PRICING FOR LARGE POSITIONS IN CONTINGENT CLAIMS

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ABSTRACT. Approximations to utility indifference prices are provided for a contingent claim in the large position size limit. Results are valid for general utility functions on the real line and semi-martingale models. It is shown that as the position size approaches infinity, the utility function's decay rate for large negative wealths is the primary driver of prices. For utilities with exponential decay, one may price like an exponential investor. For utilities with a power decay, one may price like a power investor after a suitable adjustment to the rate at which the position size becomes large. In a sizable class of diffusion models, limiting indifference prices are explicitly computed for an exponential investor. Furthermore, the large claim limit arises endogenously as the hedging error for the claim vanishes.

1. INTRODUCTION

The last two decades have seen an explosive growth in the financial derivatives market. According to [5], the notional size of the over-the-counter derivatives market increased from $\$94$ trillion in June of 2000, to $\$707$ trillion as of June of 2012. Due to their complexity, these contracts are often neither easily traded nor hedged. The purpose of this article is to identify the limits of utility based indifference prices as the position size becomes large, taking market incompleteness into account. In particular, conditions are sought under which investors with differing utility functions have the same limiting indifference price.

Let q denote the position size in a derivative contract which the investor holds, but can not trade. The goal is to study the (average bid) utility indifference price $p = p_U(x, q)$ in the limit as $q \to \infty$. Here, U is the investor's utility function and x is the initial capital. p is defined through the balance equation

$$u_U(x - qp, q) = u_U(x, 0), \tag{1.1}$$

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where, given (x, q), the value function $u_U(x, q)$ represents the optimal utility an investor may achieve by trading in the underlying market. The idea behind indifference pricing traces back to [24] and the topic has been extensively studied : see [9] for a comprehensive review. Clearly, to compute p, knowledge of the value function u_U is crucial. However, except for a few special utility functions and models, u_U is not explicitly known. This presents the primary challenge to obtaining indifference prices and motivates the study of their approximation.

One approximation occurs in the small claim limit (i.e. as $|q| \downarrow 0$). Here, [12, 13] obtain first order approximations in a Brownian setting, while [3, 39, 37, 28, 32, 4] obtain asymptotic results, regarding both pricing and hedging strategies, for the exponential utility (as well as for general utilities on the real line in [37, 4]) in varying degrees of generality. In [31], small claim approximations are obtained for utilities defined on the positive axis. A key feature present in all these articles is that the market is kept constant as the claim size becomes small.

A second approximation occurs by taking a sequence of markets which is becoming complete in some sense. In [13], asymptotics are provided in a basis-risk model as the correlation parameter between the hedgeable and unhedgeable shocks approaches one (this case is treated in detail in Section 5). [29] obtains results in a Brownian setting in the case of both fixed and vanishing portfolio constraints. A key feature of these papers is that as the market changes, the claim size remains fixed.

In contrast to the above asymptotics, for large positions it is desirable to allow both the position size and market to vary. This follows by considering the relationship between owning and hedging a claim. Indeed, as shown in Section 5, in a Brownian setting with exponential utility, given the opportunity to purchase claims for an arbitrage free price, the optimal position size (see [25]) satisfies the heuristic relationship

$$\text{risk aversion} \times \text{position size} \times \text{hedging error} \approx \text{constant}. \tag{1.2}$$

Thus, under the assumption of optimal investment, (1.2) implies that when large market sizes are observed, either investor risk aversion is approximately zero or hedging errors are vanishingly small. In this article, the primary focus is on markets, like those considered in [13], where the latter case holds. Therefore, with risk aversion fixed, the market is allowed to vary in conjunction with position size, and pricing results can be thought of as treating the regime where position size $\times$ hedging error $\approx$ constant. For the sake of completeness, however, in Section 5, pricing results are also obtained for vanishing risk aversion if the investor has exponential utility: see Remarks 5.4, 5.7 and 5.10.

In view of the above, for a given sequence of markets $\mathbf{M}^n$, claims h^n, and position sizes q_n, the "large claim limit" is defined through two requirements. First (clearly), that $q_n \to \infty$. Second, that asymptotically the $\mathbf{M}^n$ do not permit arbitrage: see Assumptions 3.2, 4.2 and Proposition 6.1 for a precise formulation of this statement. Note that this certainly includes the regime where additionally, (1.2) holds, but does not require it. The idea is that for limiting prices to have any meaning, the sequence of markets asymptotically should not allow for risk-less profit.

To avoid cumbersome admissibility restrictions, the claims h^n are assumed to be uniformly bounded and utility functions U on the whole real line are considered (see [42] for related results on optimal position taking for utilities on $(0, \infty)$ in a fixed market). The main results of the paper, Theorem 3.3 and Proposition 4.3 state that the decay of U for large negative wealths is the primary investor-specific determinant of the limiting indifference price. For exponential decay, Theorem 3.3 shows prices come together for all utilities U with the same rate of decay. This theorem has the simple, practical message that an investor with utility function U should ascertain if there is some $\alpha > 0$ such that $\lim_{x\downarrow-\infty} -(1/x)\log(-U(x)) = \alpha$ (see Examples 2.5 and 2.6 for two important, non exponential utilities), and if so, price like an exponential investor with risk aversion α when the position size is large. Furthermore, under the additional restriction that $\log(U(x)/(-e^{-\alpha x}))$ remains bounded as $x \downarrow -\infty$ (see Definition 3.6) Theorem 3.8 shows that the total monetary error incurred by approximating via exponential prices remains bounded, providing a rate of how fast prices come together.

For utilities with power-like decay, i.e. $\lim_{x\downarrow-\infty} -U(x)/(-x)^p = 1/l$ for some $p > 1$, $l > 0$, Proposition 4.3 shows that prices do come together, but only after the position size q_n has been suitably altered to $(-u_U^n(x,0))^{1/p} q_n$ where u^n is the value function in the nth market. Even though this implies that prices will typically not come together for the non-adjusted sizes q_n, Proposition 4.3 still allows for prices to be computed using the more manageable utility function $U_p(x) = -(1/p)(-x)^p, x \approx -\infty$ (more precisely, for the convex conjugate function $V_p(y) = (1/q)y^q$ where $1/p + 1/q = 1$), especially in the asymptotically complete setting where one may approximate $u_U^n(x,0)$ with the more readily computable $u_U^\infty(x,0)$: the value function in the limiting complete market.

The convergence results in Section 3 motivate the study of limiting indifference prices for an exponential investor. Using the results of [45], in Section 5 limiting prices, as well as monetary errors, are computed for a class of basis risk models which generalize those studied in [13, 23]. The appropriateness of this class for large claim analysis follows by considering the claim as an option on an asset for which no liquid market

exists, but for which there is a liquid market for some "closely related" ([13]) asset. For example, such a scenario occurs when the option is written on an asset which could be traded, but hedging with an index is preferable due to transactions costs ([23]). Proposition 5.3 identifies essentially three limiting indifference prices, depending on the rate at which q_n becomes large, although when purchasing optimal quantities one always falls into the rate at which non-trivial pricing effects occur: see Remark 5.6 and the discussion following for a financial interpretation of the three regimes. The trichotomy of limiting prices appears to be general feature of large claim analysis, and is intimately related to the theory of Large Deviations. Section 5 also motivates the relationship in (1.2), discussing its validity from an equilibrium standpoint using the notion of *partial equilibrium price quantities* introduced in [1].

Section 5 concludes by giving an important numerical example, where optimal position sizes and indifference prices are explicitly computed for a put option in the basis risk model as the correlation ρ between hedgeable and unhedgeable shocks approaches one. As Figure 1 therein shows, indifference prices converge rapidly to a limiting value which differs substantially from the complete market (i.e. $\rho = 1$) price. Furthermore, optimal position sizes are modest in size (approximately $1 - 5$ options) even when ρ is very close to one ($\rho = .99 - .999$). This example shows that even though the analysis performed within this article is zeroth order, in the sense that market frictions are not taken into account, the resultant pricing effects are non-trivial and are valid within a regime where it is reasonable to ignore such frictions.

In addition to the over-the-counter derivatives market, large claim limit pricing has applications to the insurance industry. [6] considers large positions pricing for liabilities with both financial and insurancial risks, such as revenue insurance contracts or mortality derivatives. Here, the claim is actually the sum of the contracts. Lastly, as it is well known that for exponential investors position size and risk aversion affect indifference prices only through their product, large claim analysis, if one does not consider optimal purchase quantities as in (1.2), can be thought of as treating the case where risk aversion approaches infinity. Large risk aversion asymptotics have been considered in [7, 8] and the results therein are consistent with those in the present paper when the market is fixed.

Section 2 introduces the general setup. Section 3 gives the main results for the exponential decay case. It provides examples to highlight the minimality of the assumptions and discusses extensions to random endowments and alternative risk aversions. Section 4 gives the corresponding results for the power decay case. Section 5 provides results for the class of basis risk models. Sections 6, 7 and Appendix A contain the proofs.

2. SETUP

2.1. **Assets and Martingale Measures.** Let $T > 0$ denote the horizon. For each n, let $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n)$ be a filtered probability space where the filtration $\mathbb{F}^n = (\mathcal{F}^n_t)_{0\le t\le T}$ satisfies the usual conditions of right-continuity and $\mathbb{P}^n$ completeness. Assume $\mathcal{F}^n_T = \mathcal{F}^n$ and zero interest rates so that the safe asset S_0 is identically one. The risky asset $S^n = (S^n_1, \dots, S^n_{d_n})$ is a locally bounded, $\mathbb{R}^{d_n}$-valued semi-martingale. In addition to being able to trade in S^n, the investor owns q_n units of a non-tradable, $\mathcal{F}^n$ measurable, contingent claim h^n.

For $n \in \mathbb{N}$, denote by $\mathcal{M}^n$ the set of probability measures $\mathbb{Q}^n \ll \mathbb{P}^n$ on $\mathcal{F}^n$ such that S^n is a local martingale under $\mathbb{Q}^n$. Recall that for any $\mu \ll \mathbb{P}^n$ on $\mathcal{F}^n$, the relative entropy of μ with respect to $\mathbb{P}^n$ is given by $H(\mu \mid \mathbb{P}^n) := \mathbb{E}^{\mathbb{P}^n}\left[(d\mu/d\mathbb{P}^n)\log(d\mu/d\mathbb{P}^n))\right]$. Define:

Definition 2.1. $\tilde{\mathcal{M}}^n := \{\mathbb{Q}^n \in \mathcal{M}^n \,:\, H(\mathbb{Q}^n \mid \mathbb{P}^n) < \infty\}$.

Two important examples are:

Example 2.2 (Basis Risk Model with High Correlation). There are traded and non-traded assets S and Y with dynamics

$$dS^n_t = S^n_t\left(\mu(Y_t)dt + \sigma(Y_t)(\rho_n dW_t + \sqrt{1-\rho_n^2}dB_t)\right); \qquad dY_t = b(Y_t)dt + a(Y_t)dW_t; \tag{2.1}$$

where W, B are independent Brownian motions and $\rho_n \in (-1, 1)$. When $h^n = h(Y_T)$ is an option written on the non-traded asset Y, utility-based pricing using exponential utility has been extensively studied : see [23, 12, 45, 35, 21, 22, 9, 34, 43] amongst others. Large claim pricing results for this class of models are given in Section 5. When $h^n = h(Y_T, S^n_T)$, see [26, 9, 43] for utility-based pricing results based on Partial Differential Equation methods.

Example 2.3 (Large Markets). As in [14], the claim is written on a large market consisting of infinitely many assets, but the investor is restricted to trading in only the first n assets. As an example, let the assets evolve according to $dS^i_t = S^i_t\left(\alpha^i dt + dW^i_t\right)$ for $i = 1, 2, \dots$. Here, $W^1, W^2, \dots$ is a sequence of independent Brownian motions and $\sum_{i=1}^\infty (\alpha^i)^2 < \infty$. h is a bounded claim measurable with respect to the sigma field $\sigma(W^1, W^2, ...)$. For concreteness, h can be either an index option (see [10]) or a suitably weighted sum of independent claims h^n where h^n is a function of S^n (see [33]). Pricing results for this latter case are found in [38].

2.2. **Utility Functions.** Throughout the article, a *utility function* will denote any strictly increasing, strictly concave function $U \in C^2(\mathbb{R})$. The following class is of particular importance:

Definition 2.4. For $\alpha > 0$, denote by $\mathcal{U}_\alpha$ the set of utility functions satisfying $\lim_{x\uparrow\infty} U(x) = 0$ and

$$\lim_{x\downarrow-\infty} -\frac{1}{x}\log\left(-U(x)\right) = \alpha. \tag{2.2}$$

The canonical example of $U \in \mathcal{U}_\alpha$ is the exponential utility:

$$U_\alpha(x) := -\frac{1}{\alpha}e^{-\alpha x}. \tag{2.3}$$

However, $\mathcal{U}_\alpha$ is a richer class of utility functions, as the following examples show.

Example 2.5 (Social Planner). Suppose that there are several investors with respective utilities $U_j \in \mathcal{U}_{\alpha_j}, \alpha_j > 0$ for $j = 1, \ldots, J$. The social planner considers the aggregate wealth of the individual investors, taking into account their utilities. The planner's utility function then takes the form $U(x) := \sum_{j=1}^J w_j U_j(x)$, where $\{w_j\}_{j=1}^J$ are the respective weights of individual investors. It readily follows that $U \in \mathcal{U}_\alpha$ for $\alpha := \max_{j=1,\ldots,J} \alpha_j$.

Example 2.6 (Representative Agent). This example concerns the representative agent from equilibrium theory, which dates back to [36] and has been extensively studied : see [30, 11, 17, 2] amongst others. Recall that for a utility function U, the *absolute risk aversion* is defined by

$$\alpha_U(x) := -\frac{U''(x)}{U'(x)}. \tag{2.4}$$

For $j = 1, ..., J$, let U_j be utility functions with $\lim_{x\uparrow\infty} U_j(x) = 0$, and with risk aversions that satisfy $i)$ there is a $K_j > 1$ such that $1/K_j \leq \alpha_{U_j}(x) \leq K_j$ for all $x \in \mathbb{R}$; and $ii)$ $\lim_{x\downarrow-\infty} \alpha_{U_j}(x) = \alpha_j > 0$. L'Hôpital's rule implies $U_j \in \mathcal{U}_{\alpha_j}$. For $v_1, ..., v_J$ positive coefficients, the representative agent's utility is:

$$U_v(x) := \sup_{y_1+\ldots+y_J=x} \sum_{j=1}^J v_j U_j(y_j). \tag{2.5}$$

Here, [2, Theorem 4.2] implies for all v that $U_v \in \mathcal{U}_\alpha$ where $\alpha := \left(\sum_{j=1}^J (1/\alpha_j)\right)^{-1}$.

For a general $U \in \mathcal{U}_\alpha$, Lemma A.2 below shows U satisfies the Inada conditions $\lim_{x\downarrow-\infty} U'(x) = \infty$, $\lim_{x\uparrow\infty} U'(x) = 0$ as well as the conditions of Reasonable Asymptotic Elasticity ([41]):

$$\liminf_{x\downarrow-\infty} \frac{xU'(x)}{U(x)} > 1; \qquad \limsup_{x\uparrow\infty} \frac{xU'(x)}{U(x)} < 1. \tag{2.6}$$

As $U \in \mathcal{U}_\alpha$ is bounded from above by assumption, the normalization $U(\infty) = 0$ is performed only to ensure $\log(-U(x))$ is defined for all $x \in \mathbb{R}$. Note that U being bounded from above includes both the case where the absolute risk aversion α_U from (2.4) satisfies $1/K_U \leq \alpha_U(x) \leq K_U$ and where U displays power-like behavior for large wealths (i.e. $U(x) \approx (1/p)x^p$) for some $p < 0$: two common assumptions in the optimal investment literature. However, the case $U(x) \approx (1/p)x^p, 0 < p < 1$ is not covered. Lastly, for any utility function U, denote by V the convex conjugate to U:

$$V(y) := \sup_{x \in \mathbb{R}} \{U(x) - xy\}. \tag{2.7}$$

It is straightforward to check that for $U \in \mathcal{U}_\alpha$, $V \in C^2(0, \infty)$ is strictly convex and can be continuously extended to 0 by setting $V(0) = U(\infty) = 0$. Furthermore

$$\lim_{y \uparrow \infty} \frac{V(y)}{V_\alpha(y)} = 1; \qquad V_\alpha(y) := \sup_{x \in \mathbb{R}} \{U_\alpha(x) - xy\} = \frac{1}{\alpha} y(\log(y) - 1). \tag{2.8}$$

2.3. **The Value Function and Utility Indifference Price.** Let $n \in \mathbb{N}$. A trading strategy π^n is admissible if it is predictable, S^n integrable under $\mathbb{P}^n$, and such that the gains process $(\pi^n \cdot S^n)$ remains above a constant a (which may depend upon π^n) almost surely on $[0, T]$. Denote by $\mathcal{A}^n$ the set of admissible trading strategies. Now, let $\alpha > 0$ and $U \in \mathcal{U}_\alpha$. For $x, q \in \mathbb{R}$, the value function $u_U^n(x, q; h^n)$ is defined by

$$u_U^n(x, q; h^n) := \sup_{\pi^n \in \mathcal{A}^n} \mathbb{E}^{\mathbb{P}^n} \left[U\left(x + (\pi^n \cdot S^n)_T + q h^n\right) \right]. \tag{2.9}$$

Note that n appears in two places. The superscript n outside the parentheses accounts for the dependence of S^n and $\mathcal{A}^n$ upon n. The n in h^n represents the fact that the claim may be changing with n. In the case where $h^n \equiv 0$, set

$$u_U^n(x) := u_U^n(x, q; 0). \tag{2.10}$$

Define the average utility indifference (bid) price $p_U^n(x, q; h^n)$ implicitly as the solution to the equation

$$u_U^n(x) = u_U^n(x - q p_U^n(x, q; h^n), q; h^n). \tag{2.11}$$

Thus, $p_U^n(x, q; h^n)$ is the amount an investor would pay per unit of h^n so as to be indifferent between owning and not owning q units of the claim. Note that under Assumptions 3.1 and 3.2 below, the indifference price is well defined for any utility function U satisfying the Inada and Reasonable Asymptotic Elasticity conditions : see [37, Proposition 7.1].

