} < p^\infty(0)$ and $\lim_{\ell\uparrow\infty} p^\infty(\ell) < \tilde{p} < p^\infty(0)$ respectively; the proof for the other case is the exact same. First, since $p^n(0)$ is well defined for each n, we have $0 \times \tilde{p}^n - 0 \times p^n(0) = 0$. Additionally, for $\varepsilon > 0$ so that $\limsup_{n\uparrow\infty} p^n(\infty) + \varepsilon < \tilde{p} < p^\infty(0) - \varepsilon$ we have for $q < 0$ and n large enough that

$$q\tilde{p} - qp^n(q) \geq q\tilde{p} - qp^n(0) \geq -q\varepsilon/2 > 0,$$

But, from (C.6) we see there is some $\ell > 0$ so that $\ell r_n \tilde{p}^n - \ell r_n p^n(\ell r_n) < 0$. Thus it suffices to minimize over $q > 0$ and hence Proposition C.2 yields a minimizer to the problem over $(0, \infty)$, as well as the asymptotic behavior $\hat{q}_n/r_n$ of minimizers $\hat{q}_n$ given above, finishing the result.

□

REFERENCES

[1] M. ANTHROPELOS AND G. ŽITKOVIĆ, *On agent's agreement and partial-equilibrium pricing in incomplete markets*, Math. Finance, 20 (2010), pp. 411–446.

[2] G. A. ATKESON, L. A. EISFELDT, AND P.-O. WEILL, *The market of otc derivatives*. National Bureau of Economics Reseach, 2013.

[3] G. BARLES AND B. PERTHAME, *Exit time problems in control and vanishing viscocity solutions of hamilton-jacobi-bellman equations*, SIAM Journal of Control and Opitmization, 26 (1988), pp. 1113–1148.

[4] G. BARLES AND H. M. SONER, *Option pricing with transaction costs and a nonlinear Black-Scholes equation*, Finance Stoch., 2 (1998), pp. 369–397.

[5] P. BARRIEU AND N. EL KAROUI, *Inf-convolution of risk measures and optimal risk transfer*, Finance Stoch., 9 (2005), pp. 269–298.

[6] M. BICHUCH, *Asymptotic analysis for optimal investment in finite time with transaction costs*, SIAM J. Financial Math., 3 (2012), pp. 433–458.

[7] BIS, *Amounts outstanding of over-the-counter derivatives by risk category and instrument*, Bank for International Settlements (BIS), (2014). http://www.bis.org/about/index.htm.

[8] B. BOUCHARD, R. ELIE, AND L. MOREAU, *A note on utility based pricing and asymptotic risk diversification*, Mathematics and Financial Economics, 6 (2012), pp. 59–74.

[9] R. CARMONA, *Indifference pricing*, Princeton Series in Financial Engineering, Princeton University Press, Princeton, NJ, 2009. Theory and applications.

[10] G. M. CONSTANTINIDES, *Capital market equilibrium with transaction costs*, The Journal of Political Economy, (1986), pp. 842–862.

[11] C. CZICHOWSKY AND W. SCHACHERMAYER, *Duality Theory for Portfolio Optimisation under Transaction Costs*, working paper, (2014).

[12] M. DAVIS, *Option Pricing in Incomplete Markets*, Mathematics of Derivative Securities, (1997).

[13] M. H. A. DAVIS, V. G. PANAS, AND T. ZARIPHOPOULOU, *European option pricing with transaction costs*, SIAM J. Control Optim., 31 (1993), pp. 470–493.

[14] F. DELBAEN, P. GRANDITS, T. RHEINLÄNDER, D. SAMPERI, M. SCHWEIZER, AND C. STRICKER, *Exponential hedging and entropic penalties*, Math. Finance, 12 (2002), pp. 99–123.

[15] A. DEMBO AND O. ZEITOUNI, *Large deviations techniques and applications*, vol. 38 of Applications of Mathematics (New York), Springer-Verlag, New York, second ed., 1998.

[16] W. FLEMING AND M. SONER, *Controlled Markov processes and viscosity solutions*, vol. 25 of Stochastic modelling and applied probability, Springer, New York, second ed., 2006.

[17] H. FÖLLMER AND A. SCHIED, *Stochastic finance*, vol. 27 of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, extended ed., 2004. An introduction in discrete time.

