

RECTIFIABILITY AND UNIQUENESS OF BLOW-UPS FOR POINTS WITH POSITIVE ALT-CAFFARELLI-FRIEDMAN LIMIT

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ABSTRACT. We study the regularity of the interface between the disjoint supports of a pair of nonnegative subharmonic functions. The portion of the interface where the Alt-Caffarelli-Friedman (ACF) monotonicity formula is asymptotically positive forms an \mathcal{H}^{n-1} -rectifiable set. Moreover, for \mathcal{H}^{n-1} -a.e. such point, the two functions have unique blowups, i.e. their Lipschitz rescalings converge in $W^{1,2}$ to a pair of nondegenerate truncated linear functions whose supports meet at the approximate tangent plane. The main tools used include the Naber-Valtorta framework and our recent result establishing a sharp quantitative remainder term in the ACF monotonicity formula. We also give applications of our results to free boundary problems.

1. INTRODUCTION

Recent decades have seen a significant body of research focused on understanding the interplay between the geometry of subsets of Euclidean space, their analytic properties, and the behavior of solutions to partial differential equations on these sets. A central goal of this program is to give geometric conditions that imply or characterize the regularity—often rectifiability—of a given set. Results of this type go back to Reifenberg [Rei60], and since pioneering work in the nineties [Jon90, DS91, BJ94, Tor95, Leg99], there has been a steady stream of increasingly refined results with compelling applications. Two recent and fundamental results in this direction are Jaye, Tolsa, and Villa’s resolution to Carleson’s ϵ^2 -conjecture [JTV21] and Naber and Valtorta’s Rectifiable Reifenberg Theorem [NV17] (see also [AT15, Tol15, ENV19] for related results), both of which give criteria for a set to be rectifiable in terms of geometric square functions that measure flatness at all points and scales. The latter, and the Naber-Valtorta framework more broadly, have powerful applications to singularity analysis in various nonlinear PDE, such as (approximate) harmonic maps [NV17, NV18], stationary varifolds [NV20], and free boundary problems [EE19], and will also play a key role here.

In this paper we give a different kind of rectifiability criterion, which relates the boundary behavior of nonnegative subharmonic functions to the regularity of a set. The key quantity is the *Alt-Caffarelli-Friedman (ACF) monotonicity formula*

$$(1.1) \quad J_x(r) = \left(\frac{1}{r^2} \int_{B_r(x)} \frac{|\nabla u|^2}{|x-y|^{n-2}} \right) \left(\frac{1}{r^2} \int_{B_r(x)} \frac{|\nabla v|^2}{|x-y|^{n-2}} \right)$$

for a pair of nonnegative subharmonic functions u and v with $u \cdot v = 0$. Our first main result, Theorem 1.1 below, shows that the set Γ^* of points x in the interface $\Gamma := \partial\{u > 0\} \cup \partial\{v > 0\}$ where $\lim_{r \rightarrow 0} J_x(r)$ is positive forms an \mathcal{H}^{n-1} -rectifiable set. Examples show the conclusion is in a sense optimal: this set does not need to have an approximate tangent space at *every* point, and the full interface need not be rectifiable. Our second main result, Theorem 1.2, concerns the boundary behavior of the functions u, v themselves: at \mathcal{H}^{n-1} -a.e. point x in Γ^* , the functions have unique blowups to a pair of truncated linear functions. Our hypotheses are unconditional in that they require no a priori information about the interface, and the asymptotic positivity of J is a condition that is easily checked in the context of free boundary problems. These two key features allow us to apply our result to a broad class of free boundary problems, which we outline in section 2.3.

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More precisely, fix $n \geq 2$ and let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions that satisfy

$$(1.2) \quad \begin{cases} -\Delta u \leq 0 & \text{in } \{u > 0\}, \\ -\Delta v \leq 0 & \text{in } \{v > 0\}, \\ u \cdot v = 0 & \text{in } B_{10}(0). \end{cases}$$

The simplest example of such a pair of functions is a pair of truncated linear functions supported on complementary half-planes meeting at a hyperplane interface, i.e.

$$(1.3) \quad u(y) = a((y-x) \cdot \nu)^+ \quad \text{and} \quad v(y) = b((y-x) \cdot \nu)^-$$

for $x \in \mathbb{R}^n$, constants $a, b > 0$ and $\nu \in \mathbb{S}^{n-1}$. More generally, however, the condition (1.2) is quite flexible and allows for a rich assortment of behaviors from u, v , and the interface between their supports; see the examples in section 2.1. One encounters the configuration (1.2) in various settings, for instance when u, v are phases of a solution to the two-phase Bernoulli problem in models for jets and cavities [ACF84c, ACF84a, ACF84b], when u and v are the positive and negative parts of directional derivatives of solutions to obstacle-type problems [PSU12, Sha03], or when u, v are population densities for two species in segregation models [CKL09, TVZ19]. In all of these contexts, the regularity of the interface Γ is a priori unknown, and indeed a typical objective is to understand its regularity. From the viewpoint of geometric measure theory and harmonic analysis, it is natural to use the boundary behavior of (sub-)solutions to PDE as a way to understand the regularity of an interface; this is typically studied via harmonic and elliptic measure [AHM⁺16, AHM⁺20, HMM⁺21, KP01].

The ACF monotonicity formula (1.1) was introduced by Alt, Caffarelli, and Friedman in [ACF84c] and since then has found a wide variety of applications, including to the settings described in the previous paragraph. The key feature of the quantity (1.1) is its monotonicity: given u, v as above and $x \in B_4(0)$, the function $r \mapsto J_x(r)$ is nondecreasing for $r \in (0, 4)$. Moreover, if $J_x(r_1) = J_x(r_2)$ for $r_1 < r_2$, then either one of u, v is identically zero in $B_{r_2}(x)$ (in which case $J_x(r) \equiv 0$), or else u and v are a pair of complementary truncated linear functions as in (1.3) in $B_{r_2}(x)$; see [CS05].

This rigidity statement contains stronger information when $J_x(r_1) > 0$, and in particular implies that the interface Γ is locally a hyperplane. Instead if $J_x(r_1) = 0$, the fact that u or v vanishes identically does not impose any interfacial structure. It is natural, then, that the subset of the interface Γ where one expects good structure is the set of points where the ACF formula is asymptotically positive at small scales, that is, on

$$\Gamma^* := \{x : J_x(0^+) > 0\}.$$

Here $J_x(0^+) := \lim_{r \rightarrow 0^+} J_x(r)$; monotonicity guarantees the existence of the limit. The basis of this paper is to exploit *almost* rigidity in the ACF monotonicity formula via our results in [AKNb] to prove structural properties of Γ^* and interfacial regularity of u and v on this set. Our first theorem asserts the \mathcal{H}^{n-1} -rectifiability of Γ^* .

Theorem 1.1 (Rectifiability). *Fix $n \geq 2$ and let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions that satisfy (1.2). Then $\Gamma^* \cap B_1(0)$ is \mathcal{H}^{n-1} -rectifiable.*

We will see examples in section 2.1 demonstrating that on the remaining portion of the interface where $J_x(0^+) = 0$, the interface can fail to have an approximate tangent plane at *every* point, even under the stronger assumptions that u and v are harmonic where they are positive. Furthermore, Γ^* is not better than rectifiable, and in particular the assumption $J_x(0^+) > 0$ does not imply the existence of an approximate tangent plane at x . Our second main theorem shows that the functions u and v have unique blowups at \mathcal{H}^{n-1} -a.e. point in Γ^* .

Theorem 1.2 (Uniqueness of blowups). *Fix $n \geq 2$ and let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions that satisfy (1.2). For \mathcal{H}^{n-1} -a.e. $x \in \Gamma^* \cap B_{1/2}(0)$, there exist constants $a^x, b^x > 0$ and a unit vector $\nu^x \in \mathbb{S}^{n-1}$ such that the rescaled functions*

$$(1.4) \quad u^{x,r}(z) = \frac{u(x+rz)}{r} \quad v^{x,r}(z) = \frac{v(x+rz)}{r}$$

converge locally in the strong $W^{1,2}$ topology to $\ell_+(z) = a^x(z \cdot \nu)^+$ and $\ell_-(z) = b^x(z \cdot \nu)^-$, respectively. Moreover, $a^x b^x = c_ J_x(0^+)$, where $c_* > 0$ is an explicit dimensional constant.*

1.1. Discussion of the proofs. We use two main tools to prove Theorem 1.1. The first is Naber-Valtorta’s quantitative stratification and rectifiable Reifenberg framework. Introduced in [NV17], these techniques are designed to study singular sets in geometric PDE and are remarkably flexible: whenever one has a solution to a PDE with (a) a monotonicity formula with a quadratic remainder term, and (b) strong compactness properties for sequences of solutions (usually coming from a priori estimates), one can apply the Naber-Valtorta tools to prove rectifiability of the strata and estimates for the quantitative strata. If one additionally has an ϵ -regularity theorem, this automatically gives packing and Hausdorff measure estimates for the full singular set.

The present context does not fit within this standard setup. First, we do not consider solutions to a PDE, at least not in a conventional sense: in (1.2) we assume only that u and v are *subsolutions* to the Laplace equation and impose no sort of transmission condition across the interface as one would in a free boundary problem. While subharmonicity provides important structure, the setup in (1.2) is simply too weak of an “equation” to expect a real regularity theory. Nonetheless, functions satisfying (1.2) do enjoy a monotonicity formula, and this suggests some hope of applying the Naber-Valtorta ideas. However, until our recent results in [AKNb], both key ingredients (a) and (b) described above were entirely missing in this context.

The second key tool used to prove Theorem 1.1, which equips us with ingredient (a), is the sharp quantitative remainder term in the ACF monotonicity formula we proved in [AKNb]; see Theorem 3.3 below. Often, a monotonicity formula’s proof automatically yields a quantitative remainder term that detects how far an object is from a “cone” solution; this is the case, for instance, for the area ratio for minimal surfaces, the normalized energy for harmonic maps, and the Weiss formula for Bernoulli free boundary problems. Instead, it is quite challenging to glean from the proof of the ACF formula how far functions u, v satisfying (1.2) are from truncated linear functions as in (1.3). The quantitative remainder term in [AKNb] uses a new type of sharp quantitative stability estimate for the Faber-Krahn inequality on the sphere, which in turn relies on delicate free boundary regularity results in [AKNa]. Ingredient (b) in this context is false in the strength one might hope for, but again it is Theorem 3.3 that provides us with enough $(W^{1,2})$ compactness to adapt the Naber-Valtorta ideas.

Another difference between more typical applications of the Naber-Valtorta framework and the present setting is that points in Γ^* are not singular points. In classical singularity analysis, one stratifies the singular set for a solution to a given PDE according to the number of symmetries of its blowups. On the other hand, for $x \in \Gamma^*$, every blowup takes the form (1.3) and in particular is “ $(n - 1)$ -symmetric.” This allows for various simplifications. Most notably, in Section 6.3, we give a covering argument that is significantly more streamlined than the delicate good tree/bad tree constructions used in other settings. Throughout the proof of Theorem 1.1, we will attempt to emphasize which steps are standard applications of Naber-Valtorta machinery and which steps involve novel ideas or significant departures from the usual approach. As far as the global structure of the Naber-Valtorta framework, we have in particular followed the presentation in [EE19].

Theorem 1.2 does not have an analogue in the Naber-Valtorta theory, and uniqueness of blowups is generally a subtle question. Theorem 1.2 is proven through geometric measure theoretic arguments, using the rectifiability of Γ^* and the measure estimates of Theorem 1.3 below together with the precise form (1.3) of blowups and the quantitative ACF monotonicity formula from [AKNb]. For any sequence $r_k \rightarrow 0$, the rescaled functions (1.4) at $x \in \Gamma^*$ subsequentially converge to a pair of truncated linear functions as in (1.3). A priori, the slopes a and b and the direction ν might depend on the sequence of scales — and for some points they do, as we will see in Example 2.5 below. Using Theorem 1.1, we show that ν is determined up to a sign at \mathcal{H}^{n-1} -a.e. point by the unique approximate tangent plane to Γ^* . To prove the a.e. uniqueness of the slopes a, b , we use the differentiation theory for measures and Theorem 1.3 to show that the distributional Laplacians Δu and Δv have densities with respect to $\Gamma^* \llcorner \mathcal{H}^{n-1}$ at \mathcal{H}^{n-1} -a.e. point in Γ^* . The quantitative form of the ACF formula then allows us to relate these densities (which are independent of the blowup sequence) to the slopes a, b .

1.2. Quantitative estimates. Following [CN13, NV17], we split Γ^* into smaller pieces on which we can obtain quantitative control. Since the ACF formula becomes degenerate when $J(0^+) = 0$, we divide Γ^* into pieces where $J_x(0^+)$ is *uniformly* positive, and then enlarge these pieces to include points where $J_x(r)$ is uniformly positive down to a definite scale. More specifically, let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions that satisfy (1.2). Fix $\epsilon > 0$ and $r > 0$, and define the sets

$$\Gamma_\epsilon^* = \{x \in \Gamma^* : J_x(0^+) \geq \epsilon\}, \quad \text{and} \quad \Gamma_{\epsilon,r}^* = \{x \in \Gamma^* : J_x(r) \geq \epsilon\}.$$

Observe that $\Gamma_\epsilon^* = \bigcap_{r>0} \Gamma_{\epsilon,r}^*$ and $\Gamma^* = \bigcup_{\epsilon>0} \Gamma_\epsilon^*$, and that $\Gamma_{\epsilon,r}^* \subset \Gamma_{\delta,s}^*$ if $\delta < \epsilon$ and $r < s$. Following [NV17], we prove the following main quantitative estimates that will lead to Theorem 1.1.

Theorem 1.3. Fix $n \geq 2$, $\epsilon > 0$, and $\bar{J}_0 > 0$. There is a constant $C_0 = C_0(n, \epsilon, \bar{J}_0) > 0$ such that the following holds. Let $u, v : B_8(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying (1.2) with $0 \in \Gamma(u, v)$ and $\sup_{x \in B_4} J_x(4) \leq \bar{J}_0$. For every $r \in (0, 1]$, we can find a finite collection of balls $\{B_r(x_i)\}_{i=1}^N$ such that

$$(1.5) \quad \Gamma_{\epsilon, r}^* \cap B_1(0) \subset \bigcup_{i=1}^N B_r(x_i) \quad \text{with} \quad N \leq C_0 r^{1-n}.$$

In particular, for every $r \in (0, 1]$,

$$(1.6) \quad |B_r(\Gamma_{\epsilon, r}^*) \cap B_1(0)| \leq C_0 r \quad \text{and} \quad \mathcal{H}^{n-1}(\Gamma_{\epsilon}^* \cap B_1(0)) \leq C_0.$$

2. EXAMPLES, APPLICATIONS, AND FURTHER CONNECTIONS

In this section, we present examples demonstrating the sharpness of our main results, discuss connections to harmonic analysis, and give applications to free boundary problems.

2.1. Examples. Let us give some examples of different pairs of functions u and v satisfying (1.2) showing what sort of behavior is and is not possible for such functions and their interfaces. Although given in two dimensions, the examples extend to higher dimensions simply by defining $\tilde{u}(x_1, \dots, x_n) = u(x_1, x_2)$ and $\tilde{v}(x_1, \dots, x_n) = v(x_1, x_2)$. The first example shows that Theorem 1.1 cannot be strengthened to say that the full interface $\partial\{u > 0\} \cup \partial\{v > 0\}$ is rectifiable, even with the strengthened assumptions that u, v are harmonic in their supports.

