CATEGORICITY FROM ONE SUCCESSOR CARDINAL IN TAME ABSTRACT ELEMENTARY CLASSES

RAMI GROSSBERG AND MONICA VANDIEREN

ABSTRACT. We prove that from categoricity in λ^+ we can get categoricity in all cardinals $\geq \lambda^+$ in a χ-tame abstract elementary classe $\mathcal{K}$ which has arbitrarily large models and satisfies the amalgamation and joint embedding properties, provided $\lambda > \mathrm{LS}(\mathcal{K})$ and $\lambda \geq \chi$.

For the missing case when $\lambda = \mathrm{LS}(\mathcal{K})$, we prove that $\mathcal{K}$ is totally categorical provided that $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$ and $\mathrm{LS}(\mathcal{K})^+$.

1. INTRODUCTION

The benchmark of progress in the development of a model theory for abstract elementary classes (AECs) is Shelah's Categoricity Conjecture.

Conjecture 1.1. *Let $\mathcal{K}$ be an abstract elementary class. If $\mathcal{K}$ is categorical in some $\lambda > \mathrm{Hanf}(\mathcal{K})$[1], then for every $\mu \geq \mathrm{Hanf}(\mathcal{K})$, $\mathcal{K}$ is categorical in μ.*

With the exception of [MaSh], [KoSh], [Sh 576], [ShVi] and [Va] in which extra set-theoretic assumptions are made, all work towards Shelah's Categoricity Conjecture has taken place under the assumption of the amalgamation property. An AEC satisfies the *amalgamation property* if for every triple of models M_0, M_1, M_2 in which $M_0 \prec_{\mathcal{K}} M_1$ and $M_0 \prec_{\mathcal{K}} M_2$ there exist $\mathcal{K}$-mappings g_1 and g_2 and an amalgam $N \in \mathcal{K}$ such that the diagram below commutes.

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\;g_1\;} & N \\
{\scriptstyle\mathrm{id}}\big\uparrow & & \big\uparrow{\scriptstyle g_2} \\
M_0 & \xrightarrow[\;\mathrm{id}\;]{} & M_2
\end{array}
$$

Under the assumption of the amalgamation property, there is a natural generalization of first order types. However, types are no longer identified by consistent sets of formulas. Since we assume the amalgamation and joint embedding properties, we may work inside a large monster model which

---

[1]$\mathrm{Hanf}(\mathcal{K})$ is bounded above by $\beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$. $\mathrm{Hanf}(\mathcal{K})$ is introduced in Definition 2.14 below.

we denote by $\mathfrak{C}$. We use the notation $\text{Aut}_M(\mathfrak{C})$ to represent the group of automorphisms of $\mathfrak{C}$ which fix M pointwise. With the amalgamation property, we can define the Galois-type of an element a over a model M, written ga-tp(a/M). We say two elements $a, b \in \mathfrak{C}$ realize the same Galois-type over a model M iff there is an automorphism f of $\mathfrak{C}$ such that $f(a) = b$ and $f \upharpoonright M = \text{id}_M$. We abbreviate the set of all Galois-types over a model M by ga-S(M). An AEC is *Galois-stable* in μ if for every model M of $\mathcal{K}$ of cardinality μ, there are only μ many Galois-types over M. See [Gr1] or [Ba1] for a survey of the development of these concepts.

In the first author's Ph.D. thesis and [GrVa2], we isolated the notion of tameness in order to develop a stability theory for a wide spectrum of non-elementary classes. An abstract elementary class satisfying the amalgamation property is said to be χ-*tame* if for every model M in $\mathcal{K}$ of cardinality $\geq \chi$ and every $p \neq q \in$ ga-S(M), there is a submodel N of M of cardinality χ such that $p \upharpoonright N \neq q \upharpoonright N$. A class $\mathcal{K}$ is said to be *tame* if it is χ-tame for some χ. In other words, tameness captures the local character of consistency.

All families of AECs that are known to have a structural theory satisfy the amalgamation property and are tame: [2].

(1) Elementary classes.
(2) Homogeneous model theory (as Galois-types are sets of formulas).
(3) The class of atomic models of a first-order theory (from [Sh 87a]). I.e. the class introduced to study the spectrum function of $L_{\omega_1,\omega}$ sentence (under mild assumptions) is an example of a tame AEC.
(4) Let $\mathcal{K}$ be an AEC, and suppose that $\mathcal{K}$ has an axiomatization in $L_{\chi^+,\omega}$ for some cardinal $\chi \geq \aleph_0$ and there exists $\kappa > \chi$ *strongly compact cardinal* such that $\text{LS}(\mathcal{K}) < \kappa$. Let $\mu_0 := \beth_{(2^\kappa)^+}$. Makkai and Shelah prove that if $\mathcal{K}$ is categorical in some $\lambda^+ > \mu_0$ then has the AP. By further results of [MaSh] the Galois-types can be identified with sets of formulas from $L_{\kappa,\kappa}$. Thus $\mathcal{K}$ is κ-tame.
(5) The class of algebraically closed fields with pseudo-exponentiation studied by Zilber is tame.
(6) Using the method of [GrKv] Villaveces and Zambrano in [ViZa] have shown that the class of Hrushovski's fusion $\mathcal{K}_{fus}$ is $\aleph_0$-tame.
(7) It is a corollary of [GrKv] that good frames that are excellent (in the sense of [Sh 705]) are tame.

Two popular classes of AECs are homogeneous classes and excellent classes. Tameness generalizes each of these classes. There are several examples of

---

[2] While there are structural results for continuous model theory, this context is not an AEC. The classification theory for continuous model theory has much in common with model theory of homogeneous models. The types in continuous theory satisfy the tameness requirement.

tame classes that fail to be homogeneous; for example, Zilber's pseudo-exponentiation, [Zi3]. And there are many tame classes which are not excellent. In fact, there are $\aleph_0$-stable first-order theories whose classes of models are not excellent.

As further evidence to the importance of tame AECs, recent progress on Shelah's Categoricity Conjecture has been made under the assumption of tameness by combining the work of [Sh 394] with [GrVa1].