Remark 2.7. For the exponential utility U_α it is well known that the indifference price does not depend upon initial capital. Thus, write $p^n_{U_\alpha}(q;h^n)$ for $p^n_{U_\alpha}(x,q;h^n)$.

3. Indifference Prices in the Large Claim Limit

$p^n_U(x,q_n;h^n)$ is now studied in the limit as $q_n \to \infty$. Proofs of all assertions made herein are given in Section 6. The main result states that for any $x_1, x_2 \in \mathbb{R}$ and $U_1, U_2 \in \mathcal{U}_\alpha$, as $q_n \to \infty$, the difference between $p^n_{U_1}(x_1,q_n;h^n)$ and $p^n_{U_2}(x_2,q_n;h^n)$ vanishes. The intuition for this result is gained by inspecting the indifference pricing formula in [37, Proposition 7.1 (vi)], which is valid for $q_n > 0$ in the current setup:

$$p^n_U(x,q_n;h^n) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n_V} \left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \frac{1}{q_n}\alpha^n_U(\mathbb{Q}^n) \right), \tag{3.1}$$

where the *penalty functional* α^n_U is given by

$$\alpha^n_U(\mathbb{Q}^n) := \inf_{y>0} \frac{1}{y}\left(\mathbb{E}^{\mathbb{P}^n}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)\right] + xy - u^n_U(x)\right). \tag{3.2}$$

Here, V is from (2.7), and $\tilde{\mathcal{M}}^n_V$ is the subset of $\mathcal{M}^n$ such that $\mathbb{E}^{\mathbb{P}^n}\left[V(d\mathbb{Q}^n/d\mathbb{P}^n)\right] < \infty$.

For $U \in \mathcal{U}_\alpha$, Lemma A.3 below implies that $\tilde{\mathcal{M}}^n_V = \tilde{\mathcal{M}}^n$ and hence the variational problem above is taken over the same set of measures. Furthermore, as $q_n \to \infty$, the factor $(1/q_n)$ in front of $\alpha^n_U(\mathbb{Q}^n)$ means small values of $V(z)$ may be disregarded. Therefore, by (2.8), one may replace $V(z)$ with $V_\alpha(z)$ in (3.2). Calculation then shows that $\alpha^n_U(\mathbb{Q}_n) \approx x + (1/\alpha)\left(\log(-\alpha u^n_U(x)) + H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right)$, and hence

$$p^n_U(x,q_n;h^n) \approx \frac{1}{q_n\alpha}\log(-\alpha u^n_U(x)) + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n}\left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \frac{1}{\alpha q_n} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right).$$

Thus, if $\limsup_{n\uparrow\infty} u^n_U(x) < U(\infty) = 0$, the only part of $p^n_U(x,q_n;h^n)$ dependent upon either x or U vanishes, and prices come together.

3.1. Convergence of Prices.

The above argument is made precise under the following assumptions. First, it is assumed that h^n is uniformly bounded in n:

Assumption 3.1. $\|h\| := \sup_n \|h^n\|_{L^\infty(\Omega^n,\mathcal{F}^n,\mathbb{P}^n)} < \infty$.

The next assumption essentially rules out arbitrage opportunities when investing in S^n, both for each n (see [37, Assumption 1.4]) and as $n \uparrow \infty$. Recall, from Section 2.1 the definitions of $\tilde{\mathcal{M}}^n$ and the relative entropy $H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)$.

Assumption 3.2. $\tilde{\mathcal{M}}^n \neq \emptyset$ for each n and $\limsup_{n\uparrow\infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right) < \infty$.

Regarding Assumption 3.2, it is well known (see [25, 37, 15]) that for the exponential utility U_α,

$$u^n_{U_\alpha}(x, q_n; h^n) = -\frac{1}{\alpha}\exp\left(-\alpha x - \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n}\left(\alpha q_n \mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right)\right). \tag{3.3}$$

Now, suppose that $q_n \equiv 0$ and Assumption 3.2 does not hold : i.e. for each $k = 1, 2, \ldots$ there is an integer n_k such that $\inf_{\mathbb{Q}^{n_k} \in \tilde{\mathcal{M}}^{n_k}} H\left(\mathbb{Q}^{n_k} \mid \mathbb{P}^{n_k}\right) > k$. As the infimum is strictly bigger than k, (3.3) implies the existence of an admissible trading strategy π^{n_k} such that $\mathbb{P}^{n_k}\left[(\pi^{n_k} \cdot S^{n_k})_T \geq k/(2\alpha)\right] \geq 1 - e^{-k/2}$. Therefore, there exists a sequence of admissible trading strategies such that the probability that terminal wealth fails to grow like k decreases to 0 exponentially fast on the order of k. An asymptotic arbitrage of the form above is similar to a *strong arbitrage* as defined in [20]: which contains a detailed discussion on the topic.

The main result is now presented:

Theorem 3.3. *Let Assumptions 3.1 and 3.2 hold. Let $\alpha > 0$. If $q_n \to \infty$ then for all $U_1, U_2 \in \mathcal{U}_\alpha$ and $x_1, x_2 \in \mathbb{R}$*

$$\lim_{n \uparrow \infty}\left|p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n)\right| = 0. \tag{3.4}$$

Remark 3.4. As for any $U \in \mathcal{U}_\alpha$ and $x \in \mathbb{R}$, $p^n_U(x, q_n; h^n) = -p^n_U(x, -q_n; -h^n)$, the convergence in Theorem 3.3 remains valid for $q_n \to -\infty$ as well.

3.2. **Convergence of Total Quantities.** Results are now stated which, for $x_1, x_2 \in \mathbb{R}$ and $U_1, U_2 \in \mathcal{U}_\alpha$, ensure that the total monetary difference

$$q_n\left|p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n)\right|,$$

remains bounded as $n \uparrow \infty$. Thus, an investor with utility function $U \in \mathcal{U}_\alpha$ and initial capital $x \in \mathbb{R}$ may price using the exponential utility U_α and the error in the total amount of money spent remains bounded, even in the large claim limit.

This type of convergence will not take place under the general conditions of Theorem 3.3 (see Example 3.14) and requires stronger assumptions on the utility functions. However, under these stronger assumptions, it is not necessary for the claim to remain uniformly bounded as in Assumption 3.1. Therefore, assume:

Assumption 3.5. For each n, $h^n \in L^\infty\left(\Omega^n, \mathcal{F}^n, \mathbb{P}^n\right)$.

As for the class of utility functions, convergence results are proved for $\tilde{\mathcal{U}}_\alpha \subset \mathcal{U}_\alpha$ defined by:

Definition 3.6.

$$\tilde{\mathcal{U}}_\alpha := \left\{ U \in \mathcal{U}_\alpha : 0 < \liminf_{x \downarrow -\infty} \frac{U(x)}{U_\alpha(x)} \leq \limsup_{x \downarrow -\infty} \frac{U(x)}{U_\alpha(x)} < \infty \right\}. \tag{3.5}$$

Remark 3.7. In Example 2.5, if for $j = 1, ..., J$, $U_j(x) = -(1/\alpha_j)e^{-\alpha_j x}$ then $U \in \tilde{\mathcal{U}}_\alpha$. Similarly, in Example 2.6, if the U_j therein additionally satisfy $U_j \in \tilde{\mathcal{U}}_{\alpha_j}$ for $j = 1, .., J$ then $U \in \tilde{\mathcal{U}}_\alpha$.

With these definitions and assumptions, total monetary errors are bounded, where the bound only depends on the utility functions, initial capitals, and limiting minimal relative entropy as given in Assumption 3.2:

Theorem 3.8. *Let $\alpha > 0$. Let Assumptions 3.2 and 3.5 hold. If $q_n \to \infty$ then for all $U_1, U_2 \in \tilde{\mathcal{U}}_\alpha$ and $x_1, x_2 \in \mathbb{R}$ there exists a constant C depending only on U_1, U_2, x_1, x_2 and $\limsup_{n\uparrow\infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H(\mathbb{Q}^n \mid \mathbb{P}^n)$ such that*

$$\limsup_{n\uparrow\infty} q_n \left| p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n) \right| \leq C. \tag{3.6}$$

Remark 3.9. Suppose that the exponential utility price $p^n_{U_\alpha}(q_n; h^n)$ of Remark 2.7 converges to some limit p_α. Even though for $x \in \mathbb{R}$, $U \in \tilde{\mathcal{U}}_\alpha$ both $\lim_{n\uparrow\infty} p^n_U(x, q_n; h^n) = p_\alpha$ and $\limsup_{n\uparrow\infty} q_n |p^n_U(x, q_n; h^n) - p^n_{U_\alpha}(q_n; h^n)| < \infty$, it still might be that $\lim_{n\uparrow\infty} q_n |p^n_U(x, q_n; h^n) - p_\alpha| = \infty$ (see Proposition 5.8 for examples). Here, even though the monetary error incurred by using exponential utility prices (instead of the original utility U) remains bounded, the error introduced by using the *limiting* exponential price tends towards infinity.

3.3. **Discussion.**

3.3.1. *Pricing in the Presence of Random Endowments.* Theorems 3.3 and 3.8 readily extend to an investor who, rather than starting with an initial capital, starts with a random endowment. Specifically, for each n let X^n be an $\mathcal{F}^n_T$ measurable random variable representing the random endowment. In accordance with [37, Assumption 1.3] the following is assumed:

Assumption 3.10. There exists $\underline{x}^n, \bar{x}^n \in \mathbb{R}$ and a trading strategy $\bar{\pi}^n \in \mathcal{A}^n$ such that $\mathbb{P}^n$ almost surely $\underline{x}^n \leq X^n \leq \bar{x}^n + (\bar{\pi}^n \cdot S^n)_T$.

Note that Assumption 3.10 certainly holds when X^n is bounded. For $U \in \mathcal{U}_\alpha$ denote by $u^n_U(X^n, q; h^n)$ the value function holding q units of h^n as well as X^n. When $q = 0$ set $u^n_U(X^n) = u^n_U(X^n, 0; h^n)$. The indifference price (which is well defined under Assumptions 3.1, 3.2 and 3.10) is denoted by $p^n_U(X^n, q; h^n)$

and defined implicitly through $u_U^n(X^n) = u_U^n(X^n - qp_U^n(X^n, q; h^n), q; h^n)$. To connect prices in the large claim limit, the following asymptotic assumption is enforced:

Assumption 3.11. For the sequences of constants $\{\underline{x}^n\}, \{\bar{x}^n\}$ from Assumption 3.10, $\liminf_{n\uparrow\infty} \underline{x}_n > -\infty$ and $\limsup_{n\uparrow\infty} \bar{x}_n < \infty$.

Note that Assumption 3.11 holds if the random endowments $\{X^n\}$ are uniformly bounded in n. Under Assumptions 3.10 and 3.11, the results of Theorems 3.3 and 3.8 carry over:

Proposition 3.12. *Let Assumptions 3.1, 3.2 hold. Let $\alpha > 0$ and $U_1, U_2 \in \mathcal{U}_\alpha$. Let $\{X_1^n\}$ and $\{X_2^n\}$ be two sequences of random endowments such that Assumptions 3.10 and 3.11 hold for both $\{X_1^n\}$ and $\{X_2^n\}$ respectively (with associated constants $\underline{x}_1^n, \bar{x}_1^n$ and $\underline{x}_2^n, \bar{x}_2^n$ and strategies $\bar{\pi}_1^n, \bar{\pi}_2^n$). Then for all $q_n \to \infty$*

$$\lim_{n\uparrow\infty} \left| p_{U_1}^n(X_1^n, q_n; h^n) - p_{U_2}^n(X_2^n, q_n; h^n) \right| = 0.$$

Additionally, if $U_1, U_2 \in \tilde{\mathcal{U}}_\alpha$ then

$$\limsup_{n\uparrow\infty} q_n \left| p_{U_1}^n(X_1^n, q_n; h^n) - p_{U_2}^n(X_2^n, q_n; h^n) \right| < \infty.$$

3.3.2. *Pricing for $U \in \mathcal{U}_\alpha$, $\hat{U} \in \mathcal{U}_\delta$ when $\alpha \neq \delta$.* Let $\alpha, \delta > 0, \alpha \neq \delta$. As shown in [15, 27], $p_{U_\alpha}^n(q_n; h^n)$ depends on the product αq_n of the risk aversion and position size. Thus, Theorem 3.3 has the immediate corollary that the prices $p_U^n(x_1, q_n; h^n)$ and $p_{\hat{U}}^n(x_2, \alpha q_n/\delta; h^n)$ converge as $n \uparrow \infty$ for $U \in \mathcal{U}_\alpha, \hat{U} \in \mathcal{U}_\delta$, $x_1, x_2 \in \mathbb{R}$. Indeed, as $p_{U_\alpha}^n(q_n; h^n) = p_{U_\delta}^n(\alpha q_n/\delta; h^n)$:

$$p_U^n(x_1, q_n; h^n) - p_{\hat{U}}^n(x_2, \tfrac{\alpha}{\delta} q_n; h^n) = p_U^n(x_1, q_n; h^n) - p_{U_\alpha}^n(q_n; h^n) + p_{U_\delta}^n(\tfrac{\alpha}{\delta} q_n; h^n) - p_{\hat{U}}^n(x_2; \tfrac{\alpha}{\delta} q_n; h^n),$$

and hence the convergence holds. Additionally, the monetary errors from Theorem 3.8 remain bounded for $U \in \tilde{\mathcal{U}}_\alpha$ and $\tilde{\mathcal{U}}_\delta$ for position sizes q_n and $\alpha q_n/\delta$, respectively. Lastly, as seen in Proposition 5.3 below, it is even possible for the non-adjusted prices $p_U^n(x_1, q_n; h^n)$ and $p_{\hat{U}}^n(x_2, q_n; h^n)$ to come together, however, they certainly may diverge as well.

3.3.3. *Examples.* Examples are given to highlight the necessity of a) (2.2) in Definition 2.4 and b) (3.5) in Definition 3.6. Each of the examples considers the one period trinomial model*. Here the filtered space is

*This model can be embedded into a continuous time framework by requiring S to be constant until a jump at time $T = 1$ and using the filtration generated by S

$\Omega = \{1,2,3\}$, $\mathcal{F} = \mathcal{P}^{\Omega}$, $F_0 = \{\emptyset, \Omega\}$, $F_1 = \mathcal{F}$. $S_0 \equiv 1$ and S_1, h take the respective values:

$$S_1(1) = 1+u; \quad S_1(2) = 1; \quad S_1(3) = 1-u, \qquad h(1) = h(3) = h; \quad h(2) = 0. \tag{3.7}$$

where $0 < u < 1$, and $h \neq 0$. Lastly, for each n let $0 < p_n < 1/2$ and define $\mathbb{P}^n$ by $\mathbb{P}^n(1) = \mathbb{P}^n(3) = p_n$, $\mathbb{P}^n(2) = 1 - 2p_n$. It is clear that Assumptions 3.1 and 3.2 hold, and, for any utility function U satisfying the Inada conditions, the indifference price $p^n_U(x, q_n; h)$ satisfies

$$U(x) = 2p_n U(x - q_n p^n_U(x, q_n; h) + q_n h) + (1 - 2p_n) U(x - q_n p^n_U(x, q_n; h)). \tag{3.8}$$

Example 3.13 (On the Necessity of (2.2)). The first example shows that condition (2.2) is minimal, at least within the class of utility functions U such that $\log(\alpha_U)$ is bounded, to guarantee convergence of prices. Let $\hat{U}$ be a utility function satisfying i) $\lim_{x\uparrow\infty} \hat{U}(x) = 0$ and ii) for some $K_U > 1$, $(1/K_U) \leq \alpha_{\hat{U}}(x) \leq K_U$ for all $x \in \mathbb{R}$. Assume that (2.2) fails : i.e. for some $\underline{\alpha} < \overline{\alpha}$:

$$\frac{1}{K_U} \leq \underline{\alpha} = \liminf_{x\downarrow-\infty} -\frac{1}{x} \log\left(-\hat{U}(x)\right) < \limsup_{x\downarrow-\infty} -\frac{1}{x} \log\left(-\hat{U}(x)\right) = \overline{\alpha} \leq K_U. \tag{3.9}$$

Let $x_n \downarrow -\infty$ be such that $-(1/x_n)\log(-\hat{U}(x_n))$ does not converge. In the one period model with $q_n = \log(-\hat{U}(x_n))$, $p_n = (1/2)(1 - e^{-q_n})$ and $h > (1/\underline{\alpha})$:

1) For all $x \in \mathbb{R}$, $p^n_{\hat{U}}(x, q_n; h)$ does not converge as $n \uparrow \infty$.
2) For all $\alpha > 0$, all $U \in \mathcal{U}_\alpha$ and all $x \in \mathbb{R}$, $\lim_{n\uparrow\infty} p^n_U(x, q_n; h) = \min\left\{\alpha^{-1}, h\right\}$.