Example 2.1 (Koch Snowflake). Let $\Gamma \subset \mathbb{R}^2$ be the Koch snowflake, a Jordan curve that is not rectifiable, and let Ω be the bounded region it encloses. By the Riemann mapping theorem and the conformal invariance of the Laplacian in \mathbb{R}^2 , we can find positive continuous harmonic functions u on $(B_{10}(0) \cap \{x_2 > 0\}) \cap \Omega$ and v on $(B_{10}(0) \cap \{x_2 > 0\}) \setminus \Omega$ that both vanish, say, on the set $\Gamma \cap \{x_2 > 0\}$. The interface $\partial\{u > 0\} \cup \partial\{v > 0\}$ is equal to $\Gamma \cap \{x_2 > 0\}$. This set is not \mathcal{H}^1 -rectifiable and in fact *no* point on $\partial\{u > 0\} \cup \partial\{v > 0\}$ has an approximate tangent.

Remark 2.2. By Theorem 1.1 we see that the set $\Gamma^* = \{J_x(0^+) > 0\}$ has \mathcal{H}^1 -measure zero in Example 2.1 above. In fact, we can see more directly that $J_x(0^+) = 0$ for *every* $x \in \Gamma \cap \{x_2 > 0\}$. Indeed, if it were the case that $J_x(0^+) > 0$ at some $x \in \Gamma \cap \{x_2 > 0\}$, then using the rigidity in the ACF monotonicity formula, we could deduce that every sequence of scales $r_k \rightarrow 0$ has a further subsequence along which the functions u^{x, r_k} and v^{x, r_k} converge to a pair of complementary truncated linear functions as in (1.3). On the other hand, choosing a point x and sequence r_k along which $(\Gamma - x) \cap B_{r_k}(0)/r_k$ looks identical for all k , we see this cannot happen.

This example also demonstrates how the vanishing of one of the functions u, v does not impose any structure on the interface:

Example 2.3. Let Ω and u be as in Example 2.1 and let v be identically equal to zero. The ACF monotonicity formula is constantly equal to zero at every point and scale, but as we saw above $\partial\{u > 0\}$ does not have an approximate tangent at any point.

The next example demonstrates that the existence of an approximate tangent plane at x does not imply that $J_x(0^+) > 0$.

Example 2.4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be C^1 , symmetric, nonnegative, increasing for $x > 0$, and $f(0) = 0$. Let u be positive and harmonic in $\Omega = \{(x, y) \in B_1 \mid y > f(x)\}$ and vanishing on the boundary $\Gamma = \{(x, y) \in B_1 \mid y = f(x)\}$. Similarly, let v be positive and harmonic in $B_1 \setminus \Omega$ and vanishing on Γ . Then Γ will have a tangent at every point. A basic computation (see [AKNb]) shows that

$$(\log J_0)'(r) \geq \frac{1}{r}[\alpha_u(r) + \alpha_v(r) - 2],$$

where $\alpha_u(r), \alpha_v(r)$ are the characteristic constants (see Section 2.2), and in this example $\alpha_u(r) = \pi - 2 \arctan(f(r)/r)$ and $\alpha_v(r) = \pi + 2 \arctan(f(r)/r)$. For r small, we then obtain

$$(\log J_0)'(r) \geq \frac{1}{r}[\alpha_u(r) + \alpha_v(r) - 2] \geq \frac{[\arctan(f(r)/r)]^2}{2r}.$$

By choosing $f(r)/r \rightarrow 0$ but slowly enough so that $[\arctan(f(r)/r)]^2$ is not Dini-integrable, it follows that $J_0(0^+) = 0$ despite a tangent for Γ at the origin.

The next example is a converse to the example above, was constructed by the first two authors in [AK20], and shows that in Theorem 1.1, the existence of an approximate tangent plane at \mathcal{H}^{n-1} -a.e. point in Γ^* cannot be improved to an approximate tangent at *every* point in Γ^* .

Example 2.5 (Spiraling interface with $J(0^+) > 0$). In [AK20], the first two authors construct a pair of continuous, nonnegative harmonic functions u, v in the plane with disjoint positivity sets such that $J_0(0^+) > 0$ but the interface $\Gamma = \partial\{u > 0\} \cup \partial\{v > 0\}$ does not admit an approximate tangent at 0. In this example Γ is a spiral; in any annulus $B_1(0) \setminus B_r(0)$, Γ is a piecewise smooth connected hyperplane, but it spirals in such a way that the closest pair of complementary truncated linear functions (1.3) at scale r has orientation $\nu(r)$ that rotates and does not converge at $r \rightarrow 0$.

Remark 2.6. In [AKNb, Corollary 1.4], we give a pointwise criterion ruling out behavior as in Example 2.5 at a point $x \in \Gamma^*$: if $\sum_j \log(J_x(2^{-j})/J_x(0^+))^{1/2} < +\infty$, then the functions u, v have unique blowups and Γ^* has an approximate tangent at x . This criterion asks for *very* small energy drop at each dyadic scale, whereas $J_x(0^+) > 0$ imposes the weaker condition $\sum_j \log(J_x(2^{-j})/J_x(2^{-j-1})) < +\infty$ on the dyadic energy drop.

Unlike the study of singular points, when an ϵ -regularity result allows one to pass estimates for a small enough ϵ to the entire top stratum of singular points, in our situation one cannot expect to pass the measure estimates of Theorem 1.3 for a small fixed ϵ to all of Γ^* . The next example illustrates that such estimates are not available for all of Γ^* .

Example 2.7. We modify the construction of the Koch snowflake. The heuristic idea is to retain line segments while continuing the construction on others. In this manner, we obtain a boundary that contains straight line segments whose union is infinite in length. Specifically, in the first iteration of the snowflake we have an equilateral triangle whose sides have length one, so the total length of the triangle is 3 which is greater than 2. We retain two line segments whose union we label as R_1 (which has total length 2 which is greater than 1), and continue the Koch construction on the third line segment. We run enough Koch iterations on the third line segment until the length is greater than 2, and label this piece K_2 , so that the resulting figure is $R_1 \cup K_2$. We choose only one line segment of K_2 on which to continue the Koch iterations, and label the union of the remaining line segments we retain as R_2 (which has total length greater than 1). We inductively continue in the same manner as follows: suppose K_i and R_i have been chosen. We run enough Koch iterations on the one line segment of $K_i \setminus R_i$ to obtain K_{i+1} with $\mathcal{H}^1(K_{i+1}) > 2$. We then choose one line segment of K_{i+1} and retain the union of the remainder of the line segments as R_{i+1} . Continuing inductively, we obtain a limiting figure which we label Γ . We define Ω, u, v similarly to Example 2.1 (after possibly rotating Γ). We note that $\mathcal{H}^1(R_i) > 1$ for each i , and each R_i consists of the union of finitely many line segments. The interior of each line segment is contained in Γ^* . Also, $\mathcal{H}^1(R_i) > 1$, and $\mathcal{H}^1(R_i \setminus \Gamma^*) = 0$. Since $\mathcal{H}^1(R_i \cap R_j) = 0$ for $i \neq j$ and since there are infinitely many R_i , it follows that $\mathcal{H}^1(\Gamma^*) = \infty$.

2.2. Connections to harmonic analysis, Bishop's conjecture, and Carleson's ϵ^2 problem. A problem in harmonic analysis closely related to Theorem 1.1 is known as Bishop's conjecture. Given a pair of complementary domains Ω^\pm , let ω^\pm be the harmonic measure on $\partial\Omega^\pm$ (with pole somewhere in Ω^\pm). In [Bis91] Bishop proved that in the plane, if $\omega^+ \ll \omega^- \ll \omega^+$ on a set $E \subset \partial\Omega^+ \cap \partial\Omega^-$, then ω^\pm are mutually absolutely continuous with respect to (1-dimensional) Hausdorff measure on a subset $F \subset E$ with $\omega^\pm(E \setminus F) = 0$, and F is 1-rectifiable. A sequence of papers [KPT09, AMT17, AMTV19] proved similar statements in arbitrary dimensions with progressively weaker assumptions on Ω^\pm , with [AMTV19] removing all assumptions and fully establishing the conjecture.

At least if the u, v in J_x are a pair of Green's functions, the hypothesis that $J_x(0^+) > 0$ on E is different and to some extent stronger than the assumption that ω^\pm are mutually absolutely continuous. Our conclusion is also stronger in the sense that we show that the full set E is \mathcal{H}^{n-1} rectifiable, a conclusion which requires stronger assumptions than those in Bishop's conjecture. The proofs of the results in [KPT09, AMT17, AMTV19], while also exploiting the monotonicity of J , are based on deep connections between rectifiability and Riesz transforms, and are very different from the method here. Our argument is more elementary and direct, which potentially leads to more quantitative information (such as Theorem 1.3).

Also studied in the harmonic analysis literature is the "converse" Bishop's conjecture: if one knows that E is \mathcal{H}^{n-1} -rectifiable, are ω^\pm mutually absolutely continuous on E ? When $n \geq 3$, some mild density assumptions on Ω^\pm are imposed, and "rectifiable" is replaced by the existence of strong tangents, this is shown in [AMT17] (when $n = 2$ it is contained in [Bis91]). It is unclear to what extent this resolves the problem converse to Theorem 1.1, namely:

Problem 2.8. *Assume that $E = \partial\{u > 0\} \cap \partial\{v > 0\}$ is \mathcal{H}^{n-1} rectifiable. Is $J_x(0^+) > 0$ for \mathcal{H}^{n-1} -a.e. $x \in E$ (under some minimal assumptions on E or u, v)?*

Example 2.4 shows that in Problem 2.8, $J_x(0^+)$ need not be positive for every $x \in E$. From a slightly different perspective: Bishop and Jones [BJ94] proved the following striking result: for a suitably nice domain $\Omega \subset \mathbb{R}^2$ (e.g. one bounded by a Jordan curve), the set of points for which

$$\beta_\infty(x, r) = \inf_{\substack{L \text{ a line,} \\ L \cap B_r(x) \neq \emptyset}} \sup_{x \in \partial\Omega \cap B_r(x)} \frac{d(x, L)}{r}$$

satisfies the square-Dini condition

$$\int_0^1 \frac{\beta_\infty^2(x, r) dr}{r} < \infty$$

is precisely the rectifiable portion $\partial\Omega$ (up to sets of \mathcal{H}^1 measure 0). They pose a more general question along these lines: for which choices of geometric “square function” (to use the term loosely, as in [JTV21]) in place of β_∞ (as well as dimension and *a priori* assumptions on $\partial\Omega$) does one recover such a rectifiability criterion? There has been substantial study of this topic ([Tol15]), and the rectifiable Reifenberg theorems of [NV17] fit naturally into this program. A particularly striking recent result of [JTV21] is that in two dimensions β_∞ can be replaced by the quantity $\epsilon(x, r)$, defined as follows. Let $I^+(x, r)$ length of the longest arc contained in $\Omega \cap B_r(x)$, and $I^-(x, r)$ the length of the longest arc contained in $B_r(x) \setminus \bar{\Omega}$; then $\epsilon(x, r) = \max\{|I^+(x, r)/r - \pi|, |I^-(x, r)/r - \pi|\}$. This answered a question of Carleson explicitly left open in [BJ94].

The quantity ϵ is, of course, purely geometric, but we would like to suggest a spectral reinterpretation of it which exposes certain connections to Theorem 1.1 as well as to higher-dimensional versions of Carleson’s problem. A basic computation (see [AKNb]) shows that

$$(\log J_0)'(r) \geq \frac{1}{r}[\alpha_u(r) + \alpha_v(r) - 2] \geq \frac{1}{r} [|\alpha_u(r) - 1|^2 + |\alpha_v(r) - 1|^2],$$

where α_u represents the homogeneity of the unique nontrivial nonnegative homogeneous function on a cone G with cross-section $G \cap \partial B_r(0) = \{u > 0\} \cap \partial B_r(0)$ and which vanishes on the boundary of G , the “characteristic constant.” This number can be computed directly from the first Dirichlet eigenvalue of $\{u > 0\} \cap \partial B_r(0)$, which specifically in 2D depends only on the numbers $I^\pm(0, r)$. In particular, we have that

$$\frac{1}{r} [|\alpha_u(r) - 1|^2 + |\alpha_v(r) - 1|^2] \geq \frac{c}{r} \epsilon^2(0, r).$$

Now, $(\log J_x)'$ is integrable in r if $J_x(0^+) > 0$, meaning $(\log J)'$ is a natural (and stronger) square function. If $\partial\Omega$ is a Jordan curve (or under some other assumptions, still in 2D), then the results of [JTV21] and the argument sketched above imply Theorem 1.1.

The main interest of Theorem 1.1, though, is that it holds unconditionally and in any dimension, giving a rectifiability criterion which is analytic in nature. It is clear from the heuristic presented above that $J_x(0^+) > 0$ is a stronger property than the square-summability of ϵ : indeed, a more careful analysis of $(\log J)'$ shows that it provides some mild indirect control over how $\alpha_u(r)$ changes as a function of r , something that ϵ notoriously does not. On the other hand, this suggests that of the many possible generalizations of ϵ to higher dimensions, a particularly compelling one is the “spectral” square function:

$$\lambda^2(x, r) = |\lambda_1(\Omega \cap \partial B_r(x))/r^2 - (n-1)|^2 + |\lambda_1(\partial B_r(x) \setminus \Omega)/r^2 - (n-1)|^2,$$

where λ_1 stands for the first Dirichlet eigenvalue of a domain on a sphere ($n-1$ is λ_1 of a half unit sphere).

Problem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set. Under what minimal assumptions on $\partial\Omega$ does*

$$\left\{ x \in \partial\Omega : \int_0^1 \frac{\lambda^2(x, r)}{r} dr < \infty \right\}$$

coincide with the rectifiable portion of $\partial\Omega$, up to sets of \mathcal{H}^{n-1} measure 0?

2.3. Applications to free boundary problems. In this section we present informally an example of how one might apply Theorem 1.1 to gain information about a free boundary problem. We will consider a broad class of vector-valued Bernoulli problems in (2.1) below. Problems like this derive from eigenvalue optimization and segregation models, for instance, and have been studied in the literature only very recently. The theory is still incomplete and progress so far has been limited to the restrictive setting of the variational formulation. Here we apply the main results of this paper to analyze part of the free boundary without resorting to variational techniques or any special structure of the free boundary condition.

Suppose a domain Ω and a collection $\{u_i\}_{i=1}^N$ of continuous functions $u_i : \bar{\Omega} \rightarrow \mathbb{R}$ satisfy the overdetermined system

$$(2.1) \quad \begin{cases} \Delta u_i(x) = 0 & x \in \Omega \cap B_1 \\ u_i(x) = 0 & x \in \partial\Omega \cap B_1 \\ F(\{|(u_i)_\nu(x)|\}) = 1 & x \in \partial^*\Omega \cap B_1 \\ F(\{|\nabla u_i^l|\}) = F(\{|\nabla u_i^r|\}) \geq 1 & x \in \partial^c\Omega \cap B_1 \end{cases}$$

Here $\partial^*\Omega$ represents the reduced boundary of Ω , while $\partial^c\Omega$ is the set of ‘‘cusp points’’ in $\partial\Omega$ with blow-up limits $\mathbb{R}^n \setminus H$ for a hyperplane H , i.e. points where $\partial\Omega$ has tangents but with Ω on either side of those tangents; $\nabla u_i^r, \nabla u_i^l$ represent the derivative from either side.

In typical examples, the function $F : \mathbb{R}^N \rightarrow \mathbb{R}$ in (2.1) will be strictly increasing in each parameter and smooth, however the specific form is not important here. The only well-studied example is the variational one, $F(\xi) = \sum_i \mu_i \xi_i^2$ with $\mu_i > 0$, which permits viewing this problem as the stationarity condition for a certain energy. We are deliberately vague about the sense in which the last two conditions of (2.1) hold, as the point of the method here is *that they are not used directly*.

For Ω and u_1, \dots, u_N satisfying (2.1) (extended by zero on $B_1 \setminus \Omega$), let v be any rational linear combination of the u_1, \dots, u_N , and apply the ACF formula to v_+ and v_- ; set

$$(2.2) \quad G = \bigcup_v \{x \in \partial\Omega \cap B_{1/2} : J_x(v_+, v_-, 0^+) > 0\}.$$

As a consequence of Theorems 1.1 and 1.2, we have the following.