Fact 1.2. *Suppose $\mathcal{K}$ is a χ-tame abstract elementary class satisfying the amalgamation and joint embedding properties. Let $\mu_0 := \mathrm{Hanf}(\mathcal{K})$. If $\chi \leq \beth_{(2^{\mu_0})^+}$ and $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^{\mu_0})^+}$, then $\mathcal{K}$ is categorical in μ for all $\mu > \beth_{(2^{\mu_0})^+}$.*

Previous results (e.g. [Sh 87a], [Sh 87b], [MaSh], [KoSh], [Sh 472] and [Sh 705]) of Shelah in the direction of upward categoricity required not only model-theoretic assumptions but also set-theoretic assumptions. An interesting feature of our work is that it is an upward categoricity transfer theorem in ZFC. In particular it can be viewed as an improvement of the main result of [MaSh] where the assumption of existence of a strongly compact cardinal is made.

One distinction between Fact 1.2 and Conjecture 1.1 is that Fact 1.2 applies only to classes which are categorical above the second Hanf number, $\beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$. One motivation for this paper is to improve Fact 1.2 getting a better approximation to Conjecture 1.1 for tame abstract elementary classes. In fact our results extend beyond the scope of Conjecture 1.1 since we are able, for instance, to conclude that for a $\mathrm{LS}(\mathcal{K})$-tame abstract elementary class with arbitrarily large models satisfying the amalgamation and joint embedding properties if the class is categorical in $\mathrm{LS}(\mathcal{K})$ and $\mathrm{LS}(K)^+$ then the class is categorical in all $\mu \geq \mathrm{LS}(\mathcal{K})$.

The main theorem of [Sh 394] is that from categoricity in λ^+ above the second Hanf number, one could deduce categoricity below λ^+. Here is the exact statement of Shelah's theorem.

Fact 1.3 (Shelah's downward categoricity). *Let $\mathcal{K}$ be an abstract elementary class satisfying the amalgamation and joint embedding properties. If $\mathcal{K}$ is categorical in some $\lambda^+ \geq \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$ then $\mathcal{K}$ is categorical in every μ satisfying $\mathrm{Hanf}(\mathcal{K}) \leq \mu \leq \lambda$.*

Under the additional assumption of tameness, we provide an argument to transfer categoricity in λ^+ upwards in [GrVa1]. The main step in our proof is:

Fact 1.4 (Corollary 4.3 of [GrVa1]). *Suppose that $\mathcal{K}$ has arbitrarily large models, satisfies the amalgamation property and is χ-tame with $\chi \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in both λ^+ and λ with $\lambda \geq \chi$ and $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in every μ with $\mu \geq \lambda$.*

A breakthrough in [GrVa1] was to go from categoricity in λ^+ to categoricity in λ^{++} when λ^+ was above the second Hanf number of the class.

Working under the assumption of categoricity above the second Hanf number provided us the convenience of categoricity in λ with an application of [Sh 394].

Recently, Lessmann expressed interest in whether or not the upward categoricity transfer theorem (Fact 1.4) could be proved from categoricity in only one successor cardinal. He communicated to us that he could use our methods along with quasi-minimal types and countable superlimits to prove the desired result for $\aleph_0$-tame classes with $\mathrm{LS}(\mathcal{K}) = \aleph_0$ [Le], but was unable to prove it when $\mathrm{LS}(\mathcal{K})$ is uncountable. This paper answers Lessmann's question. Using the ideas and arguments from [GrVa1] along with quasi-minimal types, we deduce from categoricity in λ^+ categoricity in λ^{++} for $\lambda > \mathrm{LS}(\mathcal{K})$ with no restrictions on the size of $\mathrm{LS}(\mathcal{K})$ or the tameness cardinal. We also improve Fact 1.4 by removing the assumption that $\lambda > \mathrm{LS}(\mathcal{K})$.

Our proof that categoricity in λ^+ implies categoricity in λ^{++} under the described setting involves showing that there are nice minimal types (which we have called deep-rooted quasi-minimal) over limit models, and these quasi-minimal types have no Vaughtian pairs of cardinality λ^{++}. Then using a characterization of limit models (Theorem 4.1 from [GrVa1]), we show that this is enough to prove the model of cardinality λ^{++} is saturated.

In [HtSh] Hart and Shelah presented a very interesting example: For every $n < \omega$ there exists an $L_{\omega_1,\omega}$-sentence ψ_n (in a countable language) which is categorical in $\aleph_k$ for all $k \leq n$ but stop being categorical in a higher cardinal. This is an evidence that categoricity can't start under $\aleph_\omega$ recently Baldwin and Kolesnikov informed us that by analyzing the Hart-Shelah example they managed to establish that it has the disjoint amalgamation property (see [BaK]). This together with Fact 1.4 gives that $\mathrm{Mod}(\psi_n)$ is not $\aleph_{n-2}$-tame (Baldwin and Kolesnikov can show this by a direct computation). Thus it is reasonable to expect that a stronger conjecture than Shelah's categoricity conjecture holds and perhaps it will be easier to prove it:

Conjecture 1.5. *Suppose $\mathcal{K}$ is an AEC. If $\mathcal{K}$ is categorical in some $\lambda \geq \mathrm{Hanf}(\mathcal{K})$ then there exists $\chi < \mathrm{Hanf}(\mathcal{K})$ such that $\mathcal{K}$ is χ-tame.*

We are grateful to John Baldwin and Olivier Lessmann for asking questions without which this paper would not exist. We thank also the referee for insisting on adding certain proofs, definitions and suggesting improvements of presentation.
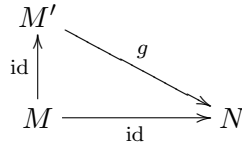
2. Preliminaries

Throughout this paper, we make the assumptions that our abstract elementary class $\mathcal{K}$ has arbitrarily large models and satisfies the joint embedding and amalgamation properties. We will also assume that the class is χ-tame. We let $\mathcal{K}_\mu$ stand for the set of all models of $\mathcal{K}$ of cardinality μ. In the natural way, we use $\mathcal{K}_{\leq \mu}$ and $\mathcal{K}_{\geq \mu}$. We will be using notation and definitions consistent with [GrVa1]. Many of the propositions can be proved

in more general settings, but we leave an exploration of those possibilities for future work.