Example 3.14 (On the necessity of $\tilde{\mathcal{U}}_\alpha$). If $U \in \mathcal{U}_\alpha, U \notin \tilde{\mathcal{U}}_\alpha$, the convergence result in Theorem 3.8 may fail for models satisfying Assumptions 3.1 and 3.2. Indeed, let $U \in \mathcal{U}_\alpha$ be such that for some $M > 0$, $U(x) = (-1/x)U_\alpha(x)$ if $x \leq -M$. Such a U can easily be constructed. Then, in the one period trinomial model with $p_n = (1/2)(1 - e^{-n})$, $h = 1$ and $q_n = n^2$, for all $x \in \mathbb{R}$:

$$\lim_{n\uparrow\infty} q_n \left(p^n_U(x, q_n; h^n) - p^n_{U_\alpha}(q_n; h^n)\right) = \infty.$$

4. Power Tails

Results similar to Theorem 3.3 hold for utility functions with power-like decay for large negative wealths. However, unlike in the exponential decay case, to obtain convergence of prices, the rate at which q_n becomes large must be suitably adjusted. Proposition 4.3 below makes the above statement precise. First, define the following class of utility functions:

Definition 4.1. Let $p > 1$ and $l > 0$. Define $\mathcal{U}_{p,l}$ to be the class of utility functions satisfying $\lim_{x\uparrow\infty} U(x) = 0$ and $\lim_{x\downarrow-\infty} -U(x)/(-x)^p = 1/l$.

Lemma A.2 below shows $U \in \mathcal{U}_{p,l}$ satisfies both the Inada and Reasonable Asymptotic Elasticity conditions. Furthermore, for $U \in \mathcal{U}_{p,l}$:

$$\lim_{y\uparrow\infty} \frac{V(y)}{V_p(y)} = 1; \qquad V_p(y) := \hat{l} y^{\gamma}, \tag{4.1}$$

where $\gamma := p/(p-1)$ is the conjugate exponential to p and $\hat{l} := (1/\gamma)(l/p)^{\gamma-1}$. Now, consider the indifference pricing formula (3.1). The factor $(1/q_n)$ again allows one to disregard small values of $V(z)$ when evaluating $\alpha_U^n(\mathbb{Q}^n)$ and hence to replace $V(z)$ with $V_p(z)$. A lengthy calculation shows $\alpha_U^n(\mathbb{Q}^n) \approx (-lu_U^n(x))^{1/p}\, \mathbb{E}^{\mathbb{P}^n}\left[(d\mathbb{Q}^n/d\mathbb{P}^n)^{\gamma}\right]^{1/\gamma}$, and hence

$$p_U^n(x, q_n; h^n) \approx \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \frac{(-lu_U^n(x))^{1/p}}{q_n} \mathbb{E}^{\mathbb{P}^n}\left[\left(\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)^{\gamma}\right]^{1/\gamma} \right).$$

Here, $\hat{\mathcal{M}}^n$ is the subset of $\mathcal{M}^n$ such that $\mathbb{E}^{\mathbb{P}^n}\left[V_p(d\mathbb{Q}^n/d\mathbb{P}^n)\right] < \infty$ and the substitution is allowed because for all $U \in \mathcal{U}_{p,l}$, $\tilde{\mathcal{M}}_V^n = \hat{\mathcal{M}}^n$ (see Lemma A.3). Therefore, by considering prices not for q_n, but rather $q_n(-u_U^n(x))^{1/p}$, one obtains

$$p_U^n(x, q_n(-u_U^n(x))^{1/p}; h^n) \approx \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \frac{l^{1/p}}{q_n} \mathbb{E}^{\mathbb{P}^n}\left[\left(\frac{d\mathbb{Q}^n}{d\mathbb{P}^n}\right)^{\gamma}\right]^{1/\gamma} \right),$$

and, as the right hand side of the above equation does not depend upon U or x, prices come together. To precisely state the above result, assume in an analogous manner to Assumption 3.2 that

Assumption 4.2. With $\hat{\mathcal{M}}^n := \left\{\mathbb{Q}^n \in \mathcal{M}^n \mid \mathbb{E}^{\mathbb{P}^n}\left[(d\mathbb{Q}^n/d\mathbb{P}^n)^{\gamma}\right] < \infty\right\}$, assume $\hat{\mathcal{M}}^n \neq \emptyset$ for each n and $\limsup_{n\uparrow\infty} \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \mathbb{E}^{\mathbb{P}^n}\left[(d\mathbb{Q}^n/d\mathbb{P}^n)^{\gamma}\right] < \infty$.

The main proposition now reads:

Proposition 4.3. *Let $p > 1$ and $l > 0$. Let Assumptions 3.1 and 4.2 hold. If $q_n \to \infty$ then for all $U_1, U_2 \in \mathcal{U}_{p,l}$ and $x_1, x_2 \in \mathbb{R}$*

$$\lim_{n\uparrow\infty} \left| p_{U_1}^n(x_1, q_n(-u_{U_1}^n(x_1))^{1/p}; h^n) - p_{U_2}^n(x_2, q_n(-u_{U_2}^n(x_2))^{1/p}; h^n) \right| = 0. \tag{4.2}$$

Remark 4.4. For $U \in \mathcal{U}_{p,l}$, $u_U^n(x)$ is typically not explicitly known. However, the above pricing results still remain valid if one can evaluate $\lim_{n\uparrow\infty} u_U^n(x)$, which, for example, is the case when the limiting market is complete: see Section 5.

5. Pricing for Basis Risk Models

The results of Section 3 are now specialized to the class of models from Example 2.2. Proofs of all assertions made herein are given in Section 7.

5.1. Model and Assumptions.

Let S^n and Y be as in (2.1). Assume $h^n = h(Y_T)$ where h is a function on the state space of Y. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is two-dimensional Wiener space and $\mathbb{F}$ is the augmentation of $\mathbb{F}^{W,B}$. Unless otherwise noted, all expectations are taken with respect to $\mathbb{P}$. Regarding h and the coefficients μ, σ, b and a in (2.1):

Assumption 5.1. $E := (l, u)$ for $-\infty \leq l < u \leq \infty$. $a, b : E \mapsto \mathbb{R}$ are continuous and $a^2(y) > 0$ for $y \in E$. Furthermore, the SDE for Y in (2.1) admits a strong solution with respect to the augmentation of $\mathbb{F}^W$ such that $\mathbb{P}[Y_t \in E, 0 \leq t \leq T] = 1$. $\mu, \sigma : E \mapsto \mathbb{R}$ are measurable such that $\sigma^2(y) > 0$ and $\lambda(y) := \mu(y)/\sigma(y)$ is bounded on E. $h : E \mapsto \mathbb{R}$ is a continuous and bounded function. Lastly, $\rho_n \in (-1, 1)$ for all n.

Assumption 5.1 implies Assumptions 3.1, 3.2 and 4.2 for any $q_n \to \infty$. The later two assumptions follow because $\hat{\mathbb{Q}}^n \in \mathcal{M}^n$ for

$$\frac{d\hat{\mathbb{Q}}^n}{d\mathbb{P}} := \mathcal{E}\left(-\int_0^{\cdot} \rho_n \lambda(Y_t) dW_t - \int_0^{\cdot} \sqrt{1-\rho_n^2}\lambda(Y_t) dB_t\right)_T, \tag{5.1}$$

and for $\gamma > 1$, $\sup_n \mathbb{E}\left[(d\hat{\mathbb{Q}}^n/d\mathbb{P})^\gamma\right] \leq \exp\left(\gamma T \sup_{y \in E} \lambda^2(y)\right)$. The following notation is used below, for $\rho \in \mathbb{R}$,

$$Z(\rho) := \mathcal{E}\left(-\rho \int_0^{\cdot} \lambda(Y_t) dW_t\right)_T; \qquad Z := Z(1). \tag{5.2}$$

Example 5.2. [23, 12] treat the case where S^n and Y are two geometric Brownian motions with instantaneous correlation ρ_n. This corresponds to $E = (0, \infty)$, $\mu(y) = \mu$, $\sigma(y) = \sigma$, $b(y) = by$ and $a(y) = ay$ for $\mu, b \in \mathbb{R}$ and $\sigma, a > 0$.

5.2. Large Claim Pricing.

Pricing results are now given in the joint limit as $q_n \to \infty$ and $\rho_n \to 1$. As indicated by Proposition 5.5 below, it is convenient to express q_n in terms of $(1-\rho_n^2)^{-1}$ and hence limiting

prices are computed for the following three regimes:

$$q_n = \frac{\gamma_n}{\alpha(1-\rho_n^2)} \quad \text{where} \quad \begin{cases} \text{(i) } \gamma_n \to 0 \text{ but } \gamma_n/(1-\rho_n^2) \to \infty \\ \text{(ii) } \gamma_n \to \gamma > 0 \\ \text{(iii) } \gamma_n \to \infty \end{cases} . \tag{5.3}$$

Define $\mathbb{Q} \sim \mathbb{P}$ via $d\mathbb{Q}/d\mathbb{P} := Z$ and $\hat{p} := \mathbb{E}^{\mathbb{Q}}[h(Y_T)]$. Note that $\mathbb{Q}$ is the unique martingale measure, and $\hat{p}$ the unique arbitrage free price for $h(Y_T)$, in the complete model where $\rho = 1$ and $\mathbb{F}$ is the augmentation of $\mathbb{F}^W$.

Proposition 5.3. *Let $\alpha > 0$, $U \in \mathcal{U}_\alpha$, $x \in \mathbb{R}$. Let Assumption 5.1 hold and assume $\rho_n \to 1$. Then,*

$$\lim_{n\uparrow\infty} p_U^n(x, q_n; h) = p_\alpha := \begin{cases} \textit{(i) :} \ \hat{p} = \mathbb{E}^{\mathbb{Q}}[h(Y_T)] \\ \textit{(ii) :} \ -(1/\gamma)\log \mathbb{E}^{\mathbb{Q}}\left[e^{-\gamma h(Y_T)}\right] \\ \textit{(iii) :} \ \inf_{y\in E} h(y) \end{cases} . \tag{5.4}$$

Remark 5.4 (Vanishing Risk Aversion). From (5.3) it is clear that large positions can also arise when the risk aversion α, rather than being constant, satisfies $\alpha_n \to 0$. Furthermore, as is discussed in the proof, the limiting prices in Proposition 5.3 depend only on the asymptotic behavior of $\gamma_n = \alpha_n q_n(1-\rho_n^2)$ and ρ_n. Thus, with γ_n converging to either $(i) : 0$, $(ii) : \gamma > 0$, or $(iii) : \infty$, the limiting prices of (5.4) remain valid as $\alpha_n \to 0$ and $\rho_n \to \rho \in [-1, 1]$: the only difference is that the expectations on the right hand side of (5.4) are now taken with respect $\mathbb{Q}^0(\rho)$, the minimal entropy martingale measure in the ρ market (note that at $\rho = 1$, $\mathbb{Q}^0(1) = \mathbb{Q}$). As such, the (suitably adjusted) limiting prices of Proposition 5.3 can be regarded as prices in the limit of large positions and vanishing risk aversion.

The economic interpretation behind Proposition 5.3 is as follows: if one ignores position size then, as $\rho_n \to 1$, the price for $h(Y_T)$ converges to the unique arbitrage free price in limiting model. In fact, this remains true even for large positions if the size grows slowly with respect to the market incompleteness factor $(1-\rho_n^2)$. However, if the position size grows in accordance with the market incompleteness factor then prices converge to the certainty equivalent of $h(Y_T)$ in the complete market for a position size of γ (see Section 5.5 for an alternative economic description of the limiting price). Lastly, if the position size grows at a rate faster than the market incompleteness factor, the investor risk aversion drives the price to the infimum of the arbitrage free prices (see (5.5)) in the pre-limiting models.

For a fixed risk aversion, the trichotomy of limiting prices is motivated by the following heuristic argument connecting prices to the theory of Large Deviations (see [16]). Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and h. For each n, assume h decomposes into $h = h_n + Z_n$ where h_n is replicable, using initial capital x_n and admissible trading strategy $\pi^n \in \mathcal{A}^n$, but Z_n is "completely unhedgeable" in that when pricing Z_n, it suffices to assume that one cannot trade in S^n. This implies $p^n_{U_\alpha}(q_n; h) = x_n - 1/(\alpha q_n) \log\left(\mathbb{E}\left[e^{-\alpha q_n Z_n}\right]\right)$ is the sum of the replicating capital for h_n and the certainty equivalent for Z_n.

Assume that h is asymptotically hedgeable in such a manner that $\{Z_n\}_{n\in\mathbb{N}}$ satisfies a Large Deviations Principle (LDP) [16] with scaling r_n and rate function I where $I(z) = 0 \Leftrightarrow z = 0$. For the sake of simplicity, assume further that the Z_n are uniformly bounded, taking values in a set E and that I is finite on E. Here, Varadhan's Integral Lemma, [16, Chapter 4] yields

$$\lim_{n\uparrow\infty} p^n_{U_\alpha}(q_n; h) - x_n = \begin{cases} \text{(i)}: 0 & q_n/r_n \to 0 \\ \text{(ii)}: \inf_{z\in E}\left(z + \frac{1}{\alpha\gamma} I(z)\right) & q_n/r_n \to \gamma \\ \text{(iii)}: \inf_{z\in E} z & q_n/r_n \to \infty \end{cases}.$$

Thus, if $q_n/r_n \to 0$ there is no large claim pricing effect. If $q_n/r_n \to \infty$ prices converge to the minimum value of the unhedgeable component. Lastly, if $q_n/r_n \to \gamma$ prices are adjusted via the rate function I.

5.3. **Optimal Quantities and Endogenous Large Positions.** The heuristic (1.2) of the Introduction is now verified. Then, using the notion of *partial-equilibrium price-quantities* (PEPQ) from [1], large positions are shown to endogenously arise even in a setting where the buyer must find a *single, risk averse* seller (as opposed to the more typical situation where there is a collection of market makers) in order to purchase the claims.

As can be seen in the proof of Proposition 5.3, the result for case (iii) holds when $\rho_n \equiv \rho$ is fixed and $\gamma_n \to \infty$. In this instance, as shown in [15] for exponential utilities, $p^n_U(x, q_n; h)$ converges to $\inf_{\mathbb{Q}\in\tilde{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[h(Y_T)]$. Therefore, for each n, $\inf_{\mathbb{Q}^n\in\mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)] = \inf_{y\in E} h(y)$ and hence (the upper bound follows in a similar manner) the interval of arbitrage free prices for $h(Y_T)$ is

$$I(h) := \left(\inf_{\mathbb{Q}^n\in\mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)], \sup_{\mathbb{Q}^n\in\mathcal{M}^n} \mathbb{E}^{\mathbb{Q}^n}[h(Y_T)]\right) = \left(\inf_{y\in E} h(y), \sup_{y\in E} h(y)\right). \tag{5.5}$$

Let $p_n \in I(h)$ and assume one can buy an arbitrary number of claims for p_n. As considered in [18, 19], this corresponds to when buyers can easily find either one another or multiple market makers. A natural

problem is to determine the utility based optimal quantity:

$$q_n^* \in \text{argmax}_{q\in\mathbb{R}}\, u_U^n(x - qp_n, q; h). \tag{5.6}$$

For exponential utility, existence of a unique maximizer q_n^* is proved for the general framework of Section 3 in [25, Theorem 3.1]. Specialized to the current model, the results in [45] enable the precise identification of q_n^*. For ease of presentation set $\Lambda := (1/2)\int_0^T \lambda(Y_t)^2 dt$, and, for any $\rho, \gamma \in \mathbb{R}$ define

$$g(\rho,\gamma) := \frac{\mathbb{E}\left[h(Y_T)Z(\rho)e^{-(1-\rho^2)\Lambda-\gamma h(Y_T)}\right]}{\mathbb{E}\left[Z(\rho)e^{-(1-\rho^2)\Lambda-\gamma h(Y_T)}\right]}. \tag{5.7}$$

Proposition 5.5. *Let Assumption 5.1 hold and let $p_n \in I(h)$. Recall $\hat{p} = \mathbb{E}^{\mathbb{Q}}[h(Y_T)]$ is the unique arbitrage free price in the $\rho = 1$ model. The unique q_n^* solving* (5.6) *satisfies $\alpha q_n^*(1-\rho_n^2) = \gamma_n^*$ where γ_n^* is uniquely determined by $p_n = g(\rho_n, \gamma_n^*)$. Let $\rho_n \to 1$. Then, for any subsequence $\{n_k\}_{k\in\mathbb{N}}$*

$$\lim_{k\uparrow\infty} |q_{n_k}^*| = \infty \Longleftrightarrow \lim_{k\uparrow\infty} \frac{|p_{n_k} - \hat{p}|}{1-\rho_{n_k}^2} = \infty. \tag{5.8}$$

Furthermore, if $p_n \to p$ for some $p \in I(h)$ then $\lim_{n\uparrow\infty} \alpha q_n^*(1-\rho_n^2) = \gamma^*$ where γ^* uniquely solves $p = g(1,\gamma^*)$. $\gamma^* \neq 0$ if and only if $p \neq \hat{p}$.