Theorem 2.10. *Consider a domain Ω and a collection of functions $\{u_i\}_{i=1}^N$ satisfying (2.1). The set G is \mathcal{H}^{n-1} -rectifiable. Moreover, for \mathcal{H}^{n-1} -a.e. $x \in G$ and for each $i = 1, \dots, N$, the blow-ups of u_i converge to a unique limiting function of the form*

$$\alpha_i(x \cdot \nu)_+ + \beta_i(x \cdot \nu)_-$$

with $\alpha_i \geq 0$ and $\beta_i \leq 0$.

In fact, Theorem 2.10 holds for functions $\{u_i\}_{i=1}^N$ satisfying only the first two conditions in (2.1).

Proof. The \mathcal{H}^{n-1} -rectifiability of G follows directly from Theorem 1.1 since the countable union of rectifiable sets is rectifiable. For each $x \in G$, there is a v to be a rational linear combination for which $J_x(v_+, v_-, 0^+) > 0$; by Theorem 1.2, outside of an \mathcal{H}^{n-1} -null set in G , the blow-ups of every such v converge to a unique limiting function (depending on v) of the form $\alpha(x \cdot \nu)_+ + \beta(x \cdot \nu)_-$ with $\alpha > 0, \beta < 0$. For each $i = 1, \dots, N$ and for ϵ small enough (and rational), $J_x((v + \epsilon u_i)_+, (v + \epsilon u_i)_-, 0^+) > 0$. This implies that the blow-ups of each u_i are also unique functions of the same form, except with $\alpha, \beta \in \mathbb{R}$ possibly 0. \square

Let us explain the relevance of the set G and of Theorem 2.10. For a solution of (2.1), let E be the set of points in $\partial\Omega$ where Ω has Lebesgue density strictly less than one, and let $D = \partial\Omega \setminus (E \cup G)$. In this way, we partition the free boundary $\partial\Omega$ into disjoint sets D, G , and E . On E , one can use blow-up and perturbative methods to classify points as in the scalar case: E is composed of the reduced boundary, which is smooth and relatively open, and a lower-dimensional set of singular points. While this problem is unstudied in the generality presented here, [KL18, KL19, MTV17, CSY18] deal with particular cases and could be adapted further. Next, Theorem 2.10 tells us that G is \mathcal{H}^{n-1} -rectifiable. For a sufficiently stable solution (as described more precisely below), the set D is expected to be empty, so in particular the entirety of $\partial\Omega = E \cup G$ is \mathcal{H}^{n-1} -rectifiable. On the other hand, for a solution where D is nonempty, then at each $x \in D$ the solution must be degenerate in one of the ways described below.

For a reasonable choice of F , one can expect to show that for a solution of (2.1), each u_i is Lipschitz continuous. This property is generally straightforward to establish for even the weakest possible interpretation of

this problem (see [CS05], for instance). Sufficiently “stable” solutions are expected to satisfy the nondegeneracy condition

$$(2.3) \quad \max_{i=1,\dots,N} \sup_{B_r(x)} |u_i| \geq cr \quad \forall x \in \partial\Omega, r < r_0;$$

for instance, this is expected for Perron solutions, or for minimizing solutions in a variational setting. Finally, the following secondary nondegeneracy property is not limited by stability but is expected to be true for any sufficiently well-behaved solution: for each $i = 1, \dots, N$, $x \in \partial\Omega$, and $r < r_0$,

$$(2.4) \quad \sup_{B_r(x)} u_i^\pm \leq C \left[\sup_{B_{2r}(x)} u_i^\mp + r \sqrt{\frac{|B_{2r}(x) \setminus \Omega|}{|B_{2r}|}} \right],$$

i.e. at points x with Lebesgue density close to 1, u^\pm must have comparable size.

Theorem 2.11. *Consider a domain Ω and a collection of functions $\{u_i\}_{i=1}^N$ satisfying (2.1). If (2.3) and (2.4) hold, then D is empty.*

The set G is not well understood, even in the case of minimizers to the variational form of this problem. In the variational case $F(\{\xi_i\}) = \sum_i \mu_i \xi_i^2$, the set G is shown to be regular in the scalar case $N = 1$ [DPSV21] and in two dimensions [SV19], or rectifiable in arbitrary dimensions [DPESV21]. Theorem 2.10 gives an alternative and rather different proof of the main result of [DPESV21], and additionally establishes the rectifiability of G in the broader context of non-variational problems.

Let us give a second example of how our main theorems above can be applied to analyze free boundaries. Consider a two-phase parabolic equation of Hele-Shaw type: a function $u : \Omega \times (0, T] \rightarrow \mathbb{R}$ solves

$$\begin{cases} \Delta_x u(x, t) = 0 & u(x, t) \neq 0 \\ \partial_t u(x, t) = |\nabla u_+|^2 - |\nabla u_-|^2 & u(x, t) = 0 \\ u(x, t) = \phi(x) & x \in \partial\Omega. \end{cases}$$

Two-phase flows of Bernoulli/Stefan type have been studied extensively in the context of heat transfer, porous media, fluid dynamics, and other models ([CS05]). In general, the results available are either local regularity theorems under strict starting assumptions on the flow to reduce to a perturbative starting point, or global theorems exploiting monotonicity of u or $\{u > 0\}$ in t by imposing special boundary or initial conditions. A major difficulty in comparison to stationary problems is the lack of geometric measure theoretic results about $\{u = 0\}$.

Here Theorem 1.1 can be applied directly to give that the set of points where $J_x(u_+(\cdot, t), u_-(\cdot, t), 0^+) > 0$ is rectifiable (on each time slice, independently). This suggests that at points of this type, one has a reasonable starting configuration for perturbative results from a linearized two-phase configuration. At points where $J_x(u_+(\cdot, t), u_-(\cdot, t), 0^+) = 0$, one instead hopes for perturbative results from a one-phase configuration (which is better understood, [CJK07, CJK09, CLGS19]), or some other more degenerate behavior.

3. PRELIMINARIES

This section contains preliminary results that will be used in the paper. Section 3.1 provides the basic notation and some initial scaling observations. In section 3.2, we recall two Reifenberg-type theorems of Naber and Valtorta. In section 3.3, we recall our main result of [AKNb] and prove some slight variants of it that will be applied in this paper. Finally, section 3.4 contains an important lemma that allows us to deduce convergence of the ACF monotonicity formulas of L^2 -convergent sequences of functions.

3.1. Notation and Basics. We use the notation $B_r(x)$ to denote a ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$ and B_r to denote $B_r(0)$, and let ω_n denote the volume of the n -dimensional Euclidean unit ball. Given a set $E \subset \mathbb{R}^n$, let $d(x, E) = \inf_{y \in E} d(x, y)$ and $B_r(E) = \cup_{x \in E} B_r(x)$.

We recall that a set $S \subset \mathbb{R}^n$ is \mathcal{H}^{n-1} -rectifiable if there exist countably many Lipschitz maps $f_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(S \setminus \bigcup_j f_j(\mathbb{R}^{n-1})) = 0$, and caution the reader that this definition is sometimes called *countable* \mathcal{H}^{n-1} -rectifiability.

It is worth noting how our quantities behave under rescalings. Suppose u, v are as in (1.2). For $r \in (0, 4)$, let $u^{x,r}(z) = u(x + rz)/r$ and $v^{x,r}(x) = v(x + rz)/r$. Then

$$\begin{aligned} J_x(\rho; u, v) &= J_0(\rho/r; u^{x,r}, v^{x,r}), & \Gamma^*(u, v) \cap B_r(x) &= \Gamma^*(u^{x,r}, v^{x,r}) \cap B_1(0), \\ \Gamma_\epsilon^*(u, v) \cap B_r(x) &= \Gamma_\epsilon^*(u^{x,r}, v^{x,r}) \cap B_1(0), & \Gamma_{\epsilon, \rho/r}^*(u, v) \cap B_r(x) &= \Gamma_{\epsilon, \rho/r}^*(u^{x,r}, v^{x,r}) \cap B_1(0). \end{aligned}$$

3.2. Reifenberg-Type Theorems. We recall two important theorems of Naber and Valtorta, the discrete Reifenberg theorem and the rectifiable Reifenberg theorem. Given a Borel measure μ , a point $x \in B_1(0)$, and a scale $r \in (0, 1)$, the $(n-1)$ -dimensional L^2 Jones' beta number $\beta_\mu(x, r)$ with respect to μ is defined by

$$(3.1) \quad \beta_\mu(x, r)^2 = \inf_L \int_{B_r(x)} \frac{d(y, L)^2}{r^2} \frac{d\mu(y)}{r^{n-1}}.$$

Here the infimum is taken over all affine hyperplanes L in \mathbb{R}^n . This quantity measures, in a scale invariant L^2 sense, how far the support of μ is to being contained in an affine hyperplane. Following Jones' use of an L^∞ analogue of (3.1), the L^p beta numbers were first introduced by David and Semmes in [DS93] and have since been used, for instance, in [DT12, AT15] as well as [NV17].

We state the following two Reifenberg-type theorems only in the $(n-1)$ -dimensional case in which we will apply them. We refer the reader to Theorems 3.4 and 3.3 in [NV17] respectively for the statements in greater generality, as well as to the paper [ENV19] for further extensions.

Theorem 3.1 (Discrete Reifenberg Theorem). *Fix $n \geq 2$. There are dimensional constants $\delta_0 = \delta_0(n) > 0$ and $C_0 = C_0(n) > 0$ such that the following holds. Let $\{B_{r_x}(x)\}$ be a collection of disjoint balls with $x \in B_1(0)$ and $r_x \in (0, 1]$, and let $\mu = \sum_x r_x^{n-1} \delta_x$ be the associated $(n-1)$ -dimensional packing measure. If*

$$\int_0^{2r} \int_{B_r(x)} \beta_\mu(z, s)^2 d\mu(z) \frac{ds}{s} \leq \delta_0 r^{n-1}$$

for all $x \in B_1(0)$ and all $r \in (0, 1]$, then

$$\sum_x r_x^{n-1} \leq C_0.$$

We will apply Theorem 3.1 in Section 4 to establish packing estimates. Later, in Section 6.3 we will apply the rectifiable Reifenberg theorem to prove that Γ_ϵ^* is \mathcal{H}^{n-1} -rectifiable.

Theorem 3.2 (Rectifiable Reifenberg Theorem). *Fix $n \geq 2$. There are constants $\delta_0 = \delta_0(n)$ and $C_0 = C_0(n)$ such that the following holds. Let $S \subset \mathbb{R}^n$ be a set and let $\mu_S = \mathcal{H}^{n-1} \llcorner S$. If*

$$\int_0^{2r} \int_{B_r(x)} \beta_\mu(z, s)^2 d\mu(z) \frac{ds}{s} \leq \delta_0 r^{n-1}$$

for all $x \in B_1(0)$ and all $r \in (0, 1]$, then $S \cap B_1(0)$ is \mathcal{H}^{n-1} -rectifiable, and for each $x \in B_1(0)$ and $0 < r < 1$,

$$\mu_S(B_r(x)) \leq C_0 r^{n-1}.$$

3.3. Quantitative remainder for the ACF monotonicity formula. In [AKNb] we proved a sharp quantitative version of the ACF monotonicity formula, which will play an important role here. The estimate will be used at various different points throughout the paper; the optimal exponent in our quantitative estimate is crucial in the L^2 subspace approximation theorem, Theorem 4.1.

Theorem 3.3. *Fix $n \geq 2$. There is a dimensional constant $C > 0$ such that the following holds. Let $u, v : B_2(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions that satisfy (1.2). For any $\rho \in [0, 1/2]$, there exist $a, b > 0$ and a direction $\nu \in \mathbb{S}^{n-1}$ such that*

$$\int_{B_1 \setminus B_\rho} (u - a(x \cdot \nu)^+)^2 + (v - b(x \cdot \nu)^-)^2 \leq C \log \left(\frac{J_0(1)}{J_0(\rho)} \right) \int_{B_1} (u^2 + v^2).$$

There is a dimensional constant κ_0 such that if $\log(J(1)/J(0^+)) < \kappa_0^2$, then a, b , and ν can be chosen independently of ρ .

In [AKNb], Theorem 3.3 is stated with the additional assumption that $u(0) = v(0) = 0$, i.e. that the point where the ACF monotonicity formula is centered lies in the mutual zero set $\{x : u(x) = v(x) = 0\}$. This assumption can be removed with the following observation and notational caveat. Suppose u, v are as in the statement of Theorem 3.3 and $v(0) > 0$. By continuity, there is a radius r_0 such that $v(x) > 0$, and thus $u(x) = 0$, for all $x \in B_{r_0}(0)$. In turn, $J_0(r) = 0$ for all $r < r_0$. Let us adopt the convention that, if $J_0(r) = 0$, then $\log(J_0(R)/J_0(r)) = +\infty$ for any $R \geq r$. With this convention in place, Theorem 4.1 holds and the proof goes through without modification without the assumption that $u(0) = v(0) = 0$.

We will use Theorem 3.3 in the form above as well as in the form of the following corollary:

Corollary 3.4. *Fix $n \geq 2$. There are dimensional constants $\rho_0(n)$, $\kappa_0(n)$, and $C = C(n)$ such that the following holds. Let $u, v : B_8(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions that satisfy (1.2) and suppose $\log(J(1)/J(\rho)) \leq \kappa_0^2$. Suppose further that $a = b$ in the conclusion of Theorem 3.3. Then for any $\rho < \rho_0$,*

$$(3.2) \quad \int_{B_1 \setminus B_\rho} \left(\frac{u}{a} - (x \cdot \nu)^+ \right)^2 + \left(\frac{v}{b} - (x \cdot \nu)^- \right)^2 \leq C \log \left(\frac{J(1)}{J(\rho)} \right).$$

Proof. Without loss of generality, suppose $\|u\|_{L^2(B_1)} \geq \|v\|_{L^2(B_1)}$. By Theorem 3.3 and the triangle inequality,

$$(3.3) \quad \left| \|u\|_{L^2(B_1 \setminus B_\rho)} - \|a(x \cdot \nu)^+\|_{L^2(B_1 \setminus B_\rho)} \right| \leq C \kappa_0 \|u\|_{L^2(B_1)}$$

for a dimensional constant C . Since u is nonnegative and subharmonic, u^2 is subharmonic and by the mean value property, $\int_{B_\rho} u^2 \leq \rho^n \int_{B_1} u^2$. As such, $\left| \|u\|_{L^2(B_1 \setminus B_\rho)} - \|u\|_{L^2(B_1)} \right| \leq \rho^{n/2} \|u\|_{L^2(B_1)}$. The analogous estimate holds for $a(x \cdot \nu)^+$ as well (since it is subharmonic, or by direct calculation). So, (3.3) implies that

$$\left| \|u\|_{L^2(B_1)} - \|a(x \cdot \nu)^+\|_{L^2(B_1)} \right| \leq C \left(\kappa_0 + \rho^{n/2} \right) \|u\|_{L^2(B_1)}.$$

Provided ρ_0 and κ_0 are chosen to be small enough, the coefficient on the right-hand side is bounded above by $1/2$. So, dividing through by a in Theorem 3.3, we have

$$\int_{B_1 \setminus B_\rho} \left(\frac{u}{a} - (x \cdot \nu)^+ \right)^2 \leq C \|(x \cdot \nu)^+\|_{L^2(B_1)}^2 \log \left(\frac{J(1)}{J(\rho)} \right) = C \log \left(\frac{J(1)}{J(\rho)} \right)$$

for a dimensional constant C . The same argument can now be repeated with v in place of u and with $\|a(x \cdot \nu)^+\|_{L^2(B_1)}$ on the right-hand side of (3.3). \square

3.4. Continuity properties for the ACF monotonicity formula. The Alt-Caffarelli-Friedman monotonicity formula is continuous in x for fixed $r > 0$, and it is upper semicontinuous in the sense that for any sequence $x_i \rightarrow x_0$ and $r_i \rightarrow 0^+$, we have

$$(3.4) \quad \limsup_{i \rightarrow \infty} J_{x_i}(r_i) \leq J_{x_0}(0^+).$$

We leave the proofs of these facts to the reader. A less immediate continuity property of the ACF monotonicity formula is given in Corollary 3.6 below: if a pair of functions (u, v) as in (1.2) converges in $L^2(B_R)$ to a pair of complementary truncated linear functions, then their ACF formulas converge as well up to scale R . This key fact will follow from the next lemma.