In abstract elementary classes saturated models have various guises. In some cases, it is more prudent to work with a limit model as opposed to a saturated model.

Definition 2.1. (1) We say N is *universal over M* iff for every $M' \in \mathcal{K}_{\|M\|}$ with $M \prec_{\mathcal{K}} M'$ there exists a $\mathcal{K}$-embedding $g : M' \to N$ such that $g \restriction M = \mathrm{id}_M$:

$$
\begin{array}{ccc}
M' & & \\
\mathrm{id} \uparrow & \searrow^{g} & \\
M & \xrightarrow[\mathrm{id}]{} & N
\end{array}
$$

(2) For $M \in \mathcal{K}_\mu$, σ a limit ordinal with $\sigma \leq \mu$ and $M' \in \mathcal{K}_\mu$ we say that M' is a *(μ, σ)-limit over M* iff there exists a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models $\langle M_i \in \mathcal{K}_\mu \mid i < \sigma \rangle$ such that
   (a) $M = M_0$,
   (b) $M' = \bigcup_{i < \sigma} M_i$ and
   (c) M_{i+1} is universal over M_i.

Using a back-and-forth argument, one can show that any two (μ, σ)-limit models are isomorphic. On the other hand, to show that all (μ, σ_1)-limit models are isomorphic to a (μ, σ_2)-limit model requires more. We will be using the following fact which is a consequence of [Va]; or see [GrVaVi] for an exposition and proof.

Fact 2.2 (Uniqueness of Limit Models). *Suppose that $\mathcal{K}$ is an abstract elementary class satisfying the amalgamation property and is categorical in some λ. If $\mathrm{LS}(\mathcal{K}) < \mu < \lambda$ and $M_0 \in \mathcal{K}_\mu$, then for every two limit models M_1 and M_2 over M_0, if M_1 and M_2 both have the same cardinality $\kappa < \lambda$, then they are isomorphic over M_0.*

Notation 2.3. In light of Fact 2.2, when the cardinality of the limit model is clear, we omit the parameters μ and σ and refer to (μ, σ)-limit models as *limit models*.

A corollary of Fact 2.2 is that

Proposition 2.4. *Assuming categoricity in λ and the joint embedding and amalgamation properties, for μ with $\mathrm{LS}(\mathcal{K}) < \mu < \lambda$, every saturated model of cardinality μ is also a (μ, σ)-limit model for any limit ordinal $\sigma < \mu^+$.*

Proof. First we show that any limit model of cardinality μ is saturated. Then by our assumptions and the uniqueness of saturated models (Lemma 0.26 of [Sh 576]), we can conclude that any saturated model of cardinality μ is isomorphic to a limit model of cardinality μ.

Suppose that M is a limit model of cardinality μ. Fix κ such that $\mathrm{LS}(\mathcal{K}) \leq \kappa < \mu$. Fix $N_1 \prec_{\mathcal{K}} M$ of cardinality κ and $p = \mathrm{ga\text{-}tp}(a/N_1, N_2)$ with $\|N_2\| = \kappa$. Since M is a limit model, we can find a continuous decomposition of M into $\langle M_i \mid i < \kappa^+ \rangle$ such that each M_i is a model of cardinality μ and M_{i+1} is universal over M_i. By the regularity of κ^+, we can find $i < \kappa^+$ such that $N_1 \prec_{\mathcal{K}} M_i$. Invoking the amalgamation property, we can amalgamate N_2 and M_i over N_1 as in the diagram below:

$$
\begin{array}{ccc}
N_2 & \xrightarrow{\ g\ } & N^* \\[2pt]
{\scriptstyle \mathrm{id}}\big\uparrow & & {\scriptstyle \mathrm{id}}\big\uparrow \\[2pt]
N_1 & \xrightarrow[\ \mathrm{id}\ ]{} & M_i
\end{array}
$$

We may assume that the amalgam N^* has cardinality μ. Since M_{i+1} is universal over M_i we can extend the commutative diagram:

$$
\begin{array}{ccc}
N_2 & \xrightarrow{\ g\ } & N^* \\[2pt]
{\scriptstyle \mathrm{id}}\big\uparrow & & {\scriptstyle \mathrm{id}}\big\uparrow \ \searrow^{f} \\[2pt]
N_1 & \xrightarrow[\ \mathrm{id}\ ]{} M_i & \xrightarrow[\ \mathrm{id}\ ]{} M_{i+1}
\end{array}
$$

Notice that $f \circ g$ witnesses that $f(g(a)) \in M_{i+1}$ realizes $\mathrm{ga\text{-}tp}(a/N_1, N_2)$.

$\dashv$

Galois-stability and the amalgamation property are enough to establish the existence of limit models (see [Sh 600] for the statement and [GrVa2] for a proof). Limit models exist in categorical AECs since categoricity implies Galois-stability:

Fact 2.5 (Claim 1.7(a) of [Sh 394] or see [Ba1] for a proof). *If $\mathcal{K}$ is categorical in $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is Galois-stable in all μ with $\mathrm{LS}(\mathcal{K}) \leq \mu < \lambda$.*

Another consequence of μ-stability is the existence of minimal types. As a replacement for first order strongly minimal types, Shelah has suggested using minimal types in [Sh 394]. We in [GrVa1] found that a more restrictive minimality condition (rooted minimal) could be used to transfer categoricity upward.

Definition 2.6. Let $M \in \mathcal{K}$ and $p \in \mathrm{ga\text{-}S}(M)$ be given.

(1) p is said to be *minimal* if it is both non-algebraic (that is, it is not realized in M) and for any $N \in \mathcal{K}$ extending M there is at most one non-algebraic extension of p to N.

(2) A minimal type p is said to be *rooted minimal* iff there is some $M_0 \prec_{\mathcal{K}} M$ with $M_0 \in \mathcal{K}_{<\|M\|}$ such that $p \upharpoonright M_0$ is also minimal. M_0 is called a *root* of p.