Remark 5.6. Proposition 5.5 implies that when purchasing optimal quantities, case (iii) in (5.3) never arises. Case (i) arises if $p_n \to \hat{p}$. For all other limiting prices p, case (ii) arises.

Remark 5.7 (Vanishing Risk Aversion). As with Proposition 5.3, Proposition 5.5 extends to when the risk aversion $\alpha_n \to 0$. Here, $q_n^*\alpha_n(1-\rho_n^2) = \gamma_n^*$ where $p_n = g(\rho_n, \gamma_n^*)$. Thus, if $p_n \to p, \rho_n \to \rho$ it follows that $\gamma_n^* \to \gamma^*$ where $p = g(\rho, \gamma^*)$. Here, $\gamma^* = 0$ if and only if $p = g(\rho, 0) = \mathbb{E}^{\mathbb{Q}^0(\rho)}[h(Y_T)]$ where as before $\mathbb{Q}^0(\rho)$ is the minimal entropy measure in the ρ market. Now, assume $\rho_n \equiv \rho$ is fixed. An analogous calculation, given in the proof below, to that which proves (5.8) shows that for all subsequences $\{n_k\}$: $\lim_{k\uparrow\infty} |q_{n_k}| = \infty$ if and only if $\lim_{k\uparrow\infty}(p_{n_k} - g(\rho,0))/\alpha_{n_k} = \infty$.

Proposition 5.5 implies the heuristic in (1.2), provided for each n one may buy claims at a price $p \neq \hat{p}$. Though it may seem unrealistic one could engage a seller at this price, it is indeed possible in the setting of PEPQ from [1]. To define a PEPQ, let X, X' be two bounded, $\mathcal{F}_T$ measurable random variables. As in Section 3.3.1, for the exponential utility U_α, denote by $u_{U_\alpha}^n(X, q_n; h)$ the value function for holding q_n claims of $h(Y_T)$ as well as X. Now, consider a second exponential investor with risk aversion $\delta > 0$. A

pair (q_n^*, p_n) where $p_n \in I(h)$ is called a PEPQ in the nth market if

$$q_n^* \in \text{argmax}_{q\in\mathbb{R}} \left(u_\alpha^n(X - q_n p_n, q_n; h)\right); \qquad q_n^* \in \text{argmax}_{q\in\mathbb{R}} \left(u_\delta^n(X' + q_n p_n, -q_n; h)\right).$$

In other words, it is optimal for the δ risk averse investor to sell q_n^* units of $h(Y_T)$ and for the α risk averse investor to buy q_n^* units of $h(Y_T)$ at the price p_n. As shown in [1, Theorem 5.8, Remark 5.0, Corollary 3.16], if $\alpha X - \delta X'$ is not replicable there exists a unique PEPQ with $q_n^* \neq 0$, otherwise there is no PEPQ.

Now, let $X \equiv 0$ and assume the seller holds a position consistent with (1.2): i.e. $X' = (\gamma/(1-\rho_n^2))h(Y_T)$ for some $\gamma > 0$. As $h(Y_T)$ is not replicable, it follows that a PEPQ (q_n^*, p_n) exists. Furthermore, (q_n^*, p_n) must satisfy the optimality conditions (recall (5.7) for $q_n^* = \gamma_n^*/(1-\rho_n^2)$):

$$p_n = g(\rho_n, \alpha\gamma_n^*) = g(\rho_n, \delta(\gamma - \gamma_n^*)).$$

Clearly, $\gamma_n^* = \gamma\delta/(\delta+\alpha)$ and $p_n = g(\rho_n, \gamma\alpha\delta/(\alpha+\delta))$. Such a p_n exists by Lemma 7.1 below, and, as $\rho_n \to 1$ it follows that $p_n \to p \neq \hat{p}$ where $p = g(1, \gamma\alpha\delta/(\alpha+\delta))$. Thus, with both buyer and seller acting optimally, the buyer enters into the regime of (1.2) and the seller is willing to sell for a price $p_n \approx p \neq \hat{p}$. The message is that, as long as there exists a single investor in the regime of (1.2), whether or not she has entered it optimally, it is possible for other investors, acting optimally, to enter into the regime (1.2) as well. Given the actual notional sizes existing in the market, it is entirely reasonable to assume some investor is in the regime of (1.2).

5.4. **Monetary Errors.** The following Proposition identifies precise conditions on γ_n from (5.3) for the monetary error introduced by using the limiting exponential utility price to remain bounded. For the sake of brevity, case (iii) is excluded.

Proposition 5.8. *Let $\alpha > 0$ and Assumption 5.1 hold. For q_n from* (5.3) *and p_α from Proposition 5.3, as $\rho_n \to 1$:*

$$\limsup_{n\uparrow\infty} q_n \left|p_{U_\alpha}^n(q_n; h) - p_\alpha\right| < \infty \iff \begin{cases} \textit{(i) :}\ \limsup_{n\uparrow\infty} \frac{\gamma_n^2}{1-\rho_n^2} < \infty \\ \textit{(ii) :}\ \limsup_{n\uparrow\infty} \frac{|\gamma_n - \gamma|}{1-\rho_n^2} < \infty \end{cases}. \tag{5.9}$$

Furthermore, if γ_n^* is chosen optimally as in Proposition 5.5 for a fixed $p \in I(h)$: i.e. γ_n^* satisfies $p = g(\rho_n, \gamma_n^*)$ then monetary errors are always bounded.

5.5. **On the Optimal Hedging Strategy.** Assume q_n takes the form in (5.3). As shown in Section 7 below, for an exponential investor, the optimal strategy $\hat{\pi}^n$ † satisfies $\hat{\pi}^n_t = (1/(\alpha\sigma(Y_t)))\left(\lambda(Y_t) + (\rho_n/(1-\rho_n^2))\theta^n_t\right)$ where θ^n satisfies (recall $\Lambda = (1/2)\int_0^T \lambda(Y_t)^2 dt$):

$$\mathcal{E}\left(\int_0^{\cdot} \theta^n_t (dW_t + \rho_n \lambda(Y_t)dt)\right)_T = \frac{e^{-(1-\rho_n^2)\Lambda - \gamma_n h(Y_T)}}{\mathbb{E}\left[Z(\rho_n)e^{-(1-\rho_n^2)\Lambda-\gamma_n h(Y_T)}\right]}. \tag{5.10}$$

That such a θ^n exists follows from the Martingale Representation Theorem. Now, if $\gamma_n \to \gamma \neq 0$ then under the given hypothesis on the model coefficients, $\theta^n \to \theta$ in the sense that $\lim_{n\uparrow\infty} \mathbb{E}\left[\int_0^T (\theta^n_t - \theta_t)^2 dt\right] = 0$, where θ solves (5.10) at $\rho_n = 1, \gamma$. Thus, even though $\hat{\pi}^n$ is taking ever larger (in magnitude) positions, the normalized trading strategy $\hat{\pi}^n/q_n$ satisfies (because $1/(1-\rho_n^2) = \alpha q_n/\gamma_n$) the limit $\lim_{n\uparrow\infty} \hat{\pi}^n_t/q_n = \theta_t/(\gamma\sigma(Y_t)) := \hat{\pi}_t$. Thus,

$$\mathcal{E}\left(\int_0^{\cdot} \gamma\hat{\pi}_t dS_t/S_t\right)_T = \mathcal{E}\left(\int_0^{\cdot} \theta_t\left(dW_t + \lambda(Y_t)dt\right)\right)_T = \frac{e^{-\gamma h(Y_T)}}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\gamma h(Y_T)}\right]} = e^{-\gamma h(Y_T) + \gamma p_\alpha},$$

where p_α is from Proposition 5.3. As $\gamma^2\hat{\pi}_t^2 d\langle S\rangle_t/S_t^2 = \theta_t^2$ it follows that $(p_\alpha, \hat{\pi})$ is a super-hedge in the $\rho = 1$ model:

$$-p_\alpha + h(Y_T) + \int_0^T \hat{\pi}_t \frac{dS_t}{S_t} = \frac{1}{2\gamma}\int_0^T \theta_t^2 dt.$$

In fact, calculation shows that $1/(2\gamma)\int_0^T \theta_t^2 dt$ is the limiting normalized (per unit claim) *residual risk*, as defined in see [35, 44], for owning q_n units of $h(Y_T)$. As such, $p_\alpha = \mathbb{E}^{\mathbb{Q}}\left[h(Y_T)\right] - 1/(2\gamma)\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \theta_t^2 dt\right]$ is the unique arbitrage free price in the complete model, less the price of the normalized residual risk.

5.6. **Asymptotic Completeness and the Local Martingale Measures.** Though $\rho_n \to 1$, the family of local martingale measures $\tilde{\mathcal{M}}^n$, even when restricted to $\mathcal{F}^W$, is not collapsing to a singleton with respect to the weak convergence of probability measures. This follows immediately from (5.5) as $h(Y_T)$ is $\mathcal{F}^W$ measurable. Indeed, setting $\mathbb{Q}^n_W := \mathbb{Q}^n\big|_{\mathcal{F}^W}$, (5.5) implies for all n that

$$\sup_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \mathbb{E}^{\mathbb{Q}^n_W}\left[h(Y_T)\right] = \sup_{y\in E} h(y); \qquad \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \mathbb{E}^{\mathbb{Q}^n_W}\left[h(Y_T)\right] = \inf_{y\in E} h(y).$$

Therefore, it cannot be that for *any two* sequences of measures $\mathbb{Q}^{n,1}, \mathbb{Q}^{2,n} \in \mathcal{M}^n$ that $\lim_{n\uparrow\infty} |\mathbb{E}^{\mathbb{Q}^{n,1}_W}\left[h(Y_T)\right] - \mathbb{E}^{\mathbb{Q}^{n,2}_W}\left[h(Y_T)\right]| = 0$. The next proposition reinforces this fact, as well as provides an alternate description of the difference between the limiting indifference and traded prices in terms of the relative entropy of two

† If the class of admissible trading strategies is enlarged to include strategies such that the resultant wealth process is a $\mathbb{Q}^n$ supermartingale for all $\mathbb{Q}^n \in \tilde{\mathcal{M}}^n$, see [27, 37]

sequences of local martingale measures in $\tilde{\mathcal{M}}^n$. To state Proposition 5.9 define $\mathbb{Q}^{n,q_n} \in \tilde{\mathcal{M}}^n$ as the solution to the dual problem in (3.3) and recall $\hat{\mathbb{Q}}^n$ from (5.1).

Proposition 5.9. *Let Assumption 5.1 hold. Let $\rho_n \to 1$, $p \in I(h)$, $p \neq \hat{p}$. Let q_n, γ be as in Proposition 5.5 and p_α be as in case (ii) of Proposition 5.3. Then*

$$\lim_{n\uparrow\infty} H\left(\mathbb{Q}^{n,q_n}_W \mid \hat{\mathbb{Q}}^n_W\right) = \gamma(p_\alpha - p). \tag{5.11}$$

Remark 5.10. The above result transfers directly to the vanishing risk aversion case where $\rho_n \equiv \rho$, $\alpha_n \to 0$, though the limiting value takes a more complicated form than $\gamma(p_\alpha - p)$: see (7.8) below. Here, the optimal purchase quantity q_n^* satisfies $q_n^*\alpha_n(1-\rho^2) = \gamma_n^* \to \gamma^* \neq 0$.

5.7. **A Numerical Example.** This section concludes with a numerical example where limiting indifference prices and optimal purchase quantities are computed for a put option in the basis risk model of Example 5.2. This example shows, first, that even for high correlations and a purchase price p away from the unique arbitrage free price $\hat{p}$ in the complete model, the optimal purchase quantity q_n^* is modest in size. Second, that despite the modest size of q_n^*, its effect upon the indifference price is non-negligible. Thus, non-trivial pricing effects occur at position sizes for which it is reasonable to ignore significant market impact effects.

Indifference prices and optimal position sizes are plotted as a function of the correlation in Figure 1. The plots make clear that non-trivial pricing effects occur at relatively small position sizes when correlations are close to one. Indeed, to reinforce this notion, the following table chooses three data points from Figure 1 and gives the correlation, optimal position size, corresponding indifference price and complete market price.

ρ	q_n^*	$p^n(q_n^*)$	$\hat{p}$
0.85	0.030	5.162	5.405
0.99	0.509	5.199	5.405
0.999	5.134	5.202	5.405

6. Proofs for Sections 3 and 4

6.1. **Preliminaries.** Unless otherwise stated, all expectations within this section are taken with respect to $\mathbb{P}^n$ and denoted by $\mathbb{E}^n$. For any $\mathbb{Q}^n \ll \mathbb{P}^n$ write $Z^{\mathbb{Q},n} := (d\mathbb{Q}^n/d\mathbb{P}^n)|_{\mathcal{F}^n}$.

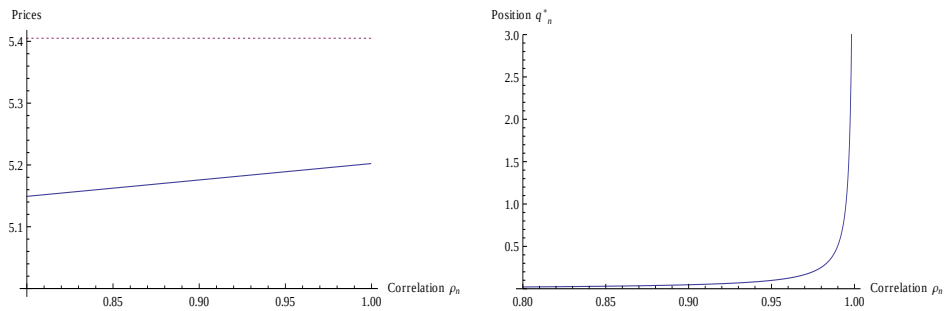

FIGURE 1. Indifference prices (left plot) and optimal position sizes (right plot) for a put option in the basis risk model of Example 5.2 as a function of the correlation. In the price plot, the indifference price $p^n(q_n)$ is given by the solid line, while $\hat{p}$, the unique arbitrage free price in the complete $\rho = 1$ model is given by the small-dashed line. q_n^*, which is shown on the left plot and used to compute $p^n(q_n^*)$, is obtained by optimally purchasing quantities of the claim for a price $p \neq \hat{p}$. Model parameter values are $\mu = 0.06, \sigma = 0.4, b = 0.05, a = 0.3, T = 1, Y_0 = 15$. The investor has risk aversion $\alpha = 3$ and the strike for the put is $K = 20$. The purchase prices is $p = 5$. Computations were performed using *Mathematica* and the code can be found on the author's website *www.math.cmu.edu/users/scottrob/research*

Let $\alpha > 0$, $U \in \mathcal{U}_\alpha$ and define V as in (2.7). As in Section 3, define $\tilde{\mathcal{M}}_V^n$ as the set of $\mathbb{Q}^n \in \mathcal{M}^n$ such that $\mathbb{E}^n\left[V(d\mathbb{Q}^n/d\mathbb{P}^n)\right] < \infty$. By applying Lemma A.3 with $Y = Z^{\mathbb{Q},n}$ and $y = 1$ it follows that $\tilde{\mathcal{M}}_V^n = \tilde{\mathcal{M}}^n$ for all $U \in \mathcal{U}_\alpha$. Therefore, the indifference pricing formula (3.1) specifies to

$$p_U^n(x, q; h^n) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^n\left[h^n Z^{\mathbb{Q},n}\right] + \frac{1}{q}\alpha_U^n\left(\mathbb{Q}^n\right)\right). \tag{6.1}$$

In a similar manner, Lemma A.3 implies $\tilde{\mathcal{M}}_V^n = \hat{\mathcal{M}}^n$ for all $U \in \mathcal{U}_{p,l}$ and hence

$$p_U^n(x, q; h^n) = \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \left(\mathbb{E}^n\left[h^n Z^{\mathbb{Q},n}\right] + \frac{1}{q}\alpha_U^n\left(\mathbb{Q}^n\right)\right). \tag{6.2}$$

6.2. **Proofs.** It is first shown that Assumptions 3.2 and 4.2 imply the "no asymptotic arbitrage" condition $\limsup_{n\uparrow\infty} u_U^n(x) < U(\infty) = 0$ for all $U \in \mathcal{U}_\alpha$ and $U \in \mathcal{U}_{p,l}$ respectively.