Lemma 3.5. *Fix $R > 0$. Let $\{u_j\} \subset W^{1,2}(B_R)$ be a sequence of nonnegative subharmonic functions and let $u \in W^{1,2}(B_R)$ satisfy $u \Delta u = 0$, i.e. u is harmonic where it is positive. If $u_j \rightarrow u$ in $W^{1,2}(B_R)$ and $u_j \rightarrow u$ in $L^2(B_R)$, then $u_j \rightarrow u$ in $W^{1,2}(B_R)$ and*

$$\lim_{j \rightarrow \infty} \int_{B_R} \frac{|\nabla u_j|^2}{|x|^{n-2}} = \int_{B_R} \frac{|\nabla u|^2}{|x|^{n-2}}.$$

Proof. Fix a nonnegative function $\phi \in C_c^1(B_R(0))$. Because u_j is subharmonic and $u_j \phi \geq 0$, we have

$$\int_{B_R} |\nabla u_j|^2 \phi = - \int_{B_R} u_j \nabla u_j \cdot \nabla \phi + \int_{B_R} \nabla u_j \cdot \nabla (u_j \phi) \leq - \int_{B_R} u_j \nabla u_j \cdot \nabla \phi.$$

Now, since $u_j \nabla \phi \rightarrow u \nabla \phi$ strongly in $L^2(B_R)$ and $\nabla u_j \rightarrow \nabla u$ weakly in $L^2(B_R)$, it follows that

$$\limsup_{j \rightarrow \infty} \int_{B_R} |\nabla u_j|^2 \phi \leq - \lim_{j \rightarrow \infty} \int_{B_R} u_j \nabla u_j \cdot \nabla \phi = - \int_{B_R} u \nabla u \cdot \nabla \phi = \int_{B_R} |\nabla u|^2 \phi.$$

Here the final identity comes from integration by parts and the fact that $u \Delta u = 0$. On the other hand, by lower semicontinuity of the energy with respect to $W^{1,2}$ weak convergence,

$$\liminf_{j \rightarrow \infty} \int_{B_R} |\nabla u_j|^2 \phi \geq \int_{B_R} |\nabla u|^2 \phi.$$

These two inequalities together tell us that, for any nonnegative $\phi \in C_c^1(B_R(0))$,

$$\lim_{j \rightarrow \infty} \int_{B_R} |\nabla u_j|^2 \phi = \int_{B_R} |\nabla u|^2 \phi.$$

The first conclusion of the lemma follows from taking a sequence of functions ϕ_k that approximate the characteristic function of B_R , while the second conclusion follows from taking $\phi_k = |x|^{n-2} \psi_k$ where $\psi_k \in C_c^1(B_R(0))$ vanishes in $B_{1/k}(0)$ and approximates the characteristic function of B_R in $L^2(B_R)$. \square

As we mentioned above, Lemma 3.5 has the following useful corollary that will allow us to upgrade L^2 convergence of sequences to convergence of their ACF formulas.

Corollary 3.6. *Let $u_j, v_j : B_2(0) \rightarrow \mathbb{R}$ be sequences of nonnegative continuous functions that satisfy (1.2) with $\sup_j J_0(1; u_j, v_j) < +\infty$ and suppose $u_j \rightarrow a(x \cdot \nu)^+$ and $v_j \rightarrow b(x \cdot \nu)^-$ in $L^2(B_1(0))$ for some $a, b > 0$ and $\nu \in \mathbb{S}^{n-1}$. Then, setting $c_* = n^2 \omega_n^2 / 16$, we have*

$$\lim_{j \rightarrow \infty} J_0(r; u_j, v_j) = c_* a^2 b^2 \quad \text{for all } r \in (0, 1].$$

The constant c_* is such that $J_0(r; a(x \cdot \nu)^+, b(x \cdot \nu)^-) = c_* a^2 b^2$ for any $r > 0$.

Proof. The assumption $\sup_j J_0(1; u_j, v_j) < +\infty$ implies that after passing to a subsequence, either

$$(3.5) \quad \sup_j \int_{B_1} \frac{|\nabla u_j|^2}{|x|^{n-2}} dx < +\infty \quad \text{or} \quad \sup_j \int_{B_1} \frac{|\nabla v_j|^2}{|x|^{n-2}} dx < +\infty$$

or both; we assume without loss of generality it holds for u_j . The weight $|x|^{2-n}$ is bounded below by 1 on $B_1(0)$, so $\sup_j \int_{B_1(0)} |\nabla u_j|^2 dx < +\infty$ and thus up to a further subsequence, $u_j \rightarrow a(x \cdot \nu)^+$ in $W^{1,2}(B_1(0))$. The truncated linear function $a(x \cdot \nu)^+$ is harmonic where it is positive, so Lemma 3.5 implies that

$$\lim_{j \rightarrow \infty} \int_{B_1(0)} \frac{|\nabla u_j|^2}{|x|^{n-2}} dx = \int_{B_1(0)} \frac{|\nabla a(x \cdot \nu)^+|^2}{|x|^{n-2}} dx = a^2 \sqrt{c_*}.$$

Here c_* is the dimensional constant defined in the corollary statement and the final equality is an elementary calculation. Now, since $a^2 \sqrt{c_*} > 0$, the assumption $\sup_j J_0(1; u_j, v_j) < +\infty$ now tells us that (3.5) holds for v along the same subsequence. Repeating the same argument shows that

$$\lim_{j \rightarrow \infty} \int_{B_1(0)} \frac{|\nabla v_j|^2}{|x|^{n-2}} dx = b^2 \sqrt{c_*}$$

and we reach the conclusion of the corollary along a subsequence. Any subsequence has a further subsequence for which the conclusion of the corollary holds, so it holds for the full sequence. \square

4. THE L^2 SUBSPACE APPROXIMATION AND PACKING ESTIMATES

This section has two main goals. First, we prove an estimate known as the L^2 subspace approximation in Theorem 4.1 below. This estimate quantitatively relates the Jones' L^2 beta numbers and the drop in the monotonicity formula from one scale to the next, and plays a key role in the Naber-Valtorta framework. The statement of Theorem 4.1 is analogous to L^2 subspace approximations in other contexts. However, our proof is different from the standard one and in particular circumvents the use of the eigenvalues and eigenvectors of the ‘‘inertia matrix’’ associated to a measure. The proof (as well as an adaptation of the standard proof) crucially relies on the sharp quantitative remainder term in for the ACF monotonicity formula of [AKNb].

Theorem 4.1 (L^2 Subspace Approximation). *Fix $n \geq 2$. There exist positive dimensional constants κ and C such that the following holds. Let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying (1.2). Let μ be a finite Borel measure. For any $r \in (0, 1)$ and $x \in B_1(0)$ such that $\log(J_x(8r)/J_x(r)) < \kappa$, we have*

$$(4.1) \quad \beta_\mu(x, r)^2 \leq \frac{C}{r^{n-1}} \int_{B_r(x)} \log \left(\frac{J_y(8r)}{J_y(r)} \right) d\mu(y).$$

Here $\beta_\mu(x, r)^2$ is the $(n-1)$ -dimensional L^2 Jones' beta number defined in (3.1). Following [NV17], Theorem 4.1 will be applied twice: when μ is the packing measure associated to a cover to prove the Proposition 4.2 below, and with $\mu = \mathcal{H}^{n-1} \llcorner \Gamma_\epsilon^*$ in section 6.3 in the proof of Theorem 1.1.

The second main goal of this section is to prove the packing estimates of the following proposition. Roughly speaking, this proposition says that if a $\Gamma_{\epsilon, r}^*$ has a Vitali cover by balls with small drop in the ACF monotonicity formula at the centers, then the cover has a uniform $(n-1)$ -dimensional packing bound. The statement and proof of Proposition 4.2 are a standard part of the Naber-Valtorta framework.

Proposition 4.2 (Packing Estimate). *Fix $n \geq 2$ and $0 < \epsilon < 1/2$. There exist $\bar{\eta} = \bar{\eta}(n, \epsilon) > 0$ and $C(n)$ such that the following holds. Let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying (1.2) and let $\bar{J} = \sup_{y \in B_1(0)} J_y(4)$. If $\{B_{2r_p}(p)\}_p$ is a collection of disjoint balls with $p \in B_1(0)$ satisfying*

$$(4.2) \quad J_p(\bar{\eta} r_p) \geq \bar{J} - \bar{\eta}, \quad p \in \Gamma_{\epsilon, R}^* \cap B_1(0), \quad R \leq r_p \leq 1,$$

then

$$(4.3) \quad \sum_p r_p^{n-1} \leq C(n).$$

4.1. Two initial lemmas. We prove two initial lemmas in preparation for the proof of Theorem 4.1. The first one will be applied when η is small, and says that if two pairs of complementary truncated linear functions are close in an L^2 sense on an annulus, then their hyperplane interfaces are also close in a quantitative way.

Lemma 4.3. *There exists a positive dimensional constant C such that the following holds. Let $y_1, y_2 \in B_1(0)$ and $\nu_1, \nu_2 \in \mathbb{S}^{n-1}$. Let*

$$\begin{aligned} \ell_1^+ &= a_1[(x - y_1) \cdot \nu_1]^+, & \ell_1^- &= b_1[(x - y_1) \cdot \nu_1]^-, \\ \ell_2^+ &= a_2[(x - y_2) \cdot \nu_2]^+, & \ell_2^- &= b_2[(x - y_2) \cdot \nu_2]^-. \end{aligned}$$

Assume that $\max\{a_1, b_1\} \geq c$. For any $\eta \leq 1/16$, if

$$\int_{B_6(0) \setminus B_2(0)} (\ell_1^+ - \ell_2^+)^2 + (\ell_1^- - \ell_2^-)^2 \leq \eta,$$

then $d(y_2, L) \leq \frac{C}{c} \sqrt{\eta}$, where $L = \{x \in \mathbb{R}^n : (x - y_1) \cdot \nu_1 = 0\}$ is the mutual zero set of ℓ_1^+ and ℓ_1^- .

Proof. Let $\sqrt{\eta} = d(y_2, L)$. We will prove the lemma by showing that

$$(4.4) \quad \int_{B_6(0) \setminus B_2(0)} (\ell_1^+ - \ell_2^+)^2 + (\ell_1^- - \ell_2^-)^2 \geq c^2 \eta.$$

Up to a rotation we may assume that $\nu_1 = e_n$, $a_1 \geq c$, and ν_2 is a linear combination of e_1 and e_n . Since $y_1, y_2 \in B_1(0)$, it suffices to replace the domain of integration in (4.4) by the smaller set $B_5(y_1) \setminus B_3(y_1) \subset B_6(0) \setminus B_2(0)$; up to a translation, we may take y_1 to be the origin and $y_2 \in B_2(0)$. After these normalizations, $L = \{x \in \mathbb{R}^n : x \cdot e_n = 0\}$. Since we are only finding $d(y_2, L) \leq C \sqrt{\eta}$, we may assume that the i -th coordinates of y_2 satisfy $y_2^i = 0$ for $1 \leq i < n$. For simplicity we will treat the case when $y_2^n \geq 0$; the case in which $y_2^n < 0$ is similar. By relabeling a_2 and b_2 if necessary, we may assume $\nu_2 \cdot e_n \geq 0$. Finally, by symmetry we will assume $\nu_2 \cdot e_1 \geq 0$. With these assumptions we have the following simplifications:

$$\begin{aligned} \ell_1^+ &= a_1 x_+^n, & \ell_1^- &= b_1 x_-^n, \\ \ell_2^+ &= a_2 [(x - \sqrt{\eta} e_n) \cdot (\gamma_1 e_1 + \gamma_2 e_n)]^+, & \ell_2^- &= b_2 [(x - \sqrt{\eta} e_n) \cdot (\gamma_1 e_1 + \gamma_2 e_n)]^-. \end{aligned}$$

So, we may find constants $c_1, c_2 \geq 0$ such that ℓ_2^+ can be expressed as

$$\ell_2^+ = a_2 [(x - \sqrt{\eta} e_n) \cdot (\gamma_1 e_1 + \gamma_2 e_n)]^+ = [c_1 a_1 (x^n - \sqrt{\eta}) + c_2 a_1 \sqrt{\eta} x^1]^+.$$

It will also suffice to integrate over the set

$$\tilde{A} = (B_5(y_1) \setminus B_3(y_1)) \cap \{x^n > \sqrt{\eta}\} \cap \{x^1 > 0\}.$$

We note on \tilde{A} that $\ell_1^- = \ell_2^- = 0$. We will show that $|\ell_1^+ - \ell_2^+| \geq a_1 \sqrt{\eta}/8$ pointwise on a subset of \tilde{A} with measure universally bounded below. To do this, it will suffice to have either

$$a_1 x^n - c_1 a_1 (x^n - \sqrt{\eta}) - c_2 a_1 \sqrt{\eta} x^1 \geq a_1 \sqrt{\eta}/8,$$

or

$$a_1x^n - c_1a_1(x^n - \sqrt{\eta}) - c_2a_1\sqrt{\eta}x^1 \leq -a_1\sqrt{\eta}/8,$$

on a sufficiently large subset of \tilde{A} . The above inequalities simplify to

$$(4.5) \quad \frac{1 - c_1}{\sqrt{\eta}}x^n \geq \frac{1}{8} - c_1 + c_2x^1,$$

and

$$(4.6) \quad \frac{1 - c_1}{\sqrt{\eta}}x^n \leq \frac{-1}{8} - c_1 + c_2x^1.$$

We again note that $x^n \geq \sqrt{\eta}$ and $x^1 \geq 0$. We break up the proof into several cases.

Case 1a: $0 \leq c_1 \leq 1$ and $c_2 \leq 3/8$. If $0 \leq x^1 \leq 1$ (and since $x^n/\sqrt{\eta} \geq 1$), then (4.5) holds.

Case 1b: $0 \leq c_1 \leq 1$ and $c_2 \geq 3/8$ and $1 - c_1 \geq c_2\sqrt{\eta}$. If $0 \leq x^1 \leq 1$ and $x^n \geq 2$, then (4.5) holds.

Case 1c: $0 \leq c_1 \leq 1$ and $c_2 \geq 3/8$ and $1 - c_1 \leq c_2\sqrt{\eta}$. If $x^1 \geq 4$ and $x^n \leq 1$, then (4.6) holds.

Case 2a: $c_1 > 1$ and $c_2 \geq 3/8$. If $x_1 \geq 3$ (and since $x^n/\sqrt{\eta} \geq 1$), then (4.6) holds.

Case 2b: $c_1 > 1$ and $c_2 \leq 3/8$ and $1 - c_1 \geq -c_1\sqrt{\eta}$. If $x_n \leq 1/2$ and $0 \leq x^1 \leq 1$, then (4.5) holds.

Case 2c: $c_1 > 1$ and $c_2 \leq 3/8$ and $1 - c_1 \leq -c_1\sqrt{\eta}$. If $x_n \geq 2$, then (4.6) holds.

In each of the above exhaustive cases, we obtain a subset of \tilde{A} with positive measure bounded below so that either (4.5) or (4.6) holds. Thus, the result is proven. \square

The next lemma shows that if an admissible pair u, v are normalized on $B_8(0)$, then the best-approximating truncated linear functions chosen with respect to any $z \in B_1(0)$ will be nondegenerate.