Fact 2.7 (Density of Minimal Types [Sh 394]). *Let $\mu > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is Galois-stable in μ, then for every $N \in \mathcal{K}_\mu$ and every $q \in \mathrm{ga\text{-}S}(N)$, there are $M \in \mathcal{K}_\mu$ and $p \in \mathrm{ga\text{-}S}(M)$ such that $N \preceq_{\mathcal{K}} M$, $q \leq p$ and p is minimal.*

The main obstacle of minimal types in this context is that while there are minimal types in stable AECs, the minimal types may be trivially minimal, meaning that the minimal type has *no* non-algebraic extensions. As in [Sh 48] and [Zi] we replace this notion of minimality with quasi-minimality.

Since a non-algebraic type may not have any non-algebraic extension, we distinguish these non-algebraic types from the well-behaved non-algebraic types. A non-algebraic type $p \in \mathrm{ga\text{-}S}(M)$ is *big* iff for every $M' \succ_{\mathcal{K}} M$ of cardinality $\|M\|$, there is a non-algebraic extensions of p to M' (see Definition 6.1 of [Sh 48]). Notice that this is equivalent to requiring that there is a big extension of p to M'.

Almost thirty years after Shelah's [Sh 48], Zilber rediscovered the notion of minimality and used perhaps the better notation quasi-minimality to distinguish it from the first order relatives. As in [Zi], we say a big type p is *quasi-minimal* iff for any $N \in \mathcal{K}$ extending M there is at most one non-algebraic extension of p to N. Analogous to the minimal case, we can define *deep-rooted quasi-minimal*. Most of the results concerning minimal types can be proved for quasi-minimal types with minimal work.

We will show that quasi-minimal types exist in Section 3. For now notice that the assumptions of the amalgamation property and no maximal models give us the following:

Remark 2.8. For any $M \in \mathcal{K}$, there exists $p \in \mathrm{ga\text{-}S}(M)$ such that p is big.

Another consequence of μ-stability is that μ-splitting is well-behaved and the notions of non-algebraic and big types over limit models are the same. We begin by reviewing some basic facts about μ-splitting.

For $M \in \mathcal{K}_{\geq\mu}$ and $N \prec_{\mathcal{K}} M$ we say that $p \in \mathrm{ga\text{-}S}(M)$ *μ-splits over N* iff there exist two models $N_1, N_2 \in \mathcal{K}_\mu$ and an isomorphism $f : N_1 \cong N_2$ such that $N \prec_{\mathcal{K}} N_l \prec_{\mathcal{K}} M$ for $l = 1, 2$; $f \restriction N = \mathrm{id}_N$ and $p \restriction N_2 \neq f(p \restriction N_1)$. Under the assumption of categoricity, non-μ-splitting has an extension property (see Lemma 6.3 of [Sh 394]) in addition to the existence property which follows from Galois-stability in μ (see Claim 3.3 of [Sh 394]):

Fact 2.9. *Suppose that $\mathcal{K}$ is categorical in some $\lambda > \mathrm{LS}(\mathcal{K})$. Let μ be a cardinal such that $\mathrm{LS}(\mathcal{K}) \leq \mu$ and let σ be a limit ordinal with $\mathrm{LS}(\mathcal{K}) \leq \sigma < \mu^+$. Then, for every (μ, σ)-limit model M and every type $p \in \mathrm{ga\text{-}S}(M)$, there exists $N \preceq_{\mathcal{K}} M$ of cardinality μ such that for every $M' \in \mathcal{K}_{\leq\lambda}$ extending M, there exists $q \in \mathrm{ga\text{-}S}(M')$ an extension of p such that q does not μ-split over N. In particular p does not μ-split over N.*

Moreover, if M is a (μ, σ)-limit model witnessed by $\langle M_i \mid i < \sigma \rangle$, then there is a $i < \sigma$ such that p does not μ-split over M_i.

The only other property of μ-splitting that we will explicitly use is an observation that non-splitting extensions of non-algebraic types remain non-algebraic.

Fact 2.10 (Corollary 2.8 of [GrVa1])**.** *Let $N, M, M' \in \mathcal{K}_\mu$ be such that M' is universal over M and M is a limit model over N. Suppose that $p \in \mathrm{ga\text{-}S}(M)$ does not μ-split over N and p is non-algebraic. For every $M' \in \mathcal{K}$ extending M of cardinality μ, if $q \in \mathrm{ga\text{-}S}(M')$ is an extension of p and does not μ-split over N, then q is non-algebraic.*

We can use non-splitting to show that

Fact 2.11. *Suppose that $\mathcal{K}$ is categorical in some $\lambda > \mathrm{LS}(\mathcal{K})$ and μ is a cardinal $< \lambda$. If M is a limit model of cardinality μ, then $p \in \mathrm{ga\text{-}S}(M)$ is non-algebraic iff p is big.*

Proof. As in the proof of Theorem I.4.10 [Va] or see Proposition 1.16[Le]. At the referee's suggestion, we have included a proof here.

Clearly every big type is non-algebraic. Suppose M is a limit model witnessed by $\langle M_i \mid i < \sigma \rangle$ and $p = \mathrm{ga\text{-}tp}(a/M)$ is non-algebraic. By Fact 2.9, there is an $i < \sigma$ such that p does not μ-split over M_i.

Let M' be a $\mathcal{K}$-extension of M of cardinality μ. We now show that p can be extended to a non-algebraic type $p' \in \mathrm{ga\text{-}S}(M')$. By the definition of limit model and our choice of $\langle M_i \mid i < \sigma \rangle$, we know that M_{i+1} is universal over M_i. Thus there is a $\mathcal{K}$-mapping $h' : M' \to M_{i+1}$ such that $h \upharpoonright M_i = \mathrm{id}_{M_i}$. Since of we are working inside a monster model, we can extend h' to $h \in \mathrm{Aut}_{M_i}(\mathfrak{C})$. Our candidate for a non-algebraic extension of p to M' will be $p' := \mathrm{ga\text{-}tp}(h^{-1}(a)/M')$. Immediately we see that p' is non-algebraic since $\mathrm{ga\text{-}tp}(a/h(M'))$ was non-algebraic.