Proposition 6.1. *Let $\alpha > 0$, $p > 1, l > 0$ and $x \in \mathbb{R}$. Then Assumption 3.2 implies $\limsup_{n\uparrow\infty} u_U^n(x) < 0$ for $U \in \mathcal{U}_\alpha$. Similarly, Assumption 4.2 implies $\limsup_{n\uparrow\infty} u_U^n(x) < 0$ for $U \in \mathcal{U}_{p,l}$.*

Proof of Proposition 6.1. In view of Assumptions 3.2 and 4.2 there exist sequences of measures $\mathbb{Q}_1^n, \mathbb{Q}_2^n \in \mathcal{M}^n$ and a constant $C > 0$ so that

$$\sup_n \mathbb{E}^n \left[V_\alpha(d\mathbb{Q}_1^n/d\mathbb{P}^n)\right] \leq C \text{ (Ass. 3.2)}; \qquad \sup_n \mathbb{E}^n \left[V_p(d\mathbb{Q}_2^n/d\mathbb{P}^n)\right] \leq C \text{ (Ass. 4.2)}. \tag{6.3}$$

Write $Z_1^n := d\mathbb{Q}_1^n/d\mathbb{P}^n$ and $Z_2^n := d\mathbb{Q}_2^n/d\mathbb{P}^n$. For $x \in \mathbb{R}$ it follows from [37, Equations (1.4), (1.5)] that

$$u_U^n(x) \leq \inf_{y>0} \left(\mathbb{E}^n \left[V\left(yZ_1^n\right)\right] + xy\right) \ (U \in \mathcal{U}_\alpha); \qquad u_U^n(x) \leq \inf_{y>0} \left(\mathbb{E}^n \left[V\left(yZ_2^n\right)\right] + xy\right) \ (U \in \mathcal{U}_{p,l}). \tag{6.4}$$

The argument below is nearly identical for $U \in \mathcal{U}_\alpha$ and $U \in \mathcal{U}_{p,l}$: thus, it will be given for $U \in \mathcal{U}_\alpha$ and only the adjustments needed for $U \in \mathcal{U}_{p,l}$ will be mentioned. Applying Lemma A.4 with $Y = Z_1^n$ shows there is a unique $y_n > 0$ solving the minimization problem in (6.4) and the first order conditions are $x = -\mathbb{E}^n \left[Z_1^n V'(y_n Z_1^n)\right]$. Assume, for now, that

$$\liminf_{n\uparrow\infty} y_n > 0. \tag{6.5}$$

Using the first order conditions for y_n and (6.4) it follows that $u_U^n(x) \leq -\mathbb{E}^n \left[(y_n Z_1^n V'(y_n Z_1^n) - V(y_n Z_1^n))\right]$. Set $f(z) := zV'(z) - V(z)$. Note that $f'(z) = zV''(z) > 0$ and $\lim_{z\downarrow 0} f(z) = 0$, as $U(\infty) = 0$. In view of (6.5), take $\delta > 0$ such that $y_n \geq \delta$ for large n. As f is increasing and non-negative, $u_U^n(x) \leq -\mathbb{E}^n \left[f(y_n Z_1^n)\right] \leq -\mathbb{E}^n \left[f(\delta Z_1^n)\right] \leq 0$. Assume, by way of contradiction, there exists a sequence (still labeled n) such that $\lim_{n\uparrow\infty} u_U^n(x) = 0$. The above inequality implies $\lim_{n\uparrow\infty} \mathbb{E}^n \left[f(\delta Z_1^n)\right] = 0$, and hence for all $\varepsilon > 0$ that $\lim_{n\uparrow\infty} \mathbb{P}^n \left[Z_1^n \geq \varepsilon\right] = 0$. In view of (6.3) the Z_1^n are "uniformly integrable" in that $\lim_{\lambda\uparrow\infty} \sup_n \mathbb{E}^n \left[Z_1^n 1_{Z_1^n \geq \lambda}\right] = 0$. This follows because for all $z > 0$ and $\lambda > 1$

$$z1_{z\geq\lambda} \leq \frac{\lambda}{V_\alpha(\lambda)} \left(V_\alpha(z) + \frac{1}{\alpha}\right); \qquad \left(\text{resp. } x1_{x\geq\lambda} \leq \frac{\lambda}{V_p(\lambda)} V_p(z)\right),$$

and because $\lim_{\lambda\uparrow\infty} \lambda/V_\alpha(\lambda) = 0$ (resp. $\lim_{\lambda\uparrow\infty} \lambda/V_p(\lambda) = 0$). Now, fix $\varepsilon > 0$ and choose λ so large that $\sup_n \mathbb{E}^n \left[Z_1^n 1_{Z_1^n \geq \lambda}\right] \leq \varepsilon$. Since $Z_1^n \in \tilde{\mathcal{M}}^n$,

$$1 = \mathbb{E}^n \left[Z_1^n\right] = \mathbb{E}^n \left[Z_1^n \left(1_{Z_1^n \leq \varepsilon} + 1_{\varepsilon < Z_1^n < \lambda} + 1_{Z_1^n \geq \lambda}\right)\right] \leq \varepsilon + \lambda \mathbb{P}^n \left[Z_1^n > \varepsilon\right] + \varepsilon.$$

Taking $n \uparrow \infty$ and then $\varepsilon \downarrow 0$ gives a contradiction and hence the result holds assuming (6.5).

To prove (6.5), recall that y_n satisfies $-x = \mathbb{E}^n \left[Z_1^n V'(y_n Z_1^n)\right]$. By way of contradiction, assume there is some sequence (still labeled n) such that $\lim_{n\uparrow\infty} y_n = 0$. Let $(M_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\uparrow\infty} M_n = \infty$ and $\lim_{n\uparrow\infty} y_n M_n = 0$. For n so large that $y_n < 1$, the strict convexity of V gives

$$-x \leq V'(y_n M_n) \mathbb{E}^n \left[Z_1^n 1_{Z_1^n \leq M_n}\right] + \mathbb{E}^n \left[Z_1^n V'(Z_1^n) 1_{Z_1^n > M_n}\right]. \tag{6.6}$$

The uniformly integrability of Z_1^n combined with $\mathbb{E}^n\left[Z_1^n\right] = 1$, $\lim_{z\downarrow 0} V'(z) = -\infty$ implies that $\lim_{n\uparrow\infty} V'(y_n M_n)\mathbb{E}^n\left[Z_1^n 1_{Z_1^n \le M_n}\right] = -\infty$. From [41, Corollary 4.2(ii)] (note: part (ii) therein does not require $U(0) > 0$) there exists some $\tilde{K} > 0$ so that $z|V'(z)| \le \tilde{K}V(z)$ for $z > 0$. Furthermore, as $M_n \to \infty$, for any $\varepsilon > 0$, (2.8) (resp. (4.1)) and the definitions of V_α (resp. V_p) imply that for large enough n:

$$V(z)1_{z\ge M_n} \le (1+\varepsilon)\left(V_\alpha(z) + \frac{1}{\alpha}\right); \qquad (\text{resp. } V(z)1_{z\ge M_n} \le (1+\varepsilon)V_p(z)).$$

In view of (6.3), for some large enough K:

$$\limsup_{n\uparrow\infty} \mathbb{E}^n\left[Z_1^n V'(y_n Z_1^n)1_{Z_1^n > M_n}\right] \le K; \qquad \left(\text{resp. } \limsup_{n\uparrow\infty} \mathbb{E}^n\left[Z_2^n V'(y_n Z_2^n)1_{Z_2^n > M_n}\right] \le K\right).$$

Therefore, (6.6) is contradicted if $y_n \to 0$, proving the result. □

Proof of Theorem 3.3. Let $\alpha > 0$, $U \in \mathcal{U}_\alpha$ and $x \in \mathbb{R}$. In view of Proposition 6.1, one may choose $\varepsilon > 0$ so that $\varepsilon < -u_U^n(x)$ for n large. Recall the definition of $\alpha_U^n(\mathbb{Q}^n)$ in (3.2) and the price $p_U^n(x, q; h^n)$ in (6.1). Lemma A.5 with $u = -u^n(x)$, $\varepsilon = \varepsilon$ and $Y = Z^{\mathbb{Q},n}$ implies there is a constant $\overline{C}(\varepsilon, U)$ such that

$$p_U^n(x, q_n; h^n) \le \frac{x + \overline{C}(\varepsilon, U) + ((1+\varepsilon)/\alpha)\log(-u_U^n(x))}{q_n} + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n}\left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \frac{1+\varepsilon}{q_n\alpha} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right).$$

Similarly, from Lemma A.6 with $u = -u_U^n(x)$, $\varepsilon = \varepsilon$ and $Y = Z^{\mathbb{Q},n}$ there exists a constant $\underline{C}(\varepsilon, U)$ such that

$$p_U^n(x, q_n; h^n) \ge \frac{x + \underline{C}(\varepsilon, U) + ((1-\varepsilon)/\alpha)\log(-u_U^n(x))}{q_n} + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n}\left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \frac{1-\varepsilon}{q_n\alpha} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right).$$

Consider the function:

$$f(\delta, n) := \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n}\left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \delta H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right), \quad \delta > 0. \tag{6.7}$$

Clearly, f is increasing with δ. Furthermore, Assumptions 3.1 and 3.2 imply for some constant $K > 0$ that $f(\delta, n) \le \|h\| + K\delta$. Let $0 < \delta < \gamma$. For any $\mathbb{Q}^n \in \tilde{\mathcal{M}}^n$

$$\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \gamma H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right) \le \frac{\gamma}{\delta}\left(\mathbb{E}^{\mathbb{Q}^n}\left[h^n\right] + \delta H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right) + \left(\frac{\gamma}{\delta} - 1\right)\|h\|.$$

Thus,

$$f(\gamma, n) - f(\delta, n) \le \left(\frac{\gamma}{\delta} - 1\right)\left(f(\delta, n) + \|h\|\right) \le \left(\frac{\gamma}{\delta} - 1\right)\left(2\|h\| + K\delta\right). \tag{6.8}$$

Now, let $U_1, U_2 \in \mathcal{U}_\alpha$ and $x_1, x_2 \in \mathbb{R}$. Choose $\varepsilon > 0$ so that for all n large enough $\varepsilon \leq -u^n_{U_1}(x_1) \leq -U_1(x_1)$ and $\varepsilon \leq -u^n_{U_2}(x_2) \leq -U_2(x_2)$. By the above calculations, there is a constant $C(n,\varepsilon)$ satisfying $C(n,\varepsilon)/q_n \to 0$ for any $q_n \to \infty$ such that

$$\begin{aligned} p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n) &\leq \frac{C(n,\varepsilon)}{q_n} + f\left(\frac{1+\varepsilon}{q_n\alpha}, n\right) - f\left(\frac{1-\varepsilon}{q_n\alpha}, n\right), \\ &\leq \frac{C(n,\varepsilon)}{q_n} + \left(\frac{1+\varepsilon}{1-\varepsilon} - 1\right)\left(2\|h\| + K\frac{1-\varepsilon}{q_n\alpha}\right). \end{aligned}$$

Therefore

$$\limsup_{n\uparrow\infty} \left(p^n_{U_1}(x_1, q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n)\right) \leq 2\|h\| \left(\frac{1+\varepsilon}{1-\varepsilon} - 1\right). \tag{6.9}$$

As the left hand side does not depend upon ε taking $\varepsilon \downarrow 0$ gives (3.4) after noting that the roles of U_1, U_2 and x_1, x_2 may be switched.

□

Proof of Theorem 3.8. Let $\alpha > 0$ and $U \in \tilde{\mathcal{U}}_\alpha$, $x \in \mathbb{R}$. Proposition 6.1 implies one can find $\varepsilon > 0$ so that $\varepsilon < -u^n_U(x)$ for large n. Using the representation for $p^n_U(x, q; h^n)$ in (6.1), it follows from Lemma A.5 applied to $u = -u^n_U(x), \varepsilon = \varepsilon, Y = Z^{\mathbb{Q},n}$ that there is a constant $\overline{C}(\varepsilon, U)$ such that

$$p^n_U(x, q_n; h^n) \leq \frac{x + \overline{C}(\varepsilon, U) + (1/\alpha)\log(-u^n_U(x))}{q_n} + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n] + \frac{1}{q_n\alpha} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right).$$

Lemma A.6 applied to $u = -u^n_U(x), \varepsilon = \varepsilon, Y = Z^{\mathbb{Q},n}$ yields the existence of a constant $\underline{C}(\varepsilon, U)$ so that

$$p^n_U(x, q_n; h^n) \geq \frac{x + \underline{C}(\varepsilon, U) + (1/\alpha)\log(-u^n_U(x))}{q_n} + \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n] + \frac{1}{q_n\alpha} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)\right).$$

Now, let $U_1, U_2 \in \tilde{\mathcal{U}}_\alpha$ and $x_1, x_2 \in \mathbb{R}$. Choose $\varepsilon > 0$ so that $\varepsilon \leq -u^n_{U_1}(x_1) \leq -U_1(x_1)$ and $\varepsilon \leq -u^n_{U_2}(x_2) \leq -U_2(x_2)$ for large n. By the above:

$$q_n\left(p^n_{U_1}(x_1; q_n; h^n) - p^n_{U_2}(x_2, q_n; h^n)\right) \leq C(n,\varepsilon),$$

where $C(n,\varepsilon) := x_1 + \underline{C}(\varepsilon, U_1) + (1/\alpha)(-\log(-u^n_{U_1}(x_1)) - x_2 - \underline{C}(\varepsilon, U_2) - (1/\alpha)(-\log(u^n_{U_2}(x_2))$. From Proposition 6.1 it follows that $\sup_n C(n,\varepsilon) < \infty$. Furthermore, for $U \in \tilde{\mathcal{U}}_\alpha$ one can show the existence of a constant $K = K(U)$ such that $-(\log(K)/\alpha)y \leq V(y) - V_\alpha(y) \leq (\log(K)/\alpha)y$ for large y. Using this, by repeating the steps in the proof of Proposition 6.1 a lengthy calculation shows for all $\varepsilon > 0, x \in \mathbb{R}$ that there exits a constant $\tilde{K} > 0$ dependent only upon U, x, ε and $\limsup_{n\uparrow\infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)$ such that $\tilde{K} \leq -u^n_U(x) \leq U(x)$. Thus, the constant $C(n,\varepsilon)$ can be made so that it only depends U_1, U_2, x_1, x_2

and $\limsup_{n\uparrow\infty} \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}^n} H\left(\mathbb{Q}^n \mid \mathbb{P}^n\right)$. This finishes the proof since the roles of U_1, x_1 and U_2, x_2 may be reversed.

□

Proof of Proposition 3.12. Similarly to (3.1) and (6.1), under Assumptions 3.1, 3.2 and 3.10 it follows from [37, Proposition 7.1] that for $q_n > 0$: [‡]

$$p_U^n(X^n, q_n; h^n) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}_n} \left(\mathbb{E}^n \left[h^n Z^{\mathbb{Q},n} \right] + \frac{1}{q_n} \inf_{y>0} \left(\frac{1}{y} \left(\mathbb{E}^n \left[V\left(y Z^{\mathbb{Q},n} \right) + y Z^{\mathbb{Q},n} X^n \right] - u_U^n(X^n) \right) \right) \right).$$

Additionally, as shown in [37, Theorems 1.1,1.2]

$$u_U^n(X^n) = \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}_n} \inf_{y>0} \mathbb{E}^n \left[V\left(y Z^{\mathbb{Q},n} \right) + y Z^{\mathbb{Q},n} X^n \right].$$

According to Assumption 3.10 for all $\mathbb{Q}^n \in \tilde{\mathcal{M}}_n$ it follows $\underline{x}_n \leq \mathbb{E}^n \left[X^n Z^{\mathbb{Q},n} \right] \leq \overline{x}_n$. Therefore, $u_U^n(\underline{x}^n) \leq u_U^n(X^n) \leq u_U^n(\overline{x}^n)$ and

$$\begin{aligned} p_U^n(X^n, q_n; h^n) &\leq \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}_n} \left(\mathbb{E}^n \left[h^n Z^{\mathbb{Q},n} \right] + \frac{1}{q_n} \inf_{y>0} \left(\frac{1}{y} \left(\mathbb{E}^n \left[V\left(y Z^{\mathbb{Q},n} \right) \right] + y \overline{x}_n - u_U^n(\underline{x}_n) \right) \right) \right), \\ &= \frac{\overline{x}_n - \underline{x}_n}{q_n} + p_U^n(\underline{x}_n, q_n; h^n). \\ p_U^n(X^n, q_n; h^n) &\geq \inf_{\mathbb{Q}^n \in \tilde{\mathcal{M}}_n} \left(\mathbb{E}^n \left[h^n Z^{\mathbb{Q},n} \right] + \frac{1}{q_n} \inf_{y>0} \left(\frac{1}{y} \left(\mathbb{E}^n \left[V\left(y Z^{\mathbb{Q},n} \right) \right] + y \underline{x}_n - u_U^n(\overline{x}_n) \right) \right) \right), \\ &= \frac{\underline{x}_n - \overline{x}_n}{q_n} + p_U^n(\overline{x}_n, q_n; h^n). \end{aligned}$$

Using Assumption 3.11 and the monotonicity of $p_U^n(x, q_n; h^n)$ in x, the result now readily follows from Theorems 3.3 and 3.8. □

Proof of Example 3.13. By Theorem 3.3, it suffices to consider the exponential utility U_α to prove convergence of $p_U^n(x, q_n; h)$ for $U \in \mathcal{U}_\alpha, x \in \mathbb{R}$. As in Remark 2.7, set $p_\alpha^n := p_{U_\alpha}^n(q_n; h^n)$. (3.8) gives $p_\alpha^n = -(1/(q_n \alpha)) \log\left((1 - e^{-q_n}) e^{-\alpha q_n h} + e^{-q_n} \right)$, and hence $\lim_{n\uparrow\infty} p_\alpha^n = \min\left\{ h, \alpha^{-1} \right\}$.