[18] M. FRITTELLI, *The minimal entropy martingale measure and the valuation problem in incomplete markets*, Math. Finance, 10 (2000), pp. 39–52.

[19] P. GRANDITS AND T. RHEINLÄNDER, *On the minimal entropy martingale measure*, Ann. Probab., 30 (2002), pp. 1003–1038.

[20] P. GUASONI AND J. MUHLE-KARBE, *Portfolio choice with transaction costs: a user's guide*, in Paris-Princeton Lectures on Mathematical Finance 2013, vol. 2081 of Lecture Notes in Math., Springer, Cham, 2013, pp. 169–201.

[21] V. HENDERSON, *Valuation of claims on nontraded assets using utility maximization*, Math. Finance, 12 (2002), pp. 351–373.

[22] S. D. HODGES AND A. NEUBERGER, *Optimal replication of contingent claims under transactions costs*, Review of Futures Markets, 8 (1989), pp. 222–239.

[23] R. HYND, *Option pricing in the large risk aversion, small transaction cost limit*, Comm. Partial Differential Equations, 39 (2014), pp. 1998–2027.

[24] A. İLHAN, M. JONSSON, AND R. SIRCAR, *Optimal investment with derivative securities*, Finance Stoch., 9 (2005), pp. 585–595.

[25] T. ISHIKAWA AND S. ROBERTSON, *Contingent claim pricing in markets with defaultable assets*, Working Paper, (2015).

[26] Y. M. KABANOV AND C. STRICKER, *On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper "Exponential hedging and entropic penalties" [Math. Finance* **12** *(2002), no. 2, 99–123; MR1891730 (2003b:91046)] by F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer and C. Stricker*, Math. Finance, 12 (2002), pp. 125–134.

[27] J. KALLSEN AND J. MUHLE-KARBE, *On using shadow prices in portfolio optimization with transaction costs*, Ann. Appl. Probab., 20 (2010), pp. 1341–1358.

[28] J. KALLSEN AND J. MUHLE-KARBE, *Option pricing and hedging with small transaction costs*, Mathematical Finance, (2013).

[29] Y. LEE AND T. RHEINLÄNDER, *Optimal martingale measures for defaultable assets*, Stochastic Process. Appl., 122 (2012), pp. 2870–2884.

[30] M. MANIA AND M. SCHWEIZER, *Dynamic exponential utility indifference valuation*, Ann. Appl. Probab., 15 (2005), pp. 2113–2143.

[31] M. MUSIELA AND T. ZARIPHOPOULOU, *An example of indifference prices under exponential preferences*, Finance Stoch., 8 (2004), pp. 229–239.

[32] M. P. OWEN AND G. ŽITKOVIĆ, *Optimal investment with an unbounded random endowment and utility-based pricing*, Math. Finance, 19 (2009), pp. 129–159.

[33] S. ROBERTSON, *Pricing for large positions in contingent claims*, Mathematical Finance, (Forthcoming).

[34] S. ROBERTSON AND K. SPILIOPOULOS, *Indifference pricing in large markets: Large deviations effects*, Submitted. In Revision, (2015).

[35] S. E. SHREVE AND H. M. SONER, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab., 4 (1994), pp. 609–692.

[36] P. SIORPAES, *Optimal investment and price dependence in a semi-static market*, Finance Stoch., 19 (2015), pp. 161–187.

[37] S. STOIKOV AND T. ZARIPHOPOULOU, *Optimal investments in the presence of unhedgeable risks and under cara preferences*, IMA Volume Series, (2005).

[38] M. TEHRANCHI, *Explicit solutions of some utility maximization problems in incomplete markets*, Stochastic Process. Appl., 114 (2004), pp. 109–125.

[39] A. V. D. VAART, *Asymptotic statistics*, vol. 22 of Statistical and Probabilistic Mathematics, Cambridge University Press, New York, first ed., 1998.

DEPARTMENT OF BANKING AND FINANCIAL MANAGEMENT, UNIVERSITY OF PIRAEUS, PIRAEUS, GREECE
E-mail address: anthropel@unipi.gr

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213
E-mail address: scottrob@andrew.cmu.edu

DEPARTMENT OF MATHEMATICS & STATISTICS, BOSTON UNIVERSITY, BOSTON, MA 02215
E-mail address: kspiliop@math.bu.edu