We claim that p' is in fact an extension of p, that is that $\mathrm{ga\text{-}tp}(h^{-1}(a)/M) = \mathrm{ga\text{-}tp}(a/M)$. By monotonicity, of non-splitting, we have that $\mathrm{ga\text{-}tp}(a/h(M')$ does not μ-split over M_i. By invariance, we have $\mathrm{ga\text{-}tp}(h^{-1}(a)/M')$ also does not μ-split over M_i. Now if $\mathrm{ga\text{-}tp}(h^{-1}(a)/M) \neq \mathrm{ga\text{-}tp}(a/M)$, we would witness that $\mathrm{ga\text{-}tp}(h^{-1}(a)/M')$ μ-splits over M_i via the mapping h. Thus $p' \upharpoonright M = p$ as required.

$\dashv$

We now go into some details of a common construction in AECs. A variation of the proposition appears in the literature as Claim 0.31(2) of [Sh 576] and in the proof of Theorem II.7.1 of [Va]. We isolate it here as Lemma 2.12. After detailing Lemma 2.12 to John Baldwin in e-mail correspondence in the Summer of 2004, we decided to include the proof here.

Lemma 2.12. *Suppose $\langle M_i \mid i < \alpha \rangle$ is an $\prec_\mathcal{K}$-increasing and continuous chain of models. Further assume that $\langle p_i \in \mathrm{ga\text{-}S}(M_i) \mid i < \alpha \rangle$ is an increasing chain of types such that there are $a_i \in \mathfrak{C}$ with $a_i \models p_i$ and $\prec_\mathcal{K}$-mappings $f_{i,j} \in \mathrm{Aut}_{M_i}(\mathfrak{C})$ with $f_{i,j}(a_i) = a_j$ for $i \leq j < \alpha$ such that for $i \leq j \leq k$ we*

have that $f_{i,k} = f_{j,k} \circ f_{i,j}$. *Then there exists $a_\alpha \in \mathfrak{C}$ realizing each p_i and there are $f_{i,\alpha} \in \mathrm{Aut}_{M_i}(\mathfrak{C})$ with $f_{i,\alpha}(a_i) = a_\alpha$.*

The proof uses direct limits which behave nicely in AECs, by way of Shelah's Presentation Theorem. One of early surprising theorems of Shelah in his first paper on AECs, [Sh 88] is a result we call *Shelah's Presentation Theorem.* In order to state it we need some notation: Let $L \subseteq L_1$ be similarity types, T_1 a first-order theory in L and Γ a set of T_1-types (over the empty set).

$$\mathrm{PC}(T_1, \Gamma, L) := \{M \upharpoonright L \ : \ M \models T_1, \text{ and } M \text{ omits all the types from } \Gamma\}.$$

A class of structures $\mathcal{K}$ is called $\mathrm{PC}_{\mu,\kappa}$ iff there are L, T_1 and Γ such that $\mathcal{K} = \mathrm{PC}(T_1, \Gamma, L)$ and $|T_1| \leq \mu$ and $|\Gamma| \leq \kappa$.

Fact 2.13 (Shelah's presentation Theorem). *Let $\mathcal{K}$ be a class of structures all of the same similarity type. If $\mathcal{K}$ is an AEC the it is a $\mathrm{PC}_{\lambda,\kappa}$-class for some $\lambda \leq \mathrm{LS}(\mathcal{K})$ and $\kappa \leq 2^{\mathrm{LS}(\mathcal{K})}$*

Shelah's theorem is a remarkable generalization of an easier result due to C. C. Chang from [Ch]: In the special case where ψ is a an $L_{\lambda^+,\omega}$-sentence (in a language of cardinality $\leq \lambda$) and L^* is fragment of cardinality $\leq \lambda$ then the AEC $\langle \mathrm{Mod}(\psi), \prec_{L^*} \rangle$ is (by Chang's theorem) a $\mathrm{PC}_{\lambda,\lambda}$-class.

Given cardinal numbers λ, κ let $\mu(\lambda, \kappa)$ be the Hanf number of of all $\mathrm{PC}_{\lambda,\kappa}$-classes, recall that this mean that: $\mu(\lambda, \kappa)$ is the smallest cardinal such that for every $\mathrm{PC}_{\lambda,\kappa}$-class $\mathcal{K}$ if $\mathcal{K}$ has a model of cardinal $\mu(\lambda, \kappa)$ then it has a model of any cardinal $\geq \lambda$. Several people contributed to this topic among them: Morley, Barwise and Kunen and Shelah. We recommend to the interested reader to read of summary of relevant facts in section 5 of Chapter VII of [Sh a]. Recall that one of the standard facts is that when $\lambda \leq \kappa$ then $\mu(\lambda, \kappa) \leq \beth_{(2^\kappa)^+}$.

Definition 2.14 (Hanf($\mathcal{K}$)). Let $\mathcal{K}$ be an AEC, let $\lambda \leq \kappa$ be the smallest pair such that $\mathcal{K}$ is a $\mathrm{PC}_{\lambda,\kappa}$-class.

$$\mathrm{Hanf}(\mathcal{K}) := \mu(\lambda, \kappa).$$

Using the axioms of AEC and Shelah's Presentation Theorem, one can show that the union axiom of the definition of AEC has an alternative formulation (see [Sh 88] or Chapter 16 of [Gr2]) with directed sets.