Now, set $p^n := p_{\hat{U}}^n(x, q_n; h)$. From (3.8) and $p_n = (1/2)(1 - e^{-q_n})$, p^n satisfies

$$\hat{U}(x) = (1 - e^{-q_n}) \hat{U}(x + q_n(h - p^n)) + e^{-q_n} \hat{U}(x - q_n p^n). \tag{6.10}$$

Assume that $p = \lim_{n\uparrow\infty} p^n$ exists. Since it is clear $0 \leq p \leq h$, first assume that $0 \leq p < h$. As $\hat{U}(\infty) = 0$, it follows that $\hat{U}(x) = \lim_{n\uparrow\infty} e^{-q_n} \hat{U}(x - q_n p^n)$. This implies $\hat{U}(x - q_n p^n) = \alpha_n \hat{U}(x) e^{q_n}$ where $\alpha_n \to 1$.

[‡]In [37] the indifference price is determined using a slightly larger class of trading strategies than the class $\mathcal{A}^n$ used here-in. However, according to Theorem 1.2 in [37] the two value functions coincide and hence the prices take the same values

Let $\hat{U}^{-1} : (0,\infty) \mapsto \mathbb{R}$ denote the inverse of $\hat{U}$. Thus, recalling $p^n \to p$

$$p = \lim_{n\uparrow\infty} -\frac{1}{q_n}\hat{U}^{-1}\left(\alpha_n \hat{U}(x)e^{q_n}\right) = \lim_{n\uparrow\infty} -\frac{1}{q_n} g\left(q_n + \log(-\alpha_n \hat{U}(x))\right), \tag{6.11}$$

where $g(z) := \hat{U}^{-1}(-e^z)$. A straightforward calculation shows that $g'(z) = \hat{U}(g(z))/\hat{U}'(g(z))$. As $g(z) \to -\infty$ as $z \to \infty$, it follows by l'Hôpital's rule and $1/K_U \leq \alpha_U(x) \leq K_U$ that $-K_U \leq \liminf_{z\uparrow\infty} g'(z) \leq \limsup_{z\uparrow\infty} g'(z) \leq -1/K_U$. Thus, for large n, $|g(q_n + \log(-\alpha_n \hat{U}(x))) - g(q_n)| \leq 2K_U|\log(-\alpha_n \hat{U}(x))|$. Therefore, (6.11) and $q_n = \log(-\hat{U}(x_n))$ imply $p = \lim_{n\uparrow\infty} -(1/q_n)g(q_n) = \lim_{n\uparrow\infty} -(1/q_n)\hat{U}^{-1}\left(-e^{q_n}\right) = \lim_{n\uparrow\infty}(-x_n)/(\log(-\hat{U}(x_n)))$, but this violates the assumption on x_n and hence p^n cannot converge to $p < h$. Next, assume $p = h$ and let $\varepsilon > 0$ be small enough so that $h - \varepsilon > 0$. For large enough n, $p^n \geq h - \varepsilon$. The negativity of $\hat{U}(x)$ and (6.10) imply $-\hat{U}(x) \geq e^{-q_n}(-\hat{U}(x - q_n p^n)) \geq e^{-q_n}\left(-\hat{U}(x - q_n(h-\varepsilon))\right)$. This gives

$$0 \geq -1 + \limsup_{n\uparrow\infty} \frac{1}{q_n}\log\left(-\hat{U}(x - q_n(h-\varepsilon))\right) \geq -1 + \underline{\alpha}(h-\varepsilon),$$

where the last inequality follows by (3.9). Taking $\varepsilon \downarrow 0$ gives that $\underline{\alpha} \leq 1/h$ but, this violates how h was constructed. Therefore, $p \neq h$ and the proof is complete. □

Proof of Example 3.14. Set $p^n_\alpha := p^n_{U_\alpha}(x, n^2; h^n)$. Specifying (3.8) for the given parameter values gives

$$n^2 p^n_\alpha = -\frac{1}{n^2\alpha}\log\left((1-e^{-n})e^{-\alpha n^2} + e^{-n}\right) = \frac{n}{\alpha} + R(n), \tag{6.12}$$

where $\lim_{n\uparrow\infty} R(n) = 0$. For the given function U and $x \in \mathbb{R}$, set $p^n := p^n_U(x, n^2; h)$. Using (3.8) again:

$$U(x) = (1-e^{-n})U(x + n^2(1-p^n)) + e^{-n}U(x - n^2 p^n). \tag{6.13}$$

Assume that $\liminf_{n\uparrow\infty} n^2(p^n - p^n_\alpha) < \infty$ and choose a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and $K > 0$ so that $n_k^2(p^{n_k} - p^{n_k}_\alpha) \leq K$ for all k. By the monotonicity of U it follows from (6.13) that

$$U(x) \geq (1-e^{-n_k})U(x + n_k^2 - n_k^2 p^{n_k}_\alpha - K) + e^{-n_k}U(x - n_k^2 p^{n_k}_\alpha - K). \tag{6.14}$$

From (6.12) and $U(\infty) = 0$ it follows that $\lim_{k\uparrow\infty}(1-e^{-n_k})U(x + n_k^2 - n_k^2 p^{n_k}_\alpha - K) = 0$. By construction of U and (6.12) again

$$\lim_{k\uparrow\infty} e^{-n_k}U(x - n_k^2 p^{n_k}_\alpha - K) = \lim_{k\uparrow\infty} -\frac{1}{\alpha}e^{-n_k}\frac{e^{-\alpha(x-K)+n_k+\alpha R(n)}}{n_k/\alpha - (x-K) + R(n)} = 0.$$

Therefore, (6.14) implies $U(x) \geq 0$, a contradiction. Thus, $\lim_{n\uparrow\infty} n^2(p^n - p^n_\alpha) = \infty$.

□

Proof of Proposition 4.3. Let $p > 1$, $l > 0$, $U \in \mathcal{U}_{p,l}$ and $x \in \mathbb{R}$. Proposition 6.1 implies for $\varepsilon > 0$ small enough, $\varepsilon < -u_U^n(x)$ for large n. Lemma A.5 with $u = -u_U^n(x)$, $\varepsilon = \varepsilon$ and $Y = Z^{\mathbb{Q},n}$ yields a constant $\overline{C}(\varepsilon, U)$ so that for all position sizes q (and not just the q_n of the Proposition)

$$p_U^n(x,q;h^n) \leq \frac{x + \overline{C}(\varepsilon,U)}{q} + \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n] + \frac{1}{q} \left(l(-u_U^n(x)) \right)^{1/p} (1+\varepsilon)^{1/\gamma} \mathbb{E}^n \left[(Z^{\mathbb{Q},n})^\gamma \right]^{1/\gamma} \right).$$

Similarly, from Lemma A.6 with $u = -u_U^n(x)$, $\varepsilon = \varepsilon$ and $Y = Z^{\mathbb{Q},n}$ there exists a constant $\underline{C}(\varepsilon, U)$ such that

$$p_U^n(x,q;h^n) \geq \frac{x + \underline{C}(\varepsilon,U)}{q} + \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n] + \frac{1}{q} \left(l(-u_U^n(x) - \varepsilon/2) \right)^{1/p} ((1-\varepsilon))^{1/\gamma} \mathbb{E}^n \left[(Z^{\mathbb{Q},n})^\gamma \right]^{1/\gamma} \right).$$

Now, consider the function:

$$\hat{f}(\delta, n) := \inf_{\mathbb{Q}^n \in \hat{\mathcal{M}}^n} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n] + \delta \mathbb{E}^n \left[\left(Z^{\mathbb{Q},n} \right)^\gamma \right]^{1/\gamma} \right), \quad \delta > 0. \tag{6.15}$$

$\hat{f}$ is increasing with δ and that Assumption 4.2 implies the existence of a constant $K > 0$ so that $\hat{f}(\delta, n) \leq \|h\| + K\delta$. Let $0 < \delta < \gamma$. For any $\mathbb{Q}^n \in \hat{\mathcal{M}}^n$

$$\mathbb{E}^{\mathbb{Q}^n}[h^n] + \gamma H(\mathbb{Q}^n \mid \mathbb{P}^n) \leq \frac{\gamma}{\delta} \left(\mathbb{E}^{\mathbb{Q}^n}[h^n] + \delta \mathbb{E}^n \left[\left(Z^{\mathbb{Q},n} \right)^\gamma \right]^{1/\gamma} \right) + \left(\frac{\gamma}{\delta} - 1 \right) \|h\|.$$

Thus,

$$\hat{f}(\gamma, n) - \hat{f}(\delta, n) \leq \left(\frac{\gamma}{\delta} - 1 \right) \left(\hat{f}(\delta, n) + \|h\| \right) \leq \left(\frac{\gamma}{\delta} - 1 \right) (K\delta + 2\|h\|). \tag{6.16}$$

Note that for any $x \in \mathbb{R}$, $U \in \mathcal{U}_{p,l}$, since $u_U^n(x) \geq U(x)$ and $\limsup_{n \uparrow \infty} u_U^n(x) < 0$ there exists some $M > 0$ so that $1/M \leq -u_U^n(x) \leq M$ for all n large enough. Now, let $U_1, U_2 \in \mathcal{U}_\alpha$ and $x_1, x_2 \in \mathbb{R}$ and consider $q_n \uparrow \infty$. Choose $M > 0, \varepsilon > 0$ so that for all n large enough $\varepsilon < 1/M \leq -u_{U_i}^n(x_i) \leq M; i = 1, 2$. By the above calculations

$$\begin{aligned} & p_{U_1}^n(x_1, q_n(-u_{U_1}^n(x_1))^{1/p}; h^n) - p_{U_2}^n(x_2, q_n(-u_{U_2}^n(x_2))^{1/p}; h^n) \\ & \qquad \leq \frac{C^+(\varepsilon,n) - C^-(\varepsilon,n)}{q_n} + \hat{f}\left(\frac{\delta^+(\varepsilon,n)}{q_n}, n \right) - \hat{f}\left(\frac{\delta^-(\varepsilon,n)}{q_n}, n \right), \end{aligned} \tag{6.17}$$

where

$$C^+(\varepsilon,n) := \frac{x_1 + \overline{C}(\varepsilon, U_1)}{(-u_{U_1}^n(x_1))^{1/p}}; \qquad C^-(\varepsilon,n) := \frac{x_2 + \underline{C}(\varepsilon, U_2)}{(-u_{U_2}^n(x_2))^{1/p}},$$

$$\delta^+(\varepsilon,n) := l^{1/p}(1+\varepsilon)^{1/\gamma}; \qquad \delta^-(\varepsilon,n) := \left(1 + \frac{\varepsilon/2}{u_{U_1}^n(x_1)} \right)^{1/p} l^{1/p}(1-\varepsilon)^{1/\gamma}.$$

As $C^{\pm}(\varepsilon,n)/q_n \to 0$ as $n \uparrow \infty$ for all $\varepsilon > 0$ they may be disregarded. Also, note that $\delta^+(\varepsilon,n) > \delta^-(\varepsilon,n)$ and

$$\frac{\delta^+(\varepsilon,n)}{\delta^-(\varepsilon,n)} \leq \frac{(1+\varepsilon)^{1/\gamma}}{(1-\varepsilon M/2)^{1/p}(1-\varepsilon)^{1/\gamma}}; \qquad \lim_{n\uparrow\infty} \frac{\delta^-(\varepsilon,n)}{q_n} = 0.$$

It thus follows from (6.18) and (6.16) that for all $\varepsilon > 0$

$$\begin{aligned} &\limsup_{n\uparrow\infty} \left(p^n_{U_1}(x_1, q_n(-lu^n_{U_1}(x_1))^{1/p}; h^n) - p^n_{U_2}(x_2, q_n(-lu^n_{U_2}(x_2))^{1/p}; h^n) \right) \\ &\qquad \leq 2\|h\| \left(\frac{(1+\varepsilon)^{1/\gamma}}{(1-\varepsilon M/2)^{1/p}(1-\varepsilon)^{1/\gamma}} - 1 \right). \end{aligned} \tag{6.18}$$

Taking $\varepsilon \downarrow 0$ gives the result after noting that the roles of U_1, U_2 and x_1, x_2 may be switched.

□

7. Proofs for Section 5

7.1. **Preliminaries.** To accommodate Remarks 5.4, 5.7 and 5.10, all proofs within this section allow for the risk aversion to change with n. By considering $\alpha_n \equiv \alpha$ the results for constant risk aversion certainly carry over. Under Assumption 5.1, the value function for the exponential utility U_{α_n} takes the form [45, Proposition 3.3]:

$$u^n_{U_{\alpha_n}}(0, q_n; h) = -\frac{1}{\alpha_n}\mathbb{E}\left[Z(\rho_n) \exp\left(-(1-\rho_n^2)\left(\alpha_n q_n h(Y_T) + \frac{1}{2}\int_0^T \lambda(Y_t)^2 dt \right)\right)\right]^{\frac{1}{1-\rho_n^2}}, \tag{7.1}$$

where $Z(\rho_n)$ is given in (5.2). Furthermore, using $q_n = \gamma_n/(\alpha_n(1-\rho_n^2))$ and θ^n from (5.10), the dual optimal element $\mathbb{Q}^{n,q_n}$ takes the form

$$\frac{d\mathbb{Q}^{n,q_n}}{d\mathbb{P}} := \mathcal{E}\left(\int_0^{\cdot} \left(-\rho_n \lambda(Y_t) + \theta^n_t\right) dW_t - \int_0^{\cdot} \left(\sqrt{1-\rho_n^2}\lambda(Y_T) - \frac{\rho_n}{\sqrt{1-\rho_n^2}}\theta^n_t \right) dB_t \right)_T. \tag{7.2}$$

It is easy to check that $\mathbb{Q}^{n,q_n} \in \mathcal{M}^n$ and a calculation shows that $H\left(\mathbb{Q}^{n,q_n} \mid \mathbb{P}\right) < \infty$. The optimality of $\mathbb{Q}^{n,q_n}$ follows by considering the (potentially non-admissible) trading strategy $\hat{\pi}^n$ from Section 5.5 and then showing that the corresponding wealth process $X^{\hat{\pi}^n}$ and $d\mathbb{Q}^{n,q_n}/d\mathbb{P}$ satisfy the first order conditions for optimality. Then, from [27, Theorem 2.1] it follows that $(\hat{\pi}^n \cdot dS^n/S^n)$ is a $\mathbb{Q}^{n,q_n}$ martingale and hence $\mathbb{Q}^{n,q_n}$ is dual optimal. Since θ^n is $\mathbb{F}^W$ adapted

$$\frac{d\mathbb{Q}^{n,q_n}}{d\mathbb{P}}\Big|_{\mathcal{F}^W_T} = \mathcal{E}\left(\int_0^{\cdot} \left(-\rho_n \lambda(Y_t) + \theta^n_t\right) dW_t \right)_T = \frac{Z(\rho_n) e^{-(1-\rho_n^2)\Lambda - \gamma_n h(Y_T)}}{\mathbb{E}\left[Z(\rho_n) e^{-(1-\rho_n^2)\Lambda - \gamma_n h(Y_T)} \right]}, \tag{7.3}$$

because $\Lambda = (1/2)\int_0^T \lambda(Y_t)^2 dt$, and the last equality follows from (5.10). Note that $Z(\rho_n)e^{-(1-\rho_n^2)\Lambda} = Z^{\rho_n}e^{-(1-\rho_n)\Lambda}$ where $Z = Z(1)$. Thus, taking the above at $\gamma_n = 0$ it follows that for the minimal entropy measure $\mathbb{Q}^0(\rho_n)$:

$$\frac{d\mathbb{Q}^0(\rho_n)}{d\mathbb{P}}\Big|_{\mathcal{F}_T^W} = \frac{Z^{\rho_n}e^{-(1-\rho_n)\Lambda}}{\mathbb{E}\left[Z^{\rho_n}e^{-(1-\rho_n)\Lambda}\right]}.$$

Using (7.1) and the definition of $p^n_{U_\alpha}(q;h^n)$ in (2.11)

$$\begin{aligned} p^n_{U_{\alpha_n}}(q_n;h) &= -\frac{1}{q_n\alpha}\log\left(\frac{u^n_{U_{\alpha_n}}(0,q_n;h)}{u^n_{U_{\alpha_n}}(0)}\right) = -\frac{1}{\gamma_n}\log\left(\frac{\mathbb{E}\left[Z^{\rho_n}e^{-(1-\rho_n)\Lambda-\gamma_n h(Y_T)}\right]}{\mathbb{E}\left[Z^{\rho_n}e^{-(1-\rho_n)\Lambda}\right]}\right), \\ &= -\frac{1}{\gamma_n}\log\left(\mathbb{E}^{\mathbb{Q}^0(\rho_n)}\left[e^{-\gamma_n h(Y_T)}\right]\right). \end{aligned} \tag{7.4}$$

Therefore, the proofs of Propositions 5.3 and 5.8 rely heavily on analysis of the function

$$f(\rho,\gamma) = -\frac{1}{\gamma}\log\left(\frac{\mathbb{E}\left[Z^{\rho}e^{-(1-\rho)\Lambda-\gamma h(Y_T)}\right]}{\mathbb{E}\left[Z^{\rho}e^{-(1-\rho)\Lambda}\right]}\right) = -\frac{1}{\gamma}\log\left(\mathbb{E}^{\mathbb{Q}^0(\rho)}\left[e^{-\gamma h(Y_T)}\right]\right); \quad \rho\in[-1,1], \gamma\in\mathbb{R}. \tag{7.5}$$

As $h(Y_T), \Lambda$ are bounded, f is jointly continuous in (ρ,γ) in $[-1,1]\times\mathbb{R}$. Indeed, continuity is clear for $\rho_n \to \rho \in [-1,1], \gamma_n \to \gamma \neq 0$ and joint continuity for $\gamma_n \to 0$ follows by the approximation $\log(1+x) \approx x$ for small x, as well as the boundedness of $h(Y_T), \Lambda$. Here, $f(\rho_n,\gamma_n) \to f(\rho,0) = \mathbb{E}^{\mathbb{Q}^0(\rho)}[h(Y_T)]$.