Lemma 4.4. *Let u, v be an admissible pair on $B_9(0)$ with $\|u + v\|_{L^2(B_8)} = 1$. Then there exist positive dimensional constants c_0 and κ such that if $\log(J_0(8)/J_0(1)) \leq \kappa$, and if $z \in B_1$ with $\log(J_z(8)/J_z(1)) \leq \kappa$ and*

$$(4.7) \quad \int_{B_7(z) \setminus B_1(z)} [u(y) - a_z((y-x) \cdot \nu_z)^+]^2 + [v(y) - b_z((y-x) \cdot \nu_z)^-]^2 dy < \kappa,$$

then $a_z + b_z \geq c$.

Proof. Since u, v are both subharmonic (and consequently u^2, v^2 are also subharmonic), we have that

$$\int_{B_8(0) \setminus B_1(0)} u^2 + v^2 \geq \frac{7}{8}.$$

From Theorem 3.3 we have the existence of $a_0, b_0 > 0$ and a direction $\nu_0 \in \mathbb{S}^{n-1}$ satisfying the conclusion of Theorem 3.3 at the origin. It then follows that for κ chosen small enough we have $\|a_0(x \cdot \nu)^+ + b_0(x \cdot \nu)^-\|_{L^2(B_8(0) \setminus B_1(0))} \geq 3/4$. From Theorem 3.3 we also have the existence of a_z, b_z and ν_z satisfying (4.7). From the structure of affine functions we have that

$$\int_{B_5(0) \setminus B_3(0)} a_0(x \cdot \nu)^+ + b_0(x \cdot \nu)^- \geq c(n),$$

and

$$\int_{B_5(0) \setminus B_3(0)} a_z(x \cdot \nu - z)^+ + b_0(x \cdot \nu - z)^- \geq c(n) \int_{B_7(z) \setminus B_1(z)} a_z(x \cdot \nu - z)^+ + b_0(x \cdot \nu - z)^-.$$

Using now the triangle inequality, we conclude that

$$\int_{B_5(0) \setminus B_3(0)} a_z(x \cdot \nu - z)^+ + b_0(x \cdot \nu - z)^- \geq c(n)$$

for a new dimensional constant $c(n)$. The conclusion then follows for a new dimensional constant c_0 . \square

4.2. Proof of the L^2 Subspace Approximation. With Lemmas 4.3 and 4.4 in hand, we can now prove Theorem 4.1. The key ingredient is the quantitative remainder term for the ACF formula, Theorem 3.3.

Proof of Theorem 4.1. By scaling and translation we may assume without loss of generality that $r = 1$ and $x = 0$. We may divide by a positive constant leaving the quotient on the right-hand side of (4.1) invariant, so we may assume that $\|u + v\|_{L^2(B_8)}^2 = 1$.

We select a good competitor hyperplane L in the definition of $\beta_\mu(0, 1)$ in the following way. Let

$$\bar{x} = \operatorname{argmin} \left\{ \log \left(\frac{J_x(8)}{J_x(1)} \right) : x \in \bar{B}_1(0) \right\}.$$

This exists because $x \mapsto J_x(r)$ is continuous and $\bar{B}_1(0)$ is compact. By assumption $\log(J_{\bar{x}}(8)/J_{\bar{x}}(1)) < \kappa$. Notice that $B_7(\bar{x}) \subset B_8$, and so our normalization implies that $\int_{B_7(\bar{x})} u^2 + v^2 \leq 1$. So, applying Theorem 3.3 on $B_7(\bar{x})$, we find a pair of truncated linear functions ℓ^\pm supported on complementary half-planes such that

$$(4.8) \quad \int_{B_7(\bar{x}) \setminus B_1(\bar{x})} (u - \ell^+) + (v - \ell^-)^2 \leq C \log \left(\frac{J_{\bar{x}}(7)}{J_{\bar{x}}(1)} \right) \int_{B_7(\bar{x})} u^2 + v^2 \leq C \log \left(\frac{J_{\bar{x}}(8)}{J_{\bar{x}}(1)} \right).$$

In the final inequality we also used the monotonicity of the ACF formula. Let $L = \{\ell^\pm = 0\}$ be the hyperplane interface between the supports of ℓ^\pm . By the same reasoning, for each $z \in \operatorname{spt} \mu \cap B_1(0)$, we apply Theorem 3.3 to obtain a pair of truncated linear functions $\ell_z^\pm(x)$ supported on complementary half-planes such that

$$(4.9) \quad \int_{B_7(z) \setminus B_1(z)} (u - \ell_z^+) + (v - \ell_z^-)^2 \leq C \log \left(\frac{J_z(7)}{J_z(1)} \right) \int_{B_7(z)} u^2 + v^2 \leq C \log \left(\frac{J_z(8)}{J_z(1)} \right).$$

Now, since the domains of integration in (4.8) and (4.9) both contain the annulus $B_6 \setminus B_2$, we use the triangle inequality and the choice of \bar{x} to deduce that

$$(4.10) \quad \begin{aligned} \int_{B_6 \setminus B_2} |(\ell^+ + \ell^-) - (\ell_z^+ + \ell_z^-)|^2 &\leq 2 \int_{B_6 \setminus B_2} (\ell^+ - u)^2 + (\ell^- - v)^2 + (\ell_z^+ - u)^2 + (\ell_z^- - v)^2 \\ &\leq C \left(\log \left(\frac{J_{\bar{x}}(8)}{J_{\bar{x}}(1)} \right) + \log \left(\frac{J_z(8)}{J_z(1)} \right) \right) \leq C \log \left(\frac{J_z(8)}{J_z(1)} \right) \end{aligned}$$

for any $z \in B_1(0)$. Now, let us split the support of μ into two pieces, letting

$$\begin{aligned} G_\mu &= \{z \in \operatorname{spt} \mu \cap B_1(0) : \log(J_z(8)/J_z(1)) \leq \kappa\}, \\ A_\mu &= \{z \in \operatorname{spt} \mu \cap B_1(0) : \log(J_z(8)/J_z(1)) > \kappa\} \end{aligned}$$

If $z \in G_\mu$, then from Lemma 4.4 and (4.9) we have that $a_z + b_z \geq c_0$, where a_z and b_z are the slopes of ℓ_z^+ and ℓ_z^- respectively. So, we can apply Lemma 4.3; together with (4.10) this tells us that $d(z, L)^2 \leq C \log(J_z(8)/J_z(1))$ for any $z \in G_\mu$. Integrating this inequality over G_μ with respect to the measure μ , we have

$$(4.11) \quad \int_{G_\mu} d(z, L)^2 d\mu(z) \leq C \int_{G_\mu} \log \left(\frac{J_z(8)}{J_z(1)} \right) d\mu(z).$$

On A_μ , the analogous estimate holds for trivial reasons: for any $z \in B_1(0)$, $d(z, L) \leq C_n$, and thus $d(z, L) \leq \frac{C_n \kappa}{\kappa} \leq \frac{C_n}{\kappa} \log(J_z(8)/J_z(1))$ for any $z \in A_\mu$. Integrating over A_μ with respect to μ tells us that

$$(4.12) \quad \int_{A_\mu} d(z, L)^2 d\mu(z) \leq C \int_{A_\mu} \log \left(\frac{J_z(8)}{J_z(1)} \right) d\mu(z).$$

The conclusion of the theorem follows by summing up (4.11) and (4.12) and using the definition of $\beta_\mu(0, 1)$:

$$\beta_\mu(0, 1)^2 \leq \int_{B_1(0)} d(z, L)^2 d\mu(z) \leq C \int_{B_1(0)} \log \left(\frac{J_z(8)}{J_z(1)} \right) d\mu(z).$$

This completes the proof. \square

4.3. Packing Estimates. Next, we apply the L^2 estimate of Theorem 4.1 to prove Proposition 4.2. The proof is an adaptation to our setting of a by-now standard induction argument using the L^2 subspace approximation and the Discrete Reifenberg Theorem.

Proof of Proposition 4.2. Step 1: Let $\eta \in (0, 1)$ be a fixed number depending on n to be specified later in the proof, and set $\bar{\eta} = \eta\epsilon/2$. Basic algebra shows that any p and r_p satisfying (4.2) will also satisfy

$$(4.13) \quad \log \left(\frac{J_p(2)}{J_p(\eta r_p)} \right) \leq \eta, \quad p \in \Gamma_{\epsilon, R}^* \cap B_1(0), \quad R \leq r_p \leq 1,$$

Indeed, our choice guarantees that $\bar{J} - \bar{\eta} \geq 2/\epsilon = \eta/\bar{\eta}$, and so from the definitions of \bar{J} and $\Gamma_{\epsilon, R}^*$,

$$\frac{J_p(2)}{J_p(\eta r_p)} \leq \frac{J_p(2)}{J_p(\bar{\eta} r_p)} \leq \frac{\bar{J}}{\bar{J} - \bar{\eta}} = 1 + \frac{\bar{\eta}}{\bar{J} - \bar{\eta}} \leq 1 + \eta;$$

then the first part of (4.13) follows from the concavity of the logarithm. The second and third parts of (4.13) are the same as (4.2) and thus hold by assumption. So, to establish the proposition, it suffices to show there exists $\eta(n)$ such that (4.3) holds for any collection of disjoint balls $\{B_{2r_p}(p)\}_p$ satisfying (4.13).

Step 2: Let $\kappa > 0$ be chosen according to Theorem 4.1, and let $\eta \leq \kappa$. We let $r_i = 2^{-i}$. For each integer $i \in \mathbb{N}$, define the packing measure

$$\mu_i = \sum_{r_p \leq r_i} r_p^{n-1} \delta_p,$$

and let $\beta_i(x, r) := \beta_{\mu_i}(x, r)$ denote the corresponding $(n-1)$ -dimensional L^2 Jones beta number as defined in (3.1). In this notation, the conclusion (4.3) of the lemma is $\mu_0(B_1(0)) \leq c(n)$. In order to prove this, we will argue inductively to prove that

$$\int_0^{2r} \int_{B_r(x)} \beta_{\mu}(z, s)^2 d\mu(z) \frac{ds}{s} \leq \delta_0 r^{n-1}$$

for all $x \in B_1(0)$ and all $r \in (0, 1]$, at which point we can apply Theorem 3.1. More specifically, we argue by induction to show that, for $r_i \leq 2^{-4}$

$$(*) \quad \sum_{r_j \leq 2r_i} \int_{B_{2r_i}(x)} \beta_i(z, r_j)^2 d\mu_i(z) \leq \delta_0 r_i^{n-1} \quad \text{for all } x \in B_1(0).$$

By the Discrete Reifenberg Theorem 3.1, whenever $(*)_i$ holds we have

$$(4.14) \quad \mu_i(B_{r_i}) \leq C_0 r_i^{n-1} \quad \text{for all } x \in B_1(0).$$

Here δ_0 and C_0 are the dimensional constants from Theorem 3.1. Notice that $(*)_i$ vacuously holds for i large enough such that $r_i < R$, as in this case $\mu_i \equiv 0$. Also, if $x \in \text{spt} \mu_j$ and $j \geq i$, then by disjointness we have

$$(4.15) \quad \beta_i(x, r_j) = \begin{cases} \beta_j(x, r_j) & \text{if } x \in \text{spt} \mu_j, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to (4.13), we can apply Theorem 4.1 to μ_i whenever $r_i < 2^{-4}$, finding that for any x that is a center of a ball in our collection, we have

$$(4.16) \quad \beta_i(x, r_i)^2 \leq \frac{C}{r_i^{n-1}} \int_{B_{r_i}(x)} \log \left(\frac{J(8r_i, y)}{J(r_i, y)} \right) d\mu_i(y).$$

Suppose now that the inductive hypothesis $(*)_j$ holds for all $j \geq i+1$. Fix $x \in B_1(0)$. We first claim that

$$(4.17) \quad \mu_{i-1}(B_{4r_i}(x)) \leq M r_i^{n-1} \quad \text{and for all } x \in B_1(0),$$

where $M = C(n)C_0$ with $C(n)$ a dimensional constant. To prove this claim, note that we have $\mu_{i-1}(B_{4r_i}(x)) = \mu_{i+1}(B_{4r_i}(x)) + \sum r_p^{n-1}$ where we sum over $p \in B_{4r_i}(x)$ having $r_{i+1} < r_p \leq r_{i-1}$. Since the $B_{2r_p}(p)$ are disjoint we sum over at most $C(n)$ points. Also, there are at most $C(n)$ points in $B_{4r_i}(x)$ that are pairwise r_{i+1} distant from each other, so we can cover $B_{4r_i}(x)$ with $C(n)$ balls with center at points p . We then use our induction hypothesis with $i+1$ and (4.14) to conclude that

$$\mu_{i+1}(B_{4r_i}(x)) \leq \sum \mu_{i+1}(B_{r_{i+1}}(y)) \leq C(n)C_0(n)r_{i+1}^{n-1},$$

which finishes the claim (4.17). Now, for any $j \geq i - 1$, i.e. for $r_j \leq 2r_i$, we have by (4.15), (4.16), and Fubini respectively that

$$\begin{aligned} \int_{B_{2r_i}(x)} \beta_i(z, r_j)^2 d\mu_i(z) &= \int_{B_{2r_i}(x)} \beta_j(z, r_j)^2 d\mu_j(z) \leq \frac{C}{r_j^{n-1}} \int_{B_{2r_i}(x)} \int_{B_{r_j}(z)} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu_j(y) d\mu_j(z) \\ &\leq \frac{C}{r_j^{n-1}} \int_{B_{2r_i+r_j}(x)} \mu_j(B_{r_j}(y)) \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu_j(y). \end{aligned}$$

Now, applying (4.17) and the ordering of the measures we have $\mu_j(B_{r_j}(y))/r_j^{n-1} \leq M$. So, summing over the expression above over all $r_j \leq 2r_i$ and recalling (4.13), we find that

$$\begin{aligned} \sum_{r_j \leq 2r_i} \int_{B_{2r_i}} \beta_i(z, r_j)^2 d\mu_i(z) &\leq CM \sum_{r_j \leq 2r_i} \int_{B_{4r_i}(x)} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu_j(y) \\ &\leq CM \int_{B_{4r_i}(x)} \sum_{r_j \leq 2r_i} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu_{i-1}(y) \\ &\leq CM \int_{B_{4r_i}(x)} \log \left(\frac{J_p(2)}{J_p(r_p)} \right) d\mu_{i-1}(y) \leq CM\eta \mu_{i-1}(B_{4r_i}(x)). \end{aligned}$$

Applying (4.17) again and then choosing η small enough, this establishes the induction step and so $(*)_i$ holds for all i . Now, recalling (4.15), from $(*)_i$ we find that for all $x \in B_1(0)$ and all $i \in \mathbb{N}$,

$$\sum_{r_j \leq 2r_i} \int_{B_{2r_i}(x)} \beta_\mu(z, r_j)^2 d\mu(z) \leq \delta_0 r_i^{n-1}.$$

In particular, we can apply the Discrete Reifenberg Theorem 3.1 to μ and conclude. \square

5. THE DICHOTOMY

In this section, we establish a key dichotomy: either all points in $\Gamma_{\epsilon, \eta}^*$ have small drop in the ACF monotonicity formula down to a small scale, or else all such points have a definite drop in their ACF formula at a smaller scale. This is simpler than the analogous dichotomy in other settings: a typical statement would say that either all points have small energy drop, or else the set of points with small energy drop looks lower dimensional in a quantitative sense. The reason behind this difference is that all blowup configurations for the ACF monotonicity formula have the same number of symmetries. The main form of the dichotomy is the following proposition:

Proposition 5.1. *Fix an integer $n \geq 2$ and positive constants $\epsilon, \bar{\eta}, \bar{\rho}$, and \bar{J}_0 . There exists η depending on these parameters such that the following holds. Fix $r \in (0, 2]$, let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying (1.2) with $x_0 \in \Gamma^* \cap B_2(0)$ and $\sup_{x \in B_{4r}(x_0)} J_x(4) \leq \bar{J} \leq \bar{J}_0$. Then at least one of the two possibilities occurs:*

- (1) *For all $x \in \Gamma_{\epsilon, \eta r}^* \cap B_{2r}(x_0)$, we have $J_x(\bar{\rho}r) > \bar{J} - \bar{\eta}$, or*
- (2) *For all $x \in \Gamma_{\epsilon, \eta r}^* \cap B_{2r}(x_0)$, we have $J_x(4\eta r) \leq \bar{J} - \eta$.*

Proposition 5.1 is a direct consequence of the following lemma, which in turn is based on compactness and the quantitative estimates of Theorem 3.3 and Corollary 3.4.