Definition 2.15. A partially ordered set $(I, \leq)$ is *directed* iff for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

Fact 2.16 (P.M. Cohn 1965). *Let $(I, \leq)$ be a directed set. If $\langle M_t \mid t \in I \rangle$ and $\{h_{t,r} \mid t \leq r \in I\}$ are such that*

   (1) *for $t \in I$, $M_t \in \mathcal{K}$*
   (2) *for $t \leq r \in I$, $h_{t,r} : M_t \to M_r$ is a $\prec_{\mathcal{K}}$-embedding and*
   (3) *for $t_1 \leq t_2 \leq t_3 \in I$, $h_{t_1,t_3} = h_{t_2,t_3} \circ h_{t_1,t_2}$ and $h_{t,t} = \mathrm{id}_{M_t}$,*

then, whenever $s = \lim_{t \in I} t$, *there exist* $M_s \in \mathcal{K}$ *and* $\prec_{\mathcal{K}}$-*mappings* $\{h_{t,s} \mid t \in I\}$ *such that*

$$h_{t,s} : M_t \to M_s, M_s = \bigcup_{t < s} h_{t,s}(M_t) \text{ and}$$

$$\text{for } t_1 \leq t_2 \leq s, h_{t_1,s} = h_{t_2,s} \circ h_{t_1,t_2} \text{ and } h_{s,s} = id_{M_s}.$$

Remark 2.17. Cohn's proof gives us that M_s is an L($\mathcal{K}$)-structure. To show that $M_s \in \mathcal{K}$ and that $h_{t,s}$ are $\mathcal{K}$-embeddings we use Shelah's presentation theorem.

In light of Fact 2.16 we make the following definition:

Definition 2.18.   (1) $(\langle M_t \mid t \in I \rangle, \{h_{t,s} \mid t \leq s \in I\})$ from Fact 2.16 is called a *directed system*.
   (2) We say that M_s together with $\langle h_{t,s} \mid t \leq s \rangle$ satisfying the conclusion of Fact 2.16 is *a direct limit of* $(\langle M_t \mid t < s \rangle, \{h_{t,r} \mid t \leq r < s\})$.

We now return to the proof of Lemma 2.12.

Proof of Lemma 2.12. Let $\langle p_i \in \text{ga-S}(M_i) \mid i < \alpha \rangle$ be an increasing chain of types and $\langle M_i \mid i < \alpha \rangle$ a $\prec_{\mathcal{K}}$-increasing chain of models with $\langle f_{i,j} \mid i \leq j < \alpha \rangle$ and a_i as in the statement of the lemma. Notice that $((\langle \mathfrak{C}_i \mid i < \alpha \rangle, \langle f_{i,j} \mid i \leq j < \alpha \rangle))$ forms a directed system where $\mathfrak{C}_i = \mathfrak{C}$ for all i. Let $\mathfrak{C}_\alpha^*$ and $\langle f_{i,\alpha}^* \mid i \leq \alpha \rangle$ be a direct limit to this system. Outright we don't have much control over this limit, but by the following claims we will be able to chose a limit $(\mathfrak{C}_\alpha, \langle f_{i,\alpha} \mid i \leq \alpha \rangle)$ so that $\bigcup_{i < \alpha} M_i \preceq_{\mathcal{K}} \mathfrak{C}_\alpha = \mathfrak{C}$ and $f_{i,\alpha} \upharpoonright M_i = id_{M_i}$.

First notice that we can take $\mathfrak{C}_\alpha$ to be $\mathfrak{C}$ since a direct limit of automorphisms is an isomorphism using the construction of direct limits from [Gra].

Claim 2.19. $\langle f_{i,\alpha}^* \upharpoonright M_i \mid i \leq \alpha \rangle$ *is increasing.*

Proof of Claim 2.19. Let $i < j < \alpha$ be given. By construction

$$f_{i,j} \upharpoonright M_i = id_{M_i}.$$

An application of $f_{j,\alpha}^*$ yields

$$f_{j,\alpha}^* \circ f_{i,j} \upharpoonright M_i = f_{j,\alpha}^* \upharpoonright M_i.$$

Since $f_{i,\alpha}^*$ and $f_{j,\alpha}^*$ come from a direct limit of the system which includes the mapping $f_{i,j}$, we have

$$f_{i,\alpha}^* \upharpoonright M_i = f_{j,\alpha}^* \circ f_{i,j} \upharpoonright M_i.$$

Combining the equalities yields

$$f_{i,\alpha}^* \upharpoonright M_i = f_{j,\alpha}^* \upharpoonright M_i.$$

This completes the proof of Claim 2.19.

$\dashv$

By the claim, we have that $f := \bigcup_{i<\alpha} f_{i,\alpha}^* \restriction M_i$ is a $\prec_{\mathcal{K}}$-mapping from $\bigcup_{i<\alpha} M_i$ onto $\bigcup_{i<\alpha} f_{i,\alpha}^*(M_i)$. Since $\mathfrak{C}$ is saturated and model homogeneous, we can extend f to $F \in \mathrm{Aut}(\mathfrak{C})$.

Now consider the direct limit defined by $\mathfrak{C}_\alpha := F^{-1}(\mathfrak{C}_\alpha^*)$ with $\langle f_{i,\alpha} := F^{-1} \circ f_{i,\alpha}^* \mid i < \alpha \rangle$ and $f_{\alpha,\alpha} = \mathrm{id}_{\mathfrak{C}_\alpha}$. Notice that $f_{i,\alpha} \restriction M_i = F^{-1} \circ f_{i,\alpha}^* \restriction M_i = \mathrm{id}_{M_i}$ for $i < \alpha$. Thus $\bigcup_{i<\alpha} M_i \preceq_{\mathcal{K}} \mathfrak{C}_\alpha$.

Let $a_\alpha := f_{0,\alpha}(a_0)$. The following argument explains why ga-tp$(a_\alpha / \bigcup_{i<\alpha} M_i)$ is an upper bound for $\langle p_i \mid i < \alpha \rangle$.

Claim 2.20. ga-tp$(a_\alpha / M_i) =$ ga-tp(a_i / M_i) *for all* $i < \alpha$.

Proof of Claim 2.20. Fix $i < \alpha$. Notice that by the definition of direct limit we have $a_\alpha = f_{0,\alpha}(a_0) = f_{i,\alpha} \circ f_{0,i}(a_0)$. But by our choice of $f_{0,i}$ we know that $f_{0,i}(a_0)$ is actually a_i. Thus $f_{i,\alpha}$ is an automorphism of $\mathfrak{C}$ fixing M_i taking a_i to a_α. So ga-tp(a_i / M_i) and ga-tp(a_α / M_i) must be the same.

$\dashv$

$\dashv$

Lemma 2.12 gives us the ability to consider unions of certain chains of types, when in general in AECs the union of a chain of types need not exist or be unique.