7.2. **Proofs.**

Proof of Proposition 5.3. From (7.4) and (7.5), $p^n_{U_{\alpha_n}}(q_n;h) = f(\rho_n,\gamma_n)$ for $\gamma_n = \alpha_n q_n(1-\rho_n^2)$. Now, consider when $\rho_n \to \rho \in [-1,1]$ and $\gamma_n \to 0$. By the joint continuity of f it follows that $f(\rho_n,\gamma_n) \to f(\rho,0) = \mathbb{E}^{\mathbb{Q}^0(\rho)}[h(Y_T)]$. This gives case (i) noting that if $\rho = 1$ then $\mathbb{Q}^0(1) = \mathbb{Q}$. Case (ii) where $\rho_n \to \rho \in [-1,1]$ and $\gamma_n \to \gamma \neq 0$ is immediate by the joint continuity of f, again noting that $\mathbb{Q}^0(1) = \mathbb{Q}$. As for case (iii), set $\underline{h} = \text{ess inf}_{\mathbb{P}}[h(Y_T)] = \inf_{y\in E} h(y)$. Clearly $\liminf_{n\uparrow\infty} f(\rho_n,\gamma_n) \geq \underline{h}$. Now, let $m > \underline{h}$, $A_m = \{h(Y_T) < m\}$ and note that $\mathbb{P}[A_m] > 0$. Then

$$\mathbb{E}\left[Z^{\rho_n}e^{(1-\rho_n)\Lambda-\gamma_n h(Y_T)}\right] \geq e^{-\gamma_n m}\mathbb{E}\left[Z^{\rho_n}e^{-(1-\rho_n)\Lambda}1_{A_m}\right],$$

so that, using (7.5) which is continuous and bounded in ρ, $\limsup_{n\uparrow\infty} f(\rho_n,\gamma_n) \leq m$. The result follows taking $m \downarrow \underline{h}$. □

The following Lemma is used in the proof of Proposition 5.5 and is a basic result on Esscher transformations. Recall $g(\rho,\gamma)$ from (5.7) and note that for $\gamma > 0$:

$$g(\rho,\gamma) = \partial_\gamma(\gamma f(\rho,\gamma)) = -\partial_\gamma\left(\log\left(\mathbb{E}^{\mathbb{Q}^0(\rho)}\left[e^{-\gamma h(Y_T)}\right]\right)\right). \tag{7.6}$$

Lemma 7.1. *Let Assumption 5.1 hold. Let $g(\rho,\gamma)$ be as in* (5.7)*. For a fixed $\rho \in [-1,1]$:*

i) g is strictly decreasing in γ with $\lim_{\gamma\downarrow-\infty} g(\rho,\gamma) = \sup_{y\in E} h(y)$ and $\lim_{\gamma\uparrow\infty} g(\rho,\gamma) = \inf_{y\in E} h(y)$.

ii) For $p \in I(h)$ from (5.5)*, there exists a unique $\gamma = \gamma(\rho,p)$ such that $p = g(\rho,\gamma(\rho,p))$. The map $(\rho,p) \mapsto \gamma(\rho,p)$ is jointly continuous in $I(h)\times[-1,1]$ and for a fixed p is C^1 in ρ.*

Proof of Lemma 7.1. Using the well known properties of cummulant generating functions (i.e. convexity and size asymptotics), part $i)$ immediately follows from (7.6), $\text{ess inf}_{\mathbb{P}}\,[h(Y_T)] = \inf_{y\in E} h(y)$, and $\text{ess sup}_{\mathbb{P}}\,[h(Y_T)] = \sup_{y\in E}$ and as $\mathbb{Q}^0(\rho)$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T^W$. As for $ii)$, part $i)$ clearly gives, for each $\rho\in\mathbb{R}$ and each $p\in I(h)$, a unique $\gamma(\rho,p)$ such that $p = g(\rho,\gamma(\rho,p))$. Now, let $p_n \to p \in I(h)$ and $\rho_n \to \rho \in [-1,1]$. If $\gamma(\rho_{n_k},p_{n_k})$ converges to a limit l for some subsequence then by joint continuity of g it follows that $p = g(\rho,l)$. Thus, by uniqueness $l = \gamma(\rho,p)$ and hence joint continuity follows. Lastly, for $p\in I(h)$ fixed the result follows by the Implicit Function Theorem [40, Theorem 9.28], since g is smooth in ρ,γ and $\partial_\gamma g(\rho,\gamma)\neq 0$. □

Proof of Proposition 5.5. Recall g from (5.7). [25, Theorem 3.1] gives that the optimal $q_n^* = \gamma_n^*/(\alpha_n(1-\rho_n^2))$ satisfies the first order conditions

$$p_n = \partial_q\left(q_n^* p^n_{U_{\alpha_n}}(q_n^*;h)\right) = \partial_\gamma(\gamma_n^* f(\rho_n,\gamma_n^*)) = g(\rho_n,\gamma_n^*).$$

That such a γ_n^* exists and is unique follows from Lemma 7.1 since $\gamma_n^* = \gamma(\rho_n,p_n)$. Now, assume $\alpha_n \equiv \alpha$ is fixed and $\rho_n \to 1$ so that $|q_n^*|\to\infty$ if and only if $|\gamma_n^*|/(1-\rho_n^2)\to\infty$. Assume that $p_n\to p$ for $p\in I(h)$. By joint continuity of both γ and g it follows that $\gamma_n^*\to\gamma^* = \gamma(1,p)$ and $p = g(1,\gamma(1,p)) = g(1,\gamma^*)$. Thus, by Lemma 7.1 it follows that $\gamma^* = 0$ if and only if $p = \hat{p}$ since $\hat{p} = g(1,0)$. Now, if $p\neq\hat{p}$ then $\gamma^*\neq 0$ and $|\gamma_n^*|/(1-\rho_n^2)\to\infty$. If $p=\hat{p}$ then $\gamma_n^*\to 0$ and by the first order Taylor approximation (higher orders may be ignored since $\rho_n\to 1, \gamma_n^*\to 0$)

$$\frac{p_n-\hat{p}}{1-\rho_n^2} = \frac{g(\rho_n,\gamma_n^*)-g(1,0)}{1-\rho_n^2} \approx \frac{-1}{1+\rho_n}\partial_\rho g(1,0) + \frac{\gamma_n^*}{1-\rho_n^2}\partial_\gamma(1,0).$$

Thus, the equivalence in (5.8) readily follow. Lastly, the conclusions in Remark 5.7 are considered. Since $p_n\to p\in I(h), \rho_n\to\rho\in[-1,1]$, the joint continuity of γ, g give $\gamma_n^*\to\gamma^* = \gamma(\rho,p)$ and $p = g(\rho,\gamma(\rho,p))$. Since $g(\rho,0) = g(\rho,\gamma(\rho,g(\rho,0)))$ it holds by uniqueness that $\gamma^* = 0$ if and only if $p = g(\rho,0)$. Lastly, assume ρ is fixed so that $|q_n^*|\to\infty$ if and only if $|\gamma_n^*|/\alpha_n\to\infty$. If $p_n\to p\neq g(\rho,0)$ then

$\gamma^* \neq 0$ and hence $|q_n^*| \to \infty$. If $p_n \to g(\rho, 0)$ then

$$\frac{p_n - g(\rho,0)}{\alpha_n} = \frac{g(\rho,\gamma_n^*) - g(\rho,0)}{\alpha_n} \approx \frac{\gamma_n^* \partial_\gamma g(\rho,0)}{\alpha_n},$$

from which the stated equivalences follow. □

Proof of Proposition 5.8. Since $p_\alpha = f(1,\gamma)$, the monetary error takes the form

$$\text{ME}_n := q_n \left| p^n_{U_\alpha}(q_n;h) - p_\alpha \right| = \frac{\gamma_n}{\alpha(1-\rho_n^2)} \left| f(\rho_n,\gamma_n) - f(1,\gamma) \right|.$$

When $\gamma_n \to \gamma > 0$, it suffices to take the first order Taylor expansion of f around $(1,\gamma)$ and

$$\text{ME}_n \approx \frac{\gamma}{\alpha(1-\rho_n^2)} \left| -(1-\rho_n)\partial_\rho f(1,\gamma) + (\gamma_n - \gamma)\partial_\gamma f(1,\gamma) \right|.$$

As $(1-\rho_n)/(1-\rho_n^2) = 1/(1+\rho_n) \to 1/2$ the first of the above two terms is finite and the equivalence in (5.9) is confirmed. Next, consider when $\gamma_n \to 0$. Here

$$\text{ME}_n \leq \frac{1}{\alpha(1-\rho_n^2)} \left| \gamma_n f(\rho_n,\gamma_n) - \gamma_n f(1,\gamma_n) \right| + \frac{\gamma_n^2}{\alpha(1-\rho_n^2)} \left| \frac{f(1,\gamma_n) - f(1,0)}{\gamma_n} \right|. \tag{7.7}$$

From (7.5), it is clear the map $(\rho,\gamma) \mapsto \gamma f(\rho,\gamma)$ is smooth with bounded first derivatives since Λ, h are bounded. Thus, $(1-\rho_n)/(1-\rho_n^2) \leq 1$ implies the first term on the right in (7.7) remains finite as $n \to \infty$. As for the second term, a direct calculation using $\mathbb{Q}^0(1) = \mathbb{Q}$ shows $\lim_{\gamma\to 0}(1/\gamma)(f(1,\gamma) - f(1,0)) = -\text{Var}^{\mathbb{Q}}[h]/2$. Thus, the equivalences in (5.9) hold. Now, let $p \in I(h)$ and assume γ_n^* is chosen optimally from Proposition 5.5 so that $p = g(\rho_n,\gamma_n^*)$. If $p = \hat{p}$ then by (5.8), since $p_n = \hat{p}$ for all n it holds that $\sup_n |q_n| < \infty$ and hence the monetary error is trivially finite. If $p \neq \hat{p}$, by Proposition 5.5 again, $\gamma_n^* \to \gamma^* \neq 0$ where $p = g(1,\gamma^*)$. Therefore, γ_n^* is in regime (ii) of (5.3) and hence the monetary error is finite if and only if $\sup_n |\gamma_n^* - \gamma^*|/(1-\rho_n^2) < \infty$. Using the notation in part $ii)$ of Lemma 7.1, $\gamma_n^* = \gamma(\rho_n,p)$ and $\gamma^* = \gamma(1,p)$. Since the map $\rho \mapsto \gamma(\rho,p)$ is C^1, for any $\varepsilon > 0$ by taking n large enough

$$\frac{|\gamma_n^* - \gamma^*|}{1-\rho_n^2} = \frac{|\gamma(\rho_n,p) - \gamma(1,p)|}{1-\rho_n^2} = \frac{1}{1+\rho_n}\frac{1}{1-\rho_n}\left| \int_{\rho_n}^1 \partial_\rho \gamma(\tau,p)d\tau \right| \leq |\partial_\rho \gamma(1,p)| + \varepsilon,$$

and hence the monetary error is bounded.

□

Proof of Proposition 5.9. A direct calculation using (5.1), (7.3) gives

$$\frac{d\mathbb{Q}_W^{n,q_n}}{d\hat{\mathbb{Q}}_W^n} = \frac{e^{-(1-\rho_n^2)\Lambda - \gamma_n h(Y_T)}}{\mathbb{E}\left[Z(\rho_n)e^{-(1-\rho_n^2)\Lambda - \gamma_n h(Y_T)}\right]}.$$

Therefore, using (7.4), (5.7) and that $p = g(\rho_n, \gamma_n)$ for g from (5.7):

$$H\left(\mathbb{Q}_W^{n,q_n} \mid \hat{\mathbb{Q}}_W^n\right) = \gamma_n(p_{U_\alpha}^n(q_n;h) - p) - (1-\rho_n^2)\frac{\mathbb{E}\left[\Lambda Z(\rho_n)e^{-(1-\rho_n^2)\Lambda-\gamma_n h(Y_T)}\right]}{\mathbb{E}\left[Z(\rho_n)e^{-(1-\rho_n^2)\Lambda-\gamma_n h(Y_T)}\right]} - \log\left(\mathbb{E}\left[Z(\rho_n)e^{-(1-\rho_n^2)\Lambda}\right]\right), \tag{7.8}$$

and so the result follows by Proposition 5.3 since $\gamma_n \to \gamma \neq 0$.

□

Appendix A. Supporting Lemmas

Remark A.1. Throughout this section, $(\Omega, \mathcal{F}, \mathbb{P})$ represents a generic probability space and all expectations are with respect to $\mathbb{P}$. Inequalities regarding random variables are assumed to hold $\mathbb{P}$ almost surely.

Lemma A.2. *Let $\alpha > 0$, $p > 1$ and $l > 0$. Then for $U \in \mathcal{U}_\alpha$ or $U \in \mathcal{U}_{p,l}$ it follows that U satisfies the Inada conditions $\lim_{x\downarrow-\infty} U'(x) = \infty$, $\lim_{x\uparrow\infty} U'(x) = 0$ as well as the conditions of Reasonable Asymptotic Elasticity in* (2.6).

Proof of Lemma A.2. For either $U \in \mathcal{U}_\alpha$ or $U \in \mathcal{U}_{p,l}$ since $U(x) < 0$ for all x, the condition in (2.6) for $x \uparrow \infty$ immediately holds. Additionally, since U is concave and $U(\infty) = 0$ it follows that for $x > 0$, $xU'(x) \leq U(x) - U(0) \leq -U(0)$ and hence $\lim_{x\uparrow\infty} U'(x) = 0$. Similarly, for $x < 0$, $-xU'(x) \geq U(0) - U(x)$ and hence $\lim_{x\downarrow-\infty} U'(x) = \infty$ (for $U \in \mathcal{U}_{p,l}$ this holds because $p > 1$).

Now, let $U \in \mathcal{U}_\alpha$ and note that for $0 < \beta < 1$:

$$\lim_{x\downarrow-\infty} -\frac{1}{x}\log\left(\frac{U(\beta x)}{U(x)}\right) = (\beta-1)\alpha < 0. \tag{A.1}$$

It is claimed that $\lim_{x\downarrow-\infty} xU'(x)/U(x) = \infty$ which clearly implies (2.6). Indeed, assume there is some $K > 1$ and $x_n \downarrow -\infty$ so that $x_n U'(x_n)/U(x_n) \leq K$ for all n. Let $1 - 1/K < \beta < 1$. By concavity, for $x_n \leq y \leq \beta x_n$ one has $U'(y) \leq U'(x_n) \leq KU(x_n)/x_n$. Therefore

$$U(\beta x_n) - U(x_n) \leq \frac{KU(x_n)}{x_n}(\beta x_n - x_n) = -KU(x_n)(1-\beta).$$

This implies, since $U(x_n) < 0$ and $1 - 1/K < \beta < 1$, that $U(\beta x_n)/U(x_n) \geq 1 - K(1-\beta) > 0$. Thus, $\liminf_{x\downarrow-\infty} -(1/x)\log\left(U(\beta x_n)/U(x_n)\right) \geq 0$, but this contradicts (A.1). Now, let $U \in \mathcal{U}_{p,l}$. By definition of $\mathcal{U}_{p,l}$:

$$\lim_{x\downarrow-\infty} \frac{U(\beta x)}{U(x)} = \beta^p; \qquad 0 < \beta < 1. \tag{A.2}$$

By way of contradiction, assume for some $x_n \downarrow -\infty$ that $\lim_{n\uparrow\infty} x_n U'(x_n)/U(x_n) = 1$. Let $0 < \beta < 1$ and $0 < \varepsilon < \beta/(1-\beta)$. Take n large enough so that $U'(x_n) \leq (1+\varepsilon)U(x_n)/x_n$. By the concavity of U:

$$U(\beta x_n) - U(x_n) \leq \frac{(1+\varepsilon)U(x_n)}{x_n}(y_n - x_n) = -(1+\varepsilon)U(x_n)(1-\beta).$$

$U(x_n) < 0$ implies $U(\beta x_n)/U(x_n) \geq 1 - (1+\varepsilon)(1-\beta) > 0$. In view of (A.2) this implies $1 - (1+\varepsilon)(1-\beta) \leq \liminf_{n\uparrow\infty} U(\beta x_n)/U(x_n) = \beta^p$. Taking $\varepsilon \downarrow 0$ gives $\beta < \beta^p$ which is a contradiction since $\beta < 1$ and $p > 1$. □

Lemma A.3. *Let $Y \geq 0$. The following statements are equivalent:*

1) $\mathbb{E}[V(yY)] < \infty$ *for all* $\alpha > 0$, $U \in \mathcal{U}_\alpha$ *and* $y > 0$.