Lemma 5.2. *Fix $n \geq 2$ and positive constants $\epsilon, \bar{\eta}, \bar{\rho}$ and \bar{J}_0 . There exist $\eta > 0$ and $\tau > 0$ depending on these parameters such that the following holds. Suppose $u, v : B_8(0) \rightarrow \mathbb{R}$ are nonnegative continuous functions satisfying (1.2) with $\sup_{x \in B_4(0)} J_x(4) \leq \bar{J} \leq \bar{J}_0$. If there is some point $y \in B_2(0)$ such that*

$$(5.1) \quad J_y(4\eta) \geq \bar{J} - \eta,$$

then there is an affine hyperplane L containing y such that for every $x \in B_\tau(L) \cap B_2(0)$, we have

$$(5.2) \quad J_x(\bar{\rho}) > \bar{J} - \bar{\eta} \quad \text{and} \quad \Gamma_{\epsilon, \eta}^* \cap B_2(0) \subset B_\tau(L).$$

In particular, $J_x(\bar{\rho}) > \bar{J} - \bar{\eta}$ for all $x \in \Gamma_{\epsilon, \eta}^ \cap B_2(0)$.*

Proof. We argue by way of contradiction. Suppose the first condition in (5.2) fails. We may thus find admissible pairs of functions (u_j, v_j) with $\sup_{x \in B_4(0)} J_x(4; u_j, v_j) \leq \bar{J}_j \leq \bar{J}_0$ and points $y_j \in B_2(0)$ satisfying (5.1) with $\eta_j \rightarrow 0$ in place of η , such that, for a sequence $\tau_j \rightarrow 0$ and for every affine hyperplane L_j containing y_j , there are points x_j in $B_2(0) \cap B_{\tau_j}(L_j)$ such that

$$(5.3) \quad 0 \leq J_{x_j}(\bar{\rho}; u_j, v_j) \leq \bar{J}_j - \bar{\eta}$$

Up to a subsequence, $\bar{J}_j \rightarrow \bar{J} \in [\bar{\eta}, \bar{J}_0]$. A basic calculation shows that $\log(J_{y_i}(1)/J_{y_i}(4\eta_j)) \leq 4\eta_j/\bar{\eta}$. So, by Theorem 3.3, for each j there exist $a_j, b_j > 0$, and $\nu_j \in \mathbb{S}^{n-1}$ such that

$$(5.4) \quad \int_{B_4(y_j) \setminus B_{2\eta_j}(y_j)} [u_j - a_j((x - y_j) \cdot \nu_j)^+]^2 + [v_j - b_j((x - y_j) \cdot \nu_j)^-]^2 \leq C \eta_j \int_{B_4(y_j)} (u_j^2 + v_j^2)$$

for a constant C depending on n and $\bar{\eta}$. Without loss of generality, we can multiply u_j and v_j by constants whose product is equal to 1 so that $a_j = b_j$, since this leaves J unchanged. We may also assume that we have precomposed u_j and v_j with a rotation so that $\nu_j = e_n$ for all j .

We want to show that u_j and v_j converge in L^2 to a pair of nondegenerate truncated linear functions, and so we must verify that the slopes a_j do not degenerate to zero or blow up to infinity along the sequence. To this end, set $\hat{u}_j = u_j/a_j$ and $\hat{v}_j = v_j/a_j$. For j sufficiently large, we have $4\eta_j \leq \max\{\rho_0, \kappa_0^2/\bar{J}\}$, where ρ_0 and κ_0 are the constants from Corollary 3.4. So, applying Corollary 3.4, (5.4) becomes

$$\int_{B_4(y_j) \setminus B_{2\eta_j}(y_j)} (\hat{u}_j - ((x - y_j) \cdot e_n)^+)^2 + (\hat{v}_j - ((x - y_j) \cdot e_n)^-)^2 \leq C \eta_j,$$

for a constant C depending on n and $\bar{\eta}$. Up to a subsequence, $y_j \rightarrow y_0 \in \bar{B}_2(0)$. So, letting $\hat{\ell}_1(x) = ((x - y_0) \cdot e_n)^+$ and $\hat{\ell}_2(x) = ((x - y_0) \cdot e_n)^-$, we see that $\hat{u}_j \rightarrow \hat{\ell}_+$ and $\hat{v}_j \rightarrow \hat{\ell}_-$ in $L^2(B_4(y_0))$. Corollary 3.6 then tells us that

$$\lim_{j \rightarrow \infty} \alpha_j^{-4} J_{y_0}(4; u_j, v_j) = \lim_{j \rightarrow \infty} J_{y_0}(4; \hat{u}_j, \hat{v}_j) = J_{y_0}(4; \hat{\ell}_+, \hat{\ell}_-) = c_*.$$

On the other hand, by (5.1) and continuity, we see that $\lim_{j \rightarrow \infty} J_{y_0}(4; u_j, v_j) = \bar{J}$, and so it follows that $\lim_{j \rightarrow \infty} a_j = a_0 := (\bar{J}/c_*)^{1/4} > 0$. Now, set $\ell_+ = a_0 \hat{\ell}_+$ and $\ell_- = a_0 \hat{\ell}_-$, so

$$(5.5) \quad \int_{B_4(y_0)} (u_j - \ell_+)^2 + (v_j - \ell_-)^2 \rightarrow 0,$$

and $J_x(r, \ell_+, \ell_-) = \bar{J}$ for each $r > 0$ and $x \in \{x \cdot e_n = 0\}$.

Now, we aim to reach a contradiction to (5.3). For each j , choose x_j corresponding to the affine hyperplane $L_j = \{(x - y_j) \cdot e_n = 0\}$. After passing to a subsequence, x_j converges to a point $x_0 \in \bar{B}_2(0) \cap \{(x - y_0) \cdot e_n = 0\}$. Since $y_0 \in \bar{B}_2(0)$ as well, we have $B_r(x_0) \subset B_4(y_0)$ for any $r \leq 1$. So, thanks to Lemma 3.5 once again, we have

$$\lim_{j \rightarrow \infty} J_{x_j}(r; u_j, v_j) \rightarrow J_{x_0}(r; \ell_+, \ell_-) = \bar{J}$$

for all $r \in (0, 1)$. On the other hand, by continuity and monotonicity, (5.3) implies that

$$\lim_{j \rightarrow \infty} J_{x_j}(r; u_j, v_j) \leq \bar{J} - \bar{\eta},$$

for $r \in (0, \bar{\rho})$, leading us to a contradiction. This establishes the first part of (5.2).

Next, let us prove that the second part of (5.2) holds, with the same $\tau > 0$ determined in the proof of (5.2) above and up to further decreasing the parameter $\eta > 0$ from the value determined above. Once again, we argue by way of contradiction and suppose that the second part of (5.2) fails for our fixed choices of $n, \epsilon, \bar{\eta}, \bar{\rho}, \bar{J}_0$ and τ . We may find sequences $(u_j, v_j), y_j$, and $\eta_j \rightarrow 0$ as above, satisfying the hypotheses of the lemma with η_j in place of η , and a sequence of points x_j violating the second part of (5.2), i.e. $x_j \in \Gamma_{\epsilon, \eta_j}^* \cap B_2(0) \setminus B_\tau(\{x \cdot e_n = 0\})$. Repeating the argument above, we find that (5.5) holds and, up to a subsequence, $x_j \rightarrow x_0$ with $\text{dist}(x_0, \{x \cdot e_n = 0\}) \geq \tau$. Without loss of generality, we can assume that $x_0 \cdot e_n > 0$ and thus $x_0 \cdot e_n \geq \tau$. By continuity and the assumption that $x_j \in \Gamma_{\epsilon, \eta_j}^*$ we have, for any fixed $r \in (0, 1)$,

$$\epsilon \leq \lim_{j \rightarrow \infty} J_{x_j}(r; u_j, v_j) = J_{x_0}(r; \ell_+, \ell_-).$$

On the other hand, $J_{x_0}(r; \ell_+, \ell_-)$ for $r \leq \tau/2$ since ℓ_- vanishes identically in $B_r(x_0)$, giving us a contradiction. We conclude that the second part of (5.2) holds. \square

Proposition 5.1 is an easy consequence of Lemma 5.2.

Proof of Proposition 5.1. Let η be chosen according to Lemma 5.2. By scaling, it suffices to prove the proposition with $r = 1$ and $x_0 = 0$. Suppose we are not in case (2) in the statement of the proposition, so there is some $y \in \Gamma_{\epsilon, \eta}^* \cap B_2(0)$ such that $J_y(4\eta) > \bar{J} - \eta$. Applying Lemma 5.2, we then find that $J_x(\bar{\rho}) > \bar{J} - \eta$ for all $x \in \Gamma_{\epsilon, \eta}^* \cap B_2(0)$, i.e. we are in case (1). \square

The following direct corollary is how Proposition 5.1 will be applied in the next section.

Corollary 5.3. *Fix an integer $n \geq 2$ and positive constants $\epsilon, \bar{J}_0, \bar{\eta}$, and $\bar{\rho}$. There exists η depending on $n, \epsilon, \bar{J}_0, \bar{\eta}$, and $\bar{\rho}$ such that the following holds. Fix $R \in (0, 1]$. For any $x \in B_2(0)$ and $r \in [R, 2]$, let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying (1.2) with $\sup_{x \in B_{4r}(x_0)} J_x(4) \leq \bar{J} \leq \bar{J}_0$. Then at least one of the two possibilities occurs:*

- (1) *For all $y \in \Gamma_{\epsilon, \eta R}^* \cap B_{2r}(x_0)$, we have $J_y(\bar{\rho}r) > \bar{J} - \bar{\eta}$, or*
- (2) *For all $y \in \Gamma_{\epsilon, \eta R}^* \cap B_{2r}(x_0)$, we have $J_y(4\eta r) \leq \bar{J} - \eta$.*

Proof. Choose $\eta > 0$ according to Proposition 5.1. Suppose we are not in case (1) of the corollary. So, there is some $y \in \Gamma_{\epsilon, \eta R}^* \cap B_{2r}(x_0)$ with $J_y(\bar{\rho}r) \leq \bar{J} - \bar{\eta}$. Since $r \geq R$ and $\Gamma_{\epsilon, \eta R}^* \subset \Gamma_{\epsilon, \eta r}^*$, we see that $y \in \Gamma_{\epsilon, \eta r}^*$. Proposition 5.1 applied at scale r then implies $J_y(4\eta r) \leq \bar{J} - \eta$ for all $y \in \Gamma_{\epsilon, \eta r}^* \cap B_{2r}(x_0)$, and thus for all $y \in \Gamma_{\epsilon, \eta R}^* \cap B_{2r}(x_0)$ thanks again to the containment $\Gamma_{\epsilon, \eta R}^* \subset \Gamma_{\epsilon, \eta r}^*$. We are thus in case (2) of the corollary. \square

6. QUANTITATIVE ESTIMATES FOR Γ_ϵ^* AND RECTIFIABILITY

The main goal of this section is to prove Theorem 1.3 and Theorem 1.1. In Section 6.1 we construct a good covering of $\Gamma_{\epsilon, \eta R}^*$ with estimates by balls of radius at least R , and in Sections 6.2 and 6.3 we prove Theorem 1.3 and 1.1 respectively. In the previous section, we saw how Proposition 5.2's dichotomy took a simple form thanks to the fact that every ‘‘cone’’ for the Alt-Caffarelli-Friedman monotonicity formula is translationally invariant along some $(n - 1)$ -dimensional affine subspace. Thanks to this fact, in Section 6.1 we give a covering construction that is substantially simpler than the covering used in many applications of the Naber-Valtorta machinery to singularity analysis. In particular, we can avoid entirely the ‘‘good tree/bad tree construction’’ and instead prove Lemma 6.1 below with a fairly straightforward stopping time argument.

6.1. Main covering construction. This section is dedicated to Lemma 6.1, which is the main covering construction that will be iterated to prove the quantitative estimates of Theorem 1.3. Given $\epsilon > 0$ and $R \in (0, 1]$, this lemma gives a covering, with estimates, of the $(\epsilon, \eta R)$ stratum for some $\eta \ll 1$, by balls of radii at least R . The two key ingredients in the proof are the key dichotomy of Proposition 5.1 and the packing estimates of Proposition 4.2. The basic idea, which of course requires some technical modification, is the following. At each point in $x \in \Gamma_{\epsilon, \eta R}^* \cap B_1(0)$, choose the smallest scale $r_x \in [R, 1]$ such that $J_x(\rho r_x)$ stays uniformly large, and take a Vitali subcover of the corresponding cover. By design, the hypotheses of Proposition 4.2 are satisfied by this cover, and this gives us the packing estimates in part (2) of the lemma below. Then, the dichotomy of the previous section gives us part (3): either the drop in the ACF monotonicity formula stays small all the way down to scale $r_x = R$, or else the drop becomes large at some scale $r_x > R$ and we can apply Proposition 5.1.

Lemma 6.1. *Fix $n \geq 2$, $\epsilon > 0$, and $\bar{J}_0 > 0$. There are positive constants C and η depending on n, ϵ , and \bar{J}_0 such that the following holds. Let $u, v : B_{10}(0) \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying (1.2) with $\bar{J} := \sup_{B_4(0)} J_x(4) \leq \bar{J}_0$, and fix $R \in (0, 1]$. There is a collection of balls $\{B_{r_x}(x)\}_{x \in \mathcal{C}}$ with $r_x \in [R, \eta]$ and $x \in \Gamma_{\epsilon, \eta R}^* \cap B_1(0)$ satisfying the following properties:*

- (1) *The balls form a covering of the $(\epsilon, \eta R)$ stratum in $B_1(0)$, that is,*

$$\Gamma_{\epsilon, \eta R}^* \cap B_1(0) \subset \bigcup_{x \in \mathcal{C}} B_{r_x}(x).$$

- (2) *The balls satisfy the packing estimates*

$$\sum_{x \in \mathcal{C}} r_x^{n-1} \leq C.$$

(3) For every $x \in \mathcal{C}$, either $r_x = R$ or

$$\sup_{y \in B_{4r_x}(x)} J_y(4r_x) \leq \bar{J} - \eta.$$

Proof. Step 0: Let us begin by fixing parameters. Let $\bar{\eta} = \bar{\eta}(n, \epsilon)$ be chosen according to Proposition 4.2 and let $\bar{\rho} = \bar{\eta}/10$. Choose η according to the dichotomy of Corollary 5.3, depending on $n, \epsilon, \bar{J}_0, \bar{\eta}$, and $\bar{\rho}$ and thus on n, ϵ , and \bar{J}_0 . Up to possibly further decreasing η , we may assume that $\eta \leq \bar{\eta}$ and $\eta \leq 1/10$.