Lemma 2.21. *Suppose* $\langle M_i \mid i < \alpha \rangle$ *is an* $\prec_{\mathcal{K}}$*-increasing and continuous chain of limit models. If* $\langle p_i \in$ ga-S$(M_i) \mid i < \alpha \rangle$ *is an increasing chain of quasi-minimal types and* α *is a limit ordinal, then we can find* $a_i \in \mathfrak{C}$ *with* $a_i \models p_i$ *and* $\prec_{\mathcal{K}}$*-mappings* $f_{i,j} \in \mathrm{Aut}_{M_i}(\mathfrak{C})$ *with* $f_{i,j}(a_i) = a_j$ *for* $i \le j < \alpha$ *such that for* $i \le j \le k$ *we have that* $f_{i,k} = f_{j,k} \circ f_{i,j}$.

Proof. We find a_i and $f_{k,i}$ by induction on i. For $i = 0$, take $a_0 \in \mathfrak{C}$ to be some realization of p_0 and $f_{0,0} := \mathrm{id}_{\mathfrak{C}}$. Suppose that we have defined a_i and $f_{k,i}$ for all $k \le i$. Let a_{i+1} be some realization of p_{i+1} in $\mathfrak{C}$. Since the types are increasing, we can find $f_{i,i+1} \in \mathrm{Aut}_{M_i} \mathfrak{C}$ with $f_{i,i+1}(a_i) = a_{i+1}$. Define $f_{k,i+1} := f_{i,i+1} \circ f_{i,i+1}$. We use quasi-minimal types to get past limit stage. Suppose that we have defined $f_{j,k}$ for all $j \le k < i$ with i a limit ordinal. By Lemma 2.12 there exists $a^* \in \mathfrak{C}$ and $f_{j,i} \in \mathrm{Aut}_{M_j} \mathfrak{C}$ with $f_{j,i} \restriction M_j = id_{M_j}$ and $f_{j,i}^*(a_j) = a^*$. This a^* comes from a direct limit construction and may not realize the same type as a_i over M_i. However, ga-tp(a^*/M_i) is a non-algebraic extension of ga-tp(a_0/M_0), which was quasi-minimal. Since M_i is also a limit model, then ga-tp(a^*/M_i) is big. So, we can actually conclude, by quasi-minimality that the types of a^* and a_i over M_i agree. So we can fix $g \in \mathrm{Aut}_{M_i}(\mathfrak{C})$ such that $g(a^*) = a_i$. Then $f_{j,i} := g \circ f_{j,i}^*$ is as required. $\dashv$

Corollary 2.22. *Suppose* $\langle M_i \mid i < \alpha \rangle$ *is an* $\prec_{\mathcal{K}}$*-increasing and continuous chain of limit models. If* $\langle p_i \in$ ga-S$(M_i) \mid i < \alpha \rangle$ *is an increasing chain of quasi-minimal types and* α *is a limit ordinal, then there is a* $p_\alpha \in$ ga-S$(\bigcup_{i<\alpha} M_i)$ *extending each of the* p_i.

Proof. Follows from Lemma 2.21 and Lemma 2.12.

$\dashv$

3. Deep-rooted minimal types

The main aim of this section is to prove the existence of deep-rooted quasi-minimal types. We will use the idea of Shelah's density of minimal types to do this. Our work generalizes Lemma 6.6 of [Sh 48] where Shelah proves the existence of (quasi)-minimal types using a rank function.

First notice that if the class is tame, then any big extension of a quasi-minimal type is also quasi-minimal:

Proposition 3.1 (Monotonicity of Minimal Types). *Suppose $\mathcal{K}$ is χ-tame for some χ with $\mu \geq \chi$. If $p \in$ ga-S(M) is quasi-minimal with $M \in \mathcal{K}_\mu$, then for all $N \in \mathcal{K}$ extending M and every $q \in$ ga-S(N) extending p, if q is big then q is quasi-minimal. If N is a limit model, then the assumption that q is big can be replaced with non-algebraic.*

Proof. The last sentence of the claim is Proposition 2.2 of [GrVa1] once we notice Fact 2.11. The proof of the rest of the claim is similar, but we include details here for completeness.

Suppose that p is a quasi-minimal type over M with a big extension q to N. For the sake of contradiction assume that q is not quasi-minimal. Then there exist distinct q_1 and q_2 non-algebraic extensions of q to some model N'. By tameness, there exists M' of cardinality μ such that $M \prec_\mathcal{K} M'$ and $q_1 \restriction M' \neq q_2 \restriction M'$. Since $q_1 \restriction M'$ and $q_2 \restriction M'$ are both non-algebraic extensions of p we have a contradiction to the quasi-minimality of p.   ⊣

Similarly to the proof of the density of minimal types, Fact 2.7, we get quasi-minimal types. Moreover, instead of a density result, we can actually find quasi-minimal types over every limit model. This is one of the obstacles in working with minimal types.

Proposition 3.2 (Existence of Quasi-Minimal Types over Limits). *Suppose $\mathcal{K}$ is Galois-stable in μ and $M \in \mathcal{K}_\mu$ is a limit model. Then there exists a quasi-minimal type over M.*

Proof. We build a tree of types, but restrict ourselves to limit models throughout the construction. Suppose for the sake of contradiction that $M \in \mathcal{K}_\mu$ is a limit model and that there are no quasi-minimal types over M. By Remark 2.8 we can fix $p \in$ ga-S(M) a big type. By induction on $i < \mu^+$ we build a $\prec_\mathcal{K}$-increasing and continuous chain of models, $\langle M_i \mid i < \mu^+ \rangle$ and a tree of types $\langle p_\eta \mid \eta \in {}^{<\mu^+}2 \rangle$ satisfying

(1) M_i is a limit model of cardinality μ
(2) M_{i+1} is a limit model over M_i
(3) $p_\eta =$ ga-tp(a_η / M_i) is big where i is the length of η
(4) $p_{\eta^\frown\langle 0 \rangle} \neq p_{\eta^\frown\langle 1 \rangle}$
(5) for all ordinals $i \leq j$ less than the length of η, we have $p_{\eta \restriction i} \leq p_\eta$, and there exist $f_{\eta \restriction i, \eta} \in \mathrm{Aut}_{M_{\eta \restriction i}} \mathfrak{C}$ such that $f_{\eta \restriction i, \eta}(a_{\eta \restriction i}) = a_\eta$ and $f_{\eta \restriction i, \eta} = f_{\eta \restriction j, \eta} \circ f_{\eta \restriction i, \eta \restriction j}$
(6) $p_{\langle \rangle} = p$

(7) $M_0 = M$.