2) $\mathbb{E}[Y \log Y] < \infty$.

Furthermore, let $p > 1$ and set $\gamma = p/(p-1)$. Then the following statements are equivalent:

A) $\mathbb{E}[V(yY)] < \infty$ *for all* $l > 0$, $U \in \mathcal{U}_{p,l}$ *and* $y > 0$.

B) $\mathbb{E}[Y^\gamma] < \infty$.

Proof of Lemma A.3. Let $\alpha > 0$ and $U \in \mathcal{U}_\alpha$. In view of (2.8), for any $\varepsilon > 0$ there is some constant $M = M(\varepsilon, U) > 0$ such that on $z \geq M$, $((1-\varepsilon)/\alpha)z(\log(z) - 1) \leq V(z) \leq ((1+\varepsilon)/\alpha)z(\log(z) - 1)$. Since both $|V(z)|$ and $|z\log(z) - 1|$ are bounded on $[0, M]$, there is some $C = C(\varepsilon, M) > 0$ so that $-C + \frac{1-\varepsilon}{\alpha} z\log(z) \leq V(z) \leq C + \frac{1+\varepsilon}{\alpha} z\log(z)$. The equivalences $1) \Leftrightarrow 2)$ now readily follow. Similarly, in view of (4.1) for any $\varepsilon > 0$ there is some constant $M = M(\varepsilon, U)$ so that on $z \geq M$, $(1-\varepsilon)(1/\gamma)\hat{l}z^\gamma \leq V(z) \leq (1+\varepsilon)\hat{l}z^\gamma$. Again, since $|V(z)|$ and z^γ are bounded on $[0, M]$ there is a constant $C = C(\varepsilon, U)$ so that $-C + (1-\varepsilon)\hat{l}z^\gamma \leq V(z) \leq C + (1+\varepsilon)\hat{l}z^\gamma$. The equivalences $A) \Leftrightarrow B)$ readily follow.

□

Lemma A.4. *Let $\alpha > 0$, $p > 1$, $l > 0$. Let $U \in \mathcal{U}_\alpha$ or $U \in \mathcal{U}_{p,l}$. Let $Y \geq 0$ be such that $\mathbb{E}[Y] = 1$ and such that $\mathbb{E}[V(Y)] < \infty$. Then the map $y \mapsto \mathbb{E}[V(yY)]$ is differentiable with derivative $\mathbb{E}[YV'(yY)]$. Furthermore, for any $x \in \mathbb{R}$ there exists a unique y such that $\mathbb{E}[YV'(yY)] = x$.*

Proof. Consider the function $f(\varepsilon, z) := (1/\varepsilon)(V((y+\varepsilon)z) - V(yz)) - (1/y)V(yz)$ for $\varepsilon > 0, z \geq 0$. Note that $f(\varepsilon, 0) = 0$ and, as V' is strictly increasing, $\partial_z f(\varepsilon, z) \geq 0$. The convexity of V implies $f(\varepsilon, Y) \leq V((1+y)Y) - V(yY) - V(yY)/y$. That $\partial_y \mathbb{E}[V(yY)] = \mathbb{E}[YV'(yY)]$ now follows by applying the dominated convergence theorem to $f(\varepsilon, Y)$, since Lemma A.3 implies $\mathbb{E}[V(yY)] < \infty$ for all $y > 0$.

Now, consider the map $g(y) := \mathbb{E}\left[YV'(yY)\right]$. The strict convexity of V implies g is strictly increasing. Since $\lim_{z\downarrow} zV'(z) = 0$ (because U is bounded from above), there is some constant $C > 0$ such that $YV'(yY) > -C$ for $y > 1$. Thus, since $\lim_{z\uparrow\infty} V'(z) = \infty$ (because of the Inada conditions) it follows by Fatou's Lemma that $\lim_{y\uparrow\infty} g(y) = \infty$. For the limit as $y \downarrow 0$, assume $y < 1$. Denote by $\hat{y}$ by the unique number such that $V'(\hat{y}) = 0$. Clearly $g(y) = \mathbb{E}\left[YV'(yY)1_{yY\leq\hat{y}}\right] + \mathbb{E}\left[YV'(yY)1_{yY>\hat{y}}\right]$. Fatou's Lemma and $\lim_{z\downarrow 0} V'(z) = -\infty$ imply that $\lim_{y\downarrow 0}\mathbb{E}\left[YV'(yY)1_{yY\leq\hat{y}}\right] = -\infty$. As for the second term, since $y < 1$, $V'(yY) \leq V'(Y)$. Thus, using [41, Corollary 4.2 (ii)] (note : part (ii) therein does not require $U(0) > 0$) there is a constant $C > 0$ so that $\mathbb{E}\left[YV'(yY)1_{yY\geq\hat{y}}\right] \leq \mathbb{E}\left[YV'(Y)1_{yY\geq\hat{y}}\right] \leq C\mathbb{E}\left[V(Y)\right] < \infty$. Thus, $\lim_{y\downarrow 0} g(y) = -\infty$ and the result holds.

□

Lemma A.5. *Let $\alpha > 0$, $p > 1$ and $l > 0$. Let $u > 0$ and $Y \geq 0$ be such that $\mathbb{E}\left[Y\right] = 1$. Then, for each $\varepsilon > 0$ there exists a constant $\overline{C}(\varepsilon, U)$ independent of Y and u such that*

$$\inf_{y>0}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) \leq \overline{C}(\varepsilon, U) + \begin{cases} \frac{1+\varepsilon}{\alpha}\left(\mathbb{E}\left[Y\log(Y)\right] + \log(u)\right) & U \in \mathcal{U}_\alpha \\ \frac{1}{\alpha}\mathbb{E}\left[Y\log(Y)\right] + \log(u) & U \in \tilde{\mathcal{U}}_\alpha \\ (lu)^{1/p}\left((1+\varepsilon)\mathbb{E}\left[Y^\gamma\right]\right)^{1/\gamma} & U \in \mathcal{U}_{p,l} \end{cases}. \tag{A.3}$$

Proof. Since for either $U \in \mathcal{U}_\alpha$ or $U \in \mathcal{U}_{p,l}$ it follows that $V(0) = 0, \lim_{y\downarrow 0} V'(y) = -\infty$ there is some $M = M(U)$ such that $V(z) \leq 0$ on $z < 1/M$ and $V(z) \leq V(M)$ on $1/M \leq z \leq M$. This gives

$$\frac{1}{y}\mathbb{E}\left[V(yY)\right] \leq V(M)\frac{1}{y}\mathbb{E}\left[1_{1/M\leq yY\leq M}\right] + \frac{1}{y}\mathbb{E}\left[V(yY)1_{yY>M}\right] \leq MV(M) + \frac{1}{y}\mathbb{E}\left[V(yY)1_{yY>M}\right] \tag{A.4}$$

where the last inequality holds since $(1/y) \leq MY$ on $\{1/M \leq yY \leq M\}$. Now, let $\varepsilon > 0$ and $U \in \mathcal{U}_\alpha$. Enlarge M so that $M > e$ and $V(z) \leq ((1+\varepsilon)/\alpha)z(\log(z) - 1)$ on $z > M$. (A.4) gives

$$\begin{aligned} \frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) &\leq \frac{u}{y} + V(M)M + \frac{1+\varepsilon}{\alpha}\mathbb{E}\left[Y(\log(yY) - 1)1_{yY>M}\right] \\ &\leq \frac{u}{y} + MV(M) + \frac{1+\varepsilon}{\alpha}(\log(y) - 1) + \frac{1+\varepsilon}{\alpha}\mathbb{E}\left[Z\log(Z)\right]. \end{aligned} \tag{A.5}$$

The minimum of the above over $y > 0$ occurs at $y = \alpha u/(1+\varepsilon)$. Plugging this in gives

$$\inf_{y>0}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) \leq MV(M) + \frac{1+\varepsilon}{\alpha}\log\left(\frac{\alpha}{1+\varepsilon}\right) + \frac{1+\varepsilon}{\alpha}\left(\mathbb{E}\left[Y\log(Y)\right] + \log(u)\right),$$

and (A.3) follows. Now, let $U \in \tilde{\mathcal{U}}_\alpha$ and define

$$f_U(z) := V(z) - \frac{1}{\alpha}z(\log(z) - 1). \tag{A.6}$$

Calculation shows $\limsup_{z\uparrow\infty} |f_U(z)|/z < \infty$ and hence by enlarging M (bigger than e) there is some $K = K(\varepsilon, U)$ so that $f(z) \leq Kz$ on $z > M$. Thus, (A.4) yields

$$\begin{aligned}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) &\leq \frac{u}{y} + MV(M) + \frac{1}{\alpha}\mathbb{E}\left[Y(\log(yY) - 1)1_{yY>M}\right] + K\mathbb{E}\left[Y1_{yY>M}\right] \\ &\leq \frac{u}{y} + MV(M) + \frac{1}{\alpha}(\log(y) - 1) + \frac{1}{\alpha}\mathbb{E}\left[Y\log(Y)\right] + K\end{aligned}$$

The result follows by repeating the argument after (A.5) taking ε therein to be 0. Lastly, let $U \in \mathcal{U}_{p,l}$ and enlarge M so that $V(z) \leq (1+\varepsilon)\hat{l}z^{\gamma}$ on $z > M$. Here, (A.4) yields

$$\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) \leq \frac{u}{y} + MV(M) + (1+\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}\left[Y^{\gamma}\right].$$

A direct calculation shows that

$$\inf_{y>0}\left(\frac{u}{y} + (1+\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}\left[Y^{\gamma}\right]\right) = \gamma\left(\frac{u}{\gamma-1}\right)^{1/p}\left((1+\varepsilon)\hat{l}\mathbb{E}\left[Y^{\gamma}\right]\right)^{1/\gamma},$$

As $\gamma(\gamma-1)^{-1/p}\hat{l}^{1/\gamma} = l^{1/p}$,

$$\inf_{y>0}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) \leq MV(M) + (lu)^{1/p}\left((1+\varepsilon)\mathbb{E}\left[Y^{\gamma}\right]\right)^{1/\gamma},$$

which gives (A.3). □

Lemma A.6. *Let $\alpha > 0$, $p > 1$ and $l > 0$. Let $u > 0$ and $Y \geq 0$ be such that $\mathbb{E}[Y] = 1$. Then for each $0 < \varepsilon < \min\{u, 1\}$ there exist a constant $\underline{C}(\varepsilon, U)$ independent of Y and u such that*

$$\inf_{y>0}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right] + u\right) \geq \underline{C}(\varepsilon, U) + \begin{cases}\frac{1-\varepsilon}{\alpha}\left(\mathbb{E}\left[Y\log(Y)\right] + \log(u)\right) & U \in \mathcal{U}_{\alpha} \\ \frac{1}{\alpha}\left(\mathbb{E}\left[Y\log(Y)\right] + \log(u)\right) & U \in \tilde{\mathcal{U}}_{\alpha} \\ \left(l(u - \frac{\varepsilon}{2})\right)^{1/p}\left((1-\varepsilon)\mathbb{E}\left[Y^{\gamma}\right]\right)^{1/\gamma} & U \in \mathcal{U}_{p,l}\end{cases}. \tag{A.7}$$

Proof of Lemma A.6. Let $0 < \varepsilon < \min\{u, 1\}$. Let $U \in \mathcal{U}_{\alpha}$ or $U \in \mathcal{U}_{p,l}$. As $V(0) = 0$ and (2.8), there is some $M = M(\varepsilon, U)$ so that $V(z) \geq -\varepsilon/2$ on $z < 1/M$ and $V(z) \geq U(0)$ on $1/M \leq z \leq M$. As $U(0) < 0$, a similar calculation to (A.4) shows

$$\frac{1}{y}\mathbb{E}\left[V(yY)\right] \geq \frac{-\varepsilon/2}{y} + U(0)M + \frac{1}{y}\mathbb{E}\left[V(yY)1_{yY>M}\right]. \tag{A.8}$$

Now, let $U \in \mathcal{U}_\alpha$ and enlarge M so that $M > e$ and $V(z) \leq ((1-\varepsilon)/\alpha)z(\log(z)-1)$ on $z > M$. Using (A.8)

(A.9)
$$\begin{aligned}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right]+u\right) &\geq \frac{u-\varepsilon/2}{y} + U(0)M + \frac{1-\varepsilon}{\alpha}\mathbb{E}\left[Y(\log(yY)-1)(1-1_{yY\leq M})\right], \\ &\geq \frac{u-\varepsilon/2}{y} + U(0)M - \frac{1-\varepsilon}{\alpha}\log(M) + \frac{1-\varepsilon}{\alpha}\log(y) + \frac{1-\varepsilon}{\alpha}\mathbb{E}\left[Y\log(Y)\right].\end{aligned}$$

As $u - \varepsilon/2 > 0$,

$$\inf_{y>0}\left(\frac{1-\varepsilon}{\alpha}\log(y) + \frac{u-\varepsilon/2}{y}\right) = \frac{1-\varepsilon}{\alpha}\left(1+\log(u)+\log\left(\frac{\alpha(u-\varepsilon/2)}{u(1-\varepsilon)}\right)\right). \tag{A.10}$$

Plugging this in above and using that $1-\varepsilon/(2u) \geq 1/2$, (A.7) holds for $U \in \mathcal{U}_\alpha$ with $\underline{C}(\varepsilon, U) = U(0)M + (1-\varepsilon)/(\alpha)\left(1+\log(\alpha/(2M(1-\varepsilon)))\right)$. Regarding (A.7) for $U \in \tilde{\mathcal{U}}_\alpha$, let f_U be as in (A.6) and recall from the previous lemma that $\limsup_{z\uparrow\infty}|f_U(z)|/z < \infty$. Thus, by enlarging M there exits some $K = K(\varepsilon, U)$ so that $f_U(z) \geq -Kz$ on $z > M$. (A.8) yields

$$\begin{aligned}\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right]+u\right) &\geq \frac{u-\varepsilon/2}{y} + U(0)M + \frac{1}{\alpha}\mathbb{E}\left[Y(\log(yY)-1)1_{yY>M}\right] - K\mathbb{E}\left[Y1_{yY>M}\right], \\ &\geq \frac{u-\varepsilon/2}{y} + U(0)M - K + \frac{1}{\alpha}\mathbb{E}\left[Y(\log(yY)-1)(1-1_{yY\leq M})\right].\end{aligned} \tag{A.11}$$

Repeating the calculation starting in (A.9) but with $1/\alpha$ replacing $(1+\varepsilon)/\alpha$ therein yields (A.7) with $\underline{C}(\varepsilon, U) = U(0)M + (1/\alpha)(1+\log(\alpha/(2M)))$. Lastly, let $U \in \mathcal{U}_{p,l}$. Here, enlarge M so that $V(z) \geq (1-\varepsilon)\hat{l}z^\gamma$ on $z > M$. (A.8) gives

$$\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right]+u\right) \geq \frac{u-\varepsilon/2}{y} + U(0)M + (1-\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}\left[Y^\gamma(1-1_{yY\leq M})\right].$$

$y^{\gamma-1}\mathbb{E}\left[Y^\gamma 1_{yY\leq M}\right] \leq M^{\gamma-1}$ since $\mathbb{E}\left[Y\right] = 1$. Thus

$$\frac{1}{y}\left(\mathbb{E}\left[V(yY)\right]+u\right) \geq \frac{u-\varepsilon/2}{y} + U(0)M - (1-\varepsilon)\hat{l}M^{\gamma-1} + (1-\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}\left[Y^\gamma\right].$$

Calculation shows that

$$\inf_{y>0}\left(\frac{u-\varepsilon/2}{y} + (1-\varepsilon)\hat{l}y^{\gamma-1}\mathbb{E}\left[Y^\gamma\right]\right) = \gamma\left(\frac{u-\varepsilon/2}{\gamma-1}\right)^{1/p}\left((1-\varepsilon)\hat{l}\mathbb{E}\left[Y^\gamma\right]\right)^{1/\gamma}.$$

As shown in the previous lemma, $\gamma(\gamma-1)^{-1/p}\hat{l}^{1/\gamma} = l^{1/p}$, so that (A.7) holds with $\underline{C}(\varepsilon, U) = U(0)M - (1-\varepsilon)\hat{l}M^{\gamma-1}$. □

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