Step 1: We construct the collection of balls and show that it forms a cover of $\Gamma_{\epsilon, \eta R}^* \cap B_1(0)$. For each $x \in \Gamma_{\epsilon, \eta R}^* \cap B_1(0)$, define the stopping time

$$(6.1) \quad \hat{r}_x := \inf \{r \in [R, 1] : J_x(\bar{\rho} r) > \bar{J} - \bar{\eta}\},$$

with the convention that $\hat{r}_x = 1$ if $J_x(\bar{\rho}) \leq \bar{J} - \bar{\eta}$. The collection $\{B_{\hat{r}_x/5}(x)\}_{x \in \Gamma_{\epsilon, \eta R}^* \cap B_1(0)}$ is a cover of $\Gamma_{\epsilon, \eta R}^* \cap B_1(0)$.

We apply the Vitali covering lemma to find a subset $\hat{\mathcal{C}} \subset \Gamma_{\epsilon, \eta R}^*$ such that the balls $\{B_{\hat{r}_x/5}(x)\}_{x \in \hat{\mathcal{C}}}$ are disjoint and the collection $\{B_{\hat{r}_x}(x)\}_{x \in \hat{\mathcal{C}}}$ forms a cover of $\Gamma_{\epsilon, \eta R}^*$. We split this set of ball centers into three subsets:

$$\mathcal{G} := \{x \in \hat{\mathcal{C}} : \hat{r}_x = R\}, \quad \hat{\mathcal{A}} := \{y \in \hat{\mathcal{C}} : \hat{r}_y \in (R, 1)\}, \quad \hat{\mathcal{V}} := \{y \in \hat{\mathcal{C}} : \hat{r}_y = 1\}.$$

For $x \in \mathcal{G}$, let $r_x = \hat{r}_x = R$. The balls $B_{r_x}(x)$ for $x \in \mathcal{G}$ will be included in the final cover. The balls $B_{\hat{r}_y}(y)$ for $y \in \hat{\mathcal{A}} \cup \hat{\mathcal{V}}$ need to be further subdivided in the following simple way. For $y \in \hat{\mathcal{A}} \cup \hat{\mathcal{V}}$, set $r_y = \max\{R, \eta \hat{r}_y\}$ and take a maximal $r_y/2$ disjoint set $\{x_{i,y}\}_{i=1}^{N_y}$ in $B_{\hat{r}_y}(y)$. There are at most $C_n \eta^{-n}$ such points, i.e. $N_y \leq C_n \eta^{-n}$, and the collection $\{B_{r_y}(x_{i,y})\}_{i=1}^{N_y}$ is a cover of $B_{\hat{r}_y}(y)$. Let

$$\mathcal{A} = \{x_{i,y} : y \in \hat{\mathcal{A}}, i = 1, \dots, N_y\}, \quad \mathcal{V} = \{x_{i,y} : y \in \hat{\mathcal{V}}, i = 1, \dots, N_y\}.$$

If $x \in \mathcal{A} \cup \mathcal{V}$, then $x = x_{i,y}$ for at least one $y \in \hat{\mathcal{A}} \cup \hat{\mathcal{V}}$; set $r_x = r_y$ for the smallest such r_y . Now, let $\mathcal{C} = \mathcal{G} \cup \mathcal{A} \cup \mathcal{V}$. The collection $\{B_{r_x}(x)\}_{x \in \mathcal{C}}$ is, by construction, a cover of $\Gamma_{\epsilon, \eta R}^* \cap B_1(0)$, so part (1) of the lemma holds.

Step 2: We now verify the condition (2) of the lemma, using Proposition 4.2's packing estimate as the main tool. First, if $x \in \mathcal{G}$, then $J_x(\bar{\rho} r_x) \geq \bar{J} - \bar{\eta}$ by the stopping time definition and the continuity of the ACF formula with respect to r . Since we have chosen our parameters so that $\bar{\rho} \leq \bar{\eta}/10$, the monotonicity of J_x guarantees that $J_x(\bar{\eta} r_x/10) \geq \bar{J} - \bar{\eta}$. The balls $\{B_{r_x/5}(x)\}_{x \in \mathcal{G}}$ are disjoint by construction. So, we may apply Proposition 4.2, with $R/10$ in place of R , to $\{B_{r_x/10}(x)\}_{x \in \mathcal{G}}$ to find that

$$(6.2) \quad \sum_{x \in \mathcal{G}} r_x^{n-1} = R^{n-1} \#(\mathcal{G}) \leq C(n).$$

We prove the packing estimate for $x \in \mathcal{A}$ in a similar way. If $y \in \hat{\mathcal{A}}$, then by continuity $J(\bar{\rho} \hat{r}_y) = \bar{J} - \bar{\eta}$. Again by the choice of $\bar{\rho}$ and $\bar{\eta}$ and by monotonicity, $J_y(\bar{\eta} \hat{r}_y/10) \geq \bar{J} - \bar{\eta}$, and the balls in the collection $\{B_{\hat{r}_y/5}(y)\}_{y \in \hat{\mathcal{A}}}$ are pairwise disjoint by construction. So, once more we apply Proposition 4.2 to $\{B_{\hat{r}_y/10}(y)\}_{y \in \hat{\mathcal{A}}}$ to find that $\sum_{y \in \hat{\mathcal{A}}} \hat{r}_y^{n-1} \leq C(n)$. Since $r_x \leq \hat{r}_y$ for any $x = x_{i,y} \in \mathcal{A}$ and $N_y \leq C(n) \eta^{-n}$ for all $y \in \hat{\mathcal{A}}$, this directly implies

$$(6.3) \quad \sum_{x \in \mathcal{A}} r_x^{n-1} \leq C(n) \eta^{-n} \sum_{y \in \hat{\mathcal{A}}} \hat{r}_y^{n-1} \leq C(n, \eta).$$

Finally, the packing estimate for \mathcal{V} is easy. The balls $\{B_{1/5}(x)\}_{x \in \hat{\mathcal{V}}}$ are pairwise disjoint, so $\#\hat{\mathcal{V}} \leq C(n)$ and $\#\mathcal{V} \leq C(n) \eta^{-n}$. Since $r_x = \eta$ for each $x \in \mathcal{V}$, this implies that

$$(6.4) \quad \sum_{x \in \mathcal{V}} r_x^{n-1} = \eta^{n-1} \#(\mathcal{V}) \leq C(n, \eta).$$

Since η depends on n, ϵ , and \bar{J}_0 , together (6.2), (6.2), and (6.4) show that condition (2) of the lemma holds with a constant C depending on n, ϵ , and \bar{J}_0 .

Step 3: Now we verify the third condition: either $r_x = R$ or there is a definite energy drop in all of $B_{4r_x}(x)$. The main tool is Corollary 5.3, which we recall states that for any $x \in B_2(0)$ and $r \in [R, 2]$, at least one of

$$\begin{aligned} J_y(\bar{\rho} r) &> \bar{J} - \bar{\eta} && \text{for all } y \in \Gamma_{\epsilon, \eta R}^* \cap B_{2r}(x), \\ \text{or } J_y(4\eta r) &\leq \bar{J} - \eta && \text{for all } y \in \Gamma_{\epsilon, \eta R}^* \cap B_{2r}(x) \end{aligned}$$

holds. If $x \in \mathcal{G}$, then $r_x = R$ and condition (3) of the lemmas holds. Next, as we observed in step 2, if $y \in \hat{\mathcal{A}}$, then by the definition of \hat{r}_y and continuity, $J_y(\bar{\rho} \hat{r}_y) = \bar{J} - \bar{\eta}$. By Corollary 5.3 applied with $r = \hat{r}_y$, we have

$$(6.5) \quad J_x(4\eta \hat{r}_y) \leq \bar{J} - \eta \quad \text{for all } x \in \Gamma_{\epsilon, \eta R}^* \cap B_{2\hat{r}_y}(y).$$

In particular, for each $x \in \mathcal{A}$ that came from subdividing $B_{\hat{r}_y}(y)$, we have $B_{4r_x}(x) = B_{4\eta \hat{r}_y}(x) \subset B_{2\hat{r}_y}(y)$. So, keeping in mind that $r_x \leq \eta \hat{r}_y$, (6.5) implies that

$$\sup_{z \in B_{4r_x}(x)} J_z(4r_x) \leq \bar{J} - \eta \quad \text{for all } x \in \mathcal{A}.$$

Thus condition (3) of the lemma holds for all $x \in \mathcal{A}$. Finally, if $y \in \hat{\mathcal{V}}$, then $J_y(\bar{\rho}) \leq \bar{J} - \bar{\eta}$ by the definition of \hat{r}_y . The exact same argument given for \mathcal{A} above shows that

$$\sup_{y \in B_{4r_x}(x)} J_y(4r_x) \leq \bar{J} - \eta \quad \text{for all } x \in \mathcal{V}.$$

So, condition (3) holds for all $x \in \mathcal{V}$. This completes the proof of the lemma. \square

6.2. Proof of the quantitative estimates. We can now iteratively apply Lemma 6.1 finitely many times to prove Theorem 1.3 in a standard way.

Proof of Theorem 1.3. First, let us note how the estimate (1.5) of Theorem 1.3 implies the two estimates in (1.6). If (1.5) holds, then for every $x \in B_r(\Gamma_{\epsilon, r}^* \cap B_1(0))$, there is an index $i \in \{1, \dots, N\}$ such that $x \in B_r(x_i)$ and thus $B_r(x) \subset B_{2r}(x_i)$, with $N \leq C r^{1-n}$ as in (1.5). So,

$$B_r(\Gamma_{\epsilon, r}^* \cap B_1(0)) \subset \bigcup_{i=1}^N B_{2r}(x_i)$$

and in particular $|B_r(\Gamma_{\epsilon, r}^* \cap B_1(0))| \leq N \omega_n (2r)^n \leq C(n, \epsilon, \bar{J}_0) r$. This proves the first estimate in (1.6). Next, since $\Gamma_\epsilon^* \subset \Gamma_{\epsilon, r}^*$, the estimate (1.5) gives the upper bound $\mathcal{H}_r^{n-1}(\Gamma_\epsilon^*) \leq C(n, \epsilon, \bar{J}_0)$ for each $r \in (0, 1]$. Passing $r \rightarrow 0$ establishes the Hausdorff measure bound in the second estimate of (1.6).

Now, fix $R \in (0, 1]$. We will construct a collection of balls satisfying (1.5) with $r = R$ by inductively applying Lemma 6.1. Let $\eta = \eta(n, \epsilon, \bar{J}_0)$ be as in Lemma 6.1. If the covering $\{B_{r_x}(x)\}_{x \in \mathcal{C}}$ provided by Lemma 6.1 has $r_x = R$ for every $x \in \mathcal{C}$, then (1.5) follows directly from the packing estimate (2), since $\Gamma_{\epsilon, \eta R}^* \subset \Gamma_{\epsilon, R}^*$. So, it suffices to construct a covering as in Lemma 6.1 where in part (3), we always have $r_x = R$. The key observation is that the definite energy drop of (3) can only occur on $\lceil \bar{J}/\eta \rceil$ scales by monotonicity, and so after $\lceil \bar{J}/\eta \rceil$ iterations of the lemma, we can only have $r_x = R$.

More specifically, we claim that there exist a (finite) sequence of covers $\{B_{r_x}(x)\}_{x \in \mathcal{C}_i}$, whose centers are $\mathcal{C}_1, \mathcal{C}_2, \dots$, satisfying the following properties:

(A_i) Covering: $\Gamma_{\epsilon, \eta R}^* \cap B_1(0) \subset \bigcup_{x \in \mathcal{C}_i} B_{r_x}(x)$,

(B_i) Packing: $\sum_{x \in \mathcal{C}_i} r_x^{n-1} \leq C \left(1 + \sum_{x \in \mathcal{C}_{i-1}} r_x^{n-1}\right)$ for a constant $C = C(n, \epsilon, \bar{J})$.

(C_i) Energy drop: for each $x \in \mathcal{C}_i$, we either have $r_x = R$ or $\sup_{B_{4r_x}(x)} J_y(4r_x) \leq \bar{J} - i\eta$.

Observe (C_i) guarantees that one of the collections $\{B_R(x)\}_{x \in \mathcal{C}_i}$ for $i \in \{1, \dots, \lceil \eta/\bar{J} \rceil\}$ satisfies (1.5), and so the theorem will follow directly from the claim.

Lemma 6.1 gives us such a covering in the base case $i = 1$. Suppose we have constructed a covering \mathcal{C}_{i-1} satisfying (A_{i-1}), (B_{i-1}), and (C_{i-1}). We construct \mathcal{C}_i satisfying (A_i), (B_i), and (C_i) in the following way. If $x \in \mathcal{C}_{i-1}$ has $r_x = R$, then we include it in \mathcal{C}_i . If $x \in \mathcal{C}_{i-1}$ has $r_x > R$, then we apply Lemma 6.1 to $B_{r_x}(x)$. This gives us a collection of balls $\{B_{r_y}(y)\}_{y \in \mathcal{C}_{x,i}}$ such that

$$\Gamma_{\epsilon, \eta R}^* \cap B_{r_x}(x) \subset \bigcup_{y \in \mathcal{C}_{x,i}} B_{r_y}(y),$$

that satisfy the packing estimates

$$(6.6) \quad \sum_{y \in \mathcal{C}_{x,i}} r_y^{n-1} \leq C r_x^{n-1}$$

for a constant $C = C(n, \epsilon, \bar{J}_0)$. Moreover, thanks to (C_{i-1}) and condition (3) of Lemma 6.1, for each $y \in \mathcal{C}_{x,i}$, either $r_y = R$, or else

$$(6.7) \quad \sup_{z \in B_{4r_y}} J_z(4r_y) \leq \sup_{B_{4r_x}(x)} J_z(4r_x) \leq \bar{J} - (i-1)\eta - \eta = \bar{J} - i\eta.$$

We let

$$\mathcal{C}_i = \{x \in \mathcal{C}_{i-1} : r_x = R\} \cup \bigcup_{x \in \mathcal{C}_{i-1}, r_x > R} \mathcal{C}_{x,i}.$$

By construction, $\{B_{r_x}(x)\}_{x \in \mathcal{C}_i}$ is a cover of $\Gamma_{\epsilon, \eta R}^* \cap B_1(0)$, and so (A_i) holds, and (C_i) follows from (6.7) and the construction. Finally, by (B_{i-1}) and (6.6), we have for a constant $C = C(n, \epsilon, \bar{J}_0)$,

$$\begin{aligned} \sum_{x \in \mathcal{C}_i} r_x^{n-1} &= \sum_{\substack{x \in \mathcal{C}_{i-1}, \\ r_x = R}} r_x^{n-1} + \sum_{\substack{x \in \mathcal{C}_{i-1}, \\ r_x > R}} \left(\sum_{y \in \mathcal{C}_{x,i}} r_y^{n-1} \right) \\ &\leq \sum_{\substack{x \in \mathcal{C}_{i-1}, \\ r_x = R}} r_x^{n-1} + C \sum_{\substack{x \in \mathcal{C}_{i-1}, \\ r_x > R}} r_x^{n-1} \leq C \sum_{x \in \mathcal{C}_{i-1}} r_x^{n-1}. \end{aligned}$$

Thus (B_i) holds as well. This proves the inductive step, and thus concludes the proof of the theorem. \square

6.3. Rectifiability. We now prove Theorem 1.1, which says that $\Gamma^* \cap B_1(0)$ is \mathcal{H}^{n-1} -rectifiable. This proof is a standard step in the Naber-Valtorta framework: combining the upper Ahlfors regularity of $\mathcal{H}^{n-1} \llcorner \Gamma_\epsilon^*$ shown in Theorem 1.3 and the L^2 subspace approximation of Theorem 4.1, we show the hypotheses of Naber-Valtorta's Rectifiable Reifenberg Theorem 3.2 hold, and thus Γ_ϵ^* is \mathcal{H}^{n-1} -rectifiable. In fact, instead of arguing on Γ_ϵ^* directly, we will break each Γ_ϵ^* into countably many smaller pieces, each of which we will show is rectifiable. Since the countable union of rectifiable sets is again rectifiable, it follows that Γ^* is rectifiable as well.

Proof of Theorem 1.1. Let $\bar{J}_0 = \sup_{x \in B_4(0)} J_x(4)$. Fix any $\epsilon > 0$. Let $\sigma = \sigma(n, \epsilon, \bar{J}_0) > 0$ be a small fixed number to be specified later in the proof, and let

$$S = S_{\epsilon, \sigma} = \{x \in B_1(0) : J(0^+) \in [\epsilon, \epsilon \exp\{\sigma\}]\} \subset \Gamma_\epsilon^* \cap B_1(0).$$

We will prove that S is \mathcal{H}^{n-1} -rectifiable. To this end, we first claim that for all $x_0 \in S$, there is a scale $r > 0$ depending on x_0 such that

$$(6.8) \quad \log \left(\frac{J_y(8r)}{J_y(0^+)} \right) \leq 2\sigma \quad \text{for all } y \in S \cap B_r(x_0).$$

Suppose by way of contradiction that we can find a sequence of points $y_j \in S$ with $y_j \rightarrow x_0$ and scales $r_j \rightarrow 0$ for which $\log(J_{y_j}(8r_j)/J_{y_j}(0^+)) > 2\sigma$. In other words, exponentiating and multiplying through by $J_{y_j}(0^+)$,

$$J_{y_j}(8r_j) > J_{y_j}(0^+) \exp\{2\sigma\} \geq \epsilon \exp\{2\sigma\},$$

where the final inequality holds because $S \subset \Gamma_\epsilon^*$. Taking the limsup of both sides as $j \rightarrow \infty$, we deduce from the upper semicontinuity property (3.4) that $J_{x_0}(0^+) \geq \epsilon \exp\{2\sigma\}$. On the other hand $J_{x_0}(0^+) \leq \epsilon \exp\{\sigma\}$ by virtue of being in S . We reach a contradiction and see that (6.8) holds for $r > 0$ sufficiently small.

To prove that S is \mathcal{H}^{n-1} -rectifiable, it suffices to show that $S \cap B_r(x_0)$ is rectifiable for any $x_0 \in S$ and for $r > 0$ such that (6.8) holds. Let $\mu = \mathcal{H}^{n-1} \llcorner S$. Applying Theorem 4.1, we then see that, letting $r_j = 2^{-j}$,

$$(6.9) \quad \sum_{r_j \leq r} \int_{B_r(x_0)} \beta_\mu(z, r_j)^2 d\mu(z) \leq \sum_{r_j \leq r} \frac{C}{r_j^{n-1}} \int_{B_r(x_0)} \int_{B_{r_j}(z)} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu(y) d\mu(z)$$

Now applying Fubini's theorem, for any $r_j \leq r$.

$$\begin{aligned} \int_{B_r(x_0)} \int_{B_{r_j}(z)} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu(y) d\mu(z) &= \int_{\mathbb{R}^n} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) \mu(B_{r_j}(y) \cap B_r(x_0)) d\mu(y) \\ &\leq C \int_{B_{2r}(x_0)} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) \mu(B_{r_j}(y)) d\mu(y). \end{aligned}$$

Next, by the Ahlfors upper bound in (1.6) in Theorem 1.3, we know that $\mu(B_s(y)) \leq C(n, \epsilon, \bar{J}_0)s^{n-1}$ for all $y \in B_{2r}(x_0)$ and $s \leq 2r$. This together with (6.9) implies

$$\begin{aligned} \sum_{r_j \leq r} \int_{B_r(x_0)} \beta_\mu(z, r_j)^2 d\mu(z) &\leq \sum_{r_j \leq r} C \int_{B_{2r}(x_0)} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu(y) \\ &= C \int_{B_{2r}(x_0)} \sum_{r_j \leq r} \log \left(\frac{J_y(8r_j)}{J_y(r_j)} \right) d\mu(y) \\ &= C \int_{B_{2r}(x_0)} \log \left(\frac{J_y(8r)}{J_y(0^+)} \right) d\mu(y) \leq C\sigma r^{n-1}. \end{aligned}$$

In the final line we used (6.8) and the Ahlfors upper bound in (1.6) once again. Now, let us choose $\sigma > 0$ small enough so that $C\sigma < \delta_0$, where δ_0 is the dimensional constant from the rectifiable Reifenberg theorem, Theorem 3.2. So, by Theorem 3.2, $S \cap B_r(x_0)$ is rectifiable. \square

7. UNIQUENESS OF FUNCTION BLOWUPS

This section is dedicated to the proof of Theorem 1.2. The basic idea is the following. Theorems 1.1 and Theorem 1.3 imply that $\mathcal{H}^{n-1} \llcorner \Gamma_\epsilon^*$ is a Radon measure and has an approximate tangent plane at \mathcal{H}^{n-1} -a.e. point. Together with the differentiation theory for measures, we prove the existence of limits of $\Delta u^{x,r}$ and $\Delta v^{x,r}$ as distributions as $r \rightarrow 0$. Using the quantitative form of the Alt-Caffarelli-Friedman monotonicity formula of Theorem 3.3, we directly relate these distributional limits to the truncated linear functions arising as the blowup limits of u^{x,r_k} and v^{x,r_k} along a sequence $r_k \rightarrow 0$. The independence of the former from the sequence $r_k \rightarrow 0$ allows us to prove the uniqueness of the latter.

Proof of Theorem 1.2. Step 1: Fix $\epsilon > 0$. We will prove the theorem under the assumption that $x \in \Gamma_\epsilon^* \cap B_{1/2}(0)$. As the statement is purely qualitative and $\Gamma^* = \cup_\epsilon \Gamma_\epsilon^*$, this will imply the conclusion.

By Theorems 1.1 and 1.3, $\nu := \mathcal{H}^{n-1} \llcorner \Gamma_\epsilon^* \cap B_1(0)$ is a Radon measure, and for \mathcal{H}^{n-1} -a.e. $x \in \Gamma_\epsilon^* \cap B_1(0)$,

$$(7.1) \quad \lim_{r \searrow 0} \frac{\nu(B_r(x))}{\omega_{n-1} r^{n-1}} = 1 \quad \text{and} \quad \nu^{x,r}(E) \xrightarrow{*} \mathcal{H}^{n-1} \llcorner \{z \cdot \nu^x = 0\} \quad \text{for some } \nu^x \in S^{n-1}.$$

Here we set $\nu^{x,r}(E) := \frac{\nu(x+rE)}{r^{n-1}}$ for $r > 0$ and convergence is in the weak-* topology for measures as $r \rightarrow 0$. See [Mag12, Theorem 10.2] for proofs of these properties of \mathcal{H}^{n-1} -rectifiable sets with locally finite \mathcal{H}^{n-1} measure.

The distributional Laplacian Δu (defined by acting on $\phi \in C_c^\infty(\mathbb{R}^n)$ by $\Delta u(\phi) = \int u \Delta \phi$) is positive in the sense that $\Delta u(\phi) \geq 0$ for any nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^n)$. It is easy to show that any such distribution is a bounded linear functional on $C_c^0(\mathbb{R}^n)$, so by the Riesz representation theorem (see for instance [Mag12, Theorem 4.7]), we may express $\Delta u(\phi) = \int \phi d\mu_u$ for a Radon measure μ_u . In the same way, the distributional Laplacian Δv of v is identified with a Radon measure μ_v .

Applying the Lebesgue-Besicovitch differentiation theorem (see for instance [Mag12, Theorem 5.8]) to μ_u (resp. μ_v) and ν , we see that for \mathcal{H}^{n-1} -a.e. $x \in \Gamma_\epsilon^*$, the limits

$$(7.2) \quad \zeta_u(x) := \lim_{r \searrow 0} \frac{\mu_u(B_r(x))}{\nu(B_r(x))}, \quad \zeta_v(x) := \lim_{r \searrow 0} \frac{\mu_v(B_r(x))}{\nu(B_r(x))} \quad \text{exist and are finite,}$$

and we may write $\mu_u = \zeta_u d\nu + \mu_u^s$ and $\mu_v = \zeta_v d\nu + \mu_v^s$, where μ_u^s (resp. μ_v^s) and ν are mutually singular. Let us restrict our attention, then, to those $x \in \Gamma_\epsilon^*$ such that both (7.1) and (7.2) hold. For any such x , we have

$$(7.3) \quad \begin{aligned} \mu_u^{x,r}(E) &:= \frac{\mu_u(x+rE)}{r} \xrightarrow{*} \zeta_u(x) \mathcal{H}^{n-1} \llcorner \{z \cdot \nu^x = 0\}, \\ \mu_v^{x,r}(E) &:= \frac{\mu_v(x+rE)}{r} \xrightarrow{*} \zeta_v(x) \mathcal{H}^{n-1} \llcorner \{z \cdot \nu^x = 0\} \end{aligned}$$

in the weak-* topology for measures. Notice that the rescaled measures correspond to the distributional Laplacians of the rescaled functions $u^{x,r}$ and $v^{x,r}$, i.e. $\mu_u^{x,r} = \Delta u^{x,r}$ and $\mu_v^{x,r} = \Delta v^{x,r}$.

Step 2: We claim that

$$(7.4) \quad \limsup_{r \searrow 0} \left\{ \|u^{x,r}\|_{L^2(B_1)} + \|v^{x,r}\|_{L^2(B_1)} \right\} < \infty.$$

To this end, set

$$q_{u,r} = \max\{\|u^{x,r}\|_{L^2(B_1)}, 1\}, \quad q_{v,r} = \max\{\|v^{x,r}\|_{L^2(B_1)}, 1\}.$$

Since $x \in \Gamma_\epsilon^*$, we have $\epsilon \leq J_x(0^+) = \lim_{r \searrow 0} J_x(r)$, and so $\lim_{r \searrow 0} \log \frac{J_x(r)}{J_x(0^+)} = 0$. So, we can apply the stability inequality, Theorem 3.3, to $\hat{u}_r := u^{x,r}/q_{u,r}$ and $\hat{v}_r := v^{x,r}/q_{v,r}$ (as usual, noting that dividing by positive constants leaves the ratio $J_0(1)/J_0(0^+)$ unchanged) to obtain

$$(7.5) \quad \int_{B_1} |\hat{u}_r - a^r(z \cdot \nu^r)^+|^2 + \int_{B_1} |\hat{v}_r - b^r(z \cdot \nu^r)^-|^2 \leq C \log \frac{J_x(r)}{J_x(0^+)}$$

for positive constants a^r, b^r and vectors $\nu^r \in \mathbb{S}^{n-1}$. (These depend on x as well as r , but we suppress the dependence in the notation since x is fixed throughout this step). Clearly $a^r, b^r < C$ uniformly for all r sufficiently small. Assume by way of contradiction that $q_{u,r_k} \rightarrow \infty$ along a sequence $r_k \searrow 0$. Up to a subsequence, $a^{r_k} \rightarrow a^* \in [0, C]$, $b^{r_k} \rightarrow b^* \in [0, C]$, and $\nu^{r_k} \rightarrow \nu^* \in \mathbb{S}^{n-1}$, and thus (7.5) tells us that $\hat{u}_{r_k} \rightarrow \hat{u}^* := a^*(z \cdot \nu^*)^+$ and $\hat{v}_{r_k} \rightarrow \hat{v}^* := b^*(z \cdot \nu^*)^-$ in $L^2(B_1)$. Moreover, $a^* > 0$ since $\|\hat{u}_{r_k}\|_{L^2(B_1)} = 1$ for all k large enough.

From (7.2) we have that $\Delta u^{x,r_k}(B_1) \rightarrow \omega_{n-1} \zeta_u(x) < \infty$ and $\Delta v^{x,r_k}(B_1) \rightarrow \omega_{n-1} \zeta_v(x) < \infty$. Since $q_{u,r}, q_{v,r} \geq 1$, we must have

$$\limsup_{r \searrow 0} \Delta \hat{u}_r(B_1) \leq \omega_{n-1} \zeta_u(x) < \infty \quad \text{and} \quad \limsup_{r \searrow 0} \Delta \hat{v}_r(B_1) \leq \zeta_v(x) < \infty.$$

Up to passing to a further subsequence, this implies that $\Delta \hat{u}_{r_k} \rightarrow \Delta \hat{u}^*$ in the sense of distributions on $B_{1/2}$. Thanks to (7.3) and the definition of \hat{u}_r , we have

$$\Delta \hat{u}^* = \left(\lim_{k \rightarrow \infty} \frac{1}{q_{u,r_k}} \right) \zeta_u(x) \mathcal{H}^{n-1} \llcorner \{z \cdot \nu^x = 0\}.$$

This implies, in particular, that $\Delta \hat{u}^* = 0$, i.e. that \hat{u}^* is harmonic on $B_{1/2}$. But this clearly contradicts the expression $\hat{u}^* = a^*(z \cdot \nu^*)_+$ with $a^* > 0$ obtained above. This proves (7.4).

Step 3: With this in mind, we may repeat the blowup argument in Step 2 above except this time without normalizing by $q_{u,r}, q_{v,r}$. This shows that, along every sequence of radii $r_k \searrow 0$, there exists a subsequence with $u^{x,r_k} \rightarrow u^* := a^*(z \cdot \nu^*)^+$ and $v^{x,r_k} \rightarrow v^* := b^*(z \cdot \nu^*)^-$ in $L^2(B_1)$, for some $a^*, b^* \in [0, C]$ and $\nu^* \in \mathbb{S}^{n-1}$, and moreover $\Delta u^{x,r_k} \rightarrow \Delta u^*$ and $\Delta v^{x,r_k} \rightarrow \Delta v^*$ as distributions. To complete the proof of the theorem, we must show that a^*, b^* , and ν^* do not depend on the sequence r_k . The convergence (7.3) implies that $a^* = \zeta_u(x)$ and $b^* = \zeta_v(x)$, and that ν^* is either ν^x or $-\nu^x$ (but either is possible). We must, then, show that ν^* is one or the other independent of subsequence. Before doing so, however, note that a direct application of Corollary 3.6 implies that $c_* \zeta_u(x) \zeta_v(x) = J_x(0^+)$, so in particular both $\zeta_u(x)$ and $\zeta_v(x)$ are nonzero.

To show that ν^* is independent of subsequence, observe that as $\pm \nu^x$ are the only two limit points for $\nu^{x,r}$, for any $\delta > 0$, there exists an $r_0 > 0$ such that $\min\{|\nu^{x,r} - \nu^x|, |\nu^{x,r} + \nu^x|\} < \delta$ for all $r < r_0$. If $\nu^{x,r}$ has more than one limit point, then there must be a sequence $r_k \searrow 0$ and another sequence $s_k < r_k$ with $1 - \frac{s_k}{r_k} \rightarrow 0$ such that $|\nu^{x,r_k} - \nu^x| < \delta$ and $|\nu^{x,s_k} + \nu^x| < \delta$. The first of these implies that

$$\limsup_{k \rightarrow \infty} \int_{B_1} |u_1^{x,r_k} - \zeta_1(x)(x \cdot \nu^x)_+|^2 \leq \limsup_{k \rightarrow \infty} \int_{B_1} |u_1^{x,r_k} - \zeta_1(x)(x \cdot \nu^{x,r_k})_+|^2 + C\delta \leq C\delta.$$

Restricting to B_{s_k/r_k} and rescaling,

$$\int_{B_1} |u_1^{x,s_k} - \zeta_1(x)(x \cdot \nu^x)_+|^2 \leq C\delta \left(\frac{r_k}{s_k} \right)^{n-2} \leq C\delta.$$

On the other hand, ν^{x,s_k} is close to $-\nu^x$, so

$$\int_{B_1} |u_1^{x,s_k} + \zeta_1(x)(x \cdot \nu^x)_+|^2 \leq C\delta.$$

This implies that $\zeta_1(x) \leq C\sqrt{\delta}$ for any $\delta > 0$, which contradicts that $\zeta_1(x) > 0$. Finally, thanks to Lemma 3.5, the L^2 convergence immediately implies $W^{1,2}$ convergence. \square

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