Suppose that M_i and $p_\eta \in \text{ga-S}(M_i)$ have been defined. Since M_i is isomorphic to M (by Fact 2.2), our assumption implies that p_η cannot be quasi-minimal. So we may fix an extension N of M_i and two distinct big extensions of p_η to N. Let $a'_{\eta^\frown\langle 0 \rangle}$ and $a'_{\eta^\frown\langle 1 \rangle}$ realize these big extensions and let $M'_1 \in \mathcal{K}_\mu$ be some extension of N containing both $a'_{\eta^\frown\langle 0 \rangle}$ and $a'_{\eta^\frown\langle 1 \rangle}$. Fix a (μ, ω)-limit model over N and call it M_{i+1}. By the definition of big types, there are $a_{\eta^\frown\langle 0 \rangle}$ and $a_{\eta^\frown\langle 1 \rangle}$ realizing big extensions of $\text{ga-tp}(a'_{\eta^\frown\langle 0 \rangle}/N)$ and $\text{ga-tp}(a'_{\eta^\frown\langle 1 \rangle}/N)$, respectively.

For the limit stage of the construction notice that $M_i := \bigcup_{j<i} M_j$ is a limit model as guaranteed by condition (2). For $\eta \in {}^i 2$ with i a limit ordinal, we choose p_η to be some (there may be more than one) non-algebraic extension of the $p_{\eta \restriction j}$ for $j < i$. This is possible by our construction of the $f_{\eta \restriction j, \eta}$'s and Lemma 2.12. This lemma also gives us the required $f_{\eta \restriction j, \eta}$'s.

To see that this construction is enough, let i be the first ordinal $< \mu^+$ such that $2^i > \mu$. Then, $\{p_\eta \in \text{ga-S}(M_i) \mid \eta \in {}^i 2\}$ witnesses that $\mathcal{K}$ is not Galois-stable in μ.

$\dashv$

We need an extension property for quasi-minimal types in order to find deep-rooted quasi-minimal types.

Proposition 3.3 (Extension Property for Quasi-Minimal Types). *Let $\mathcal{K}$ be categorical in some $\lambda > \chi$. Let μ be such that $\text{LS}(\mathcal{K}) \leq \mu \leq \lambda$. If $p \in \text{ga-S}(M)$ is quasi-minimal and M is a (μ, σ)-limit model for some limit ordinal satisfying $\text{LS}(\mathcal{K}) \leq \sigma < \mu^+$, then for every $M' \in \mathcal{K}_{\leq \lambda}$ extending M, there is a quasi-minimal $q \in \text{ga-S}(M')$ such that q extends p. Furthermore if p does not μ split over some N, then q can be chosen so that q does not μ-split over N.*

Proof. Without loss of generality M' is a limit model over M. Let $p \in \text{ga-S}(M)$ be quasi-minimal. Since M is (μ, σ)-limit model, using Fact 2.9, we can find a proper submodel $N \prec_{\mathcal{K}} M$ of cardinality μ such that for every $M' \in \mathcal{K}_{\leq \lambda}$ there exists $q \in \text{ga-S}(M')$ extending p such that q does not μ-split over N. Suppose for the sake of contradiction that q is not quasi-minimal. Then tameness and Proposition 3.1 tells us that q must be algebraic. Let $a \in M'$ realize q and $M^a \in \mathcal{K}_\mu$ contain a with $M \prec_{\mathcal{K}} M^a \prec_{\mathcal{K}} M'$. Then $q \restriction M^a$ is also algebraic. However, since $q \restriction M^a$ does not μ-split over N and extends p, by Corollary 2.10 we see that $q \restriction M^a$ is not-algebraic. This gives us a contradiction. $\dashv$

Remark 3.4. Proposition 3.3 holds for minimal types as well. Simply replace quasi-minimal with minimal in the proof. This will be used in the last section of the paper.

Propositions 3.3 and 3.2 are key to get the existence of deep-rooted quasi-minimal types.

Proposition 3.5 (Existence of deep-rooted quasi-minimal types). *Let $\mathcal{K}$ be categorical in some $\lambda > \chi$. Then for every $M' \in \mathcal{K}_\lambda$, there exists a deep-rooted quasi-minimal $q \in \mathrm{ga\text{-}S}(M')$. Furthermore, if $M \prec_\mathcal{K} M'$ is a limit model of cardinality μ with $\chi \leq \mu < \lambda$ and $p \in \mathrm{ga\text{-}S}(M)$ is quasi-minimal, then we can find $q \in \mathrm{ga\text{-}S}(M')$ a deep-rooted quasi-minimal extension of p with root M.*

Proof. Fix μ with $\chi \leq \mu < \lambda$. Notice that by Fact 2.5, $\mathcal{K}$ is Galois-stable in μ. Choose $M \in \mathcal{K}_\mu$ to be some $\mathcal{K}$-substructure of M'. Since $\mathcal{K}$ is stable in μ and categorical in λ, we may take M to be a (μ, σ)-limit model for some limit ordinal σ with $\mathrm{LS}(\mathcal{K}) \leq \sigma < \mu^+$. By Proposition 3.2 and monotonicity of quasi-minimal types, we can choose M such that there is a quasi-minimal type $p \in \mathrm{ga\text{-}S}(M)$. Then by Proposition 3.3, there exists a quasi-minimal $q \in \mathrm{ga\text{-}S}(M')$ extending p. q is rooted with root M_i.

$\dashv$

4. Vaughtian Pairs

We will show that for deep-rooted quasi-minimal types, there are no true Vaughtian-pairs. This is a variation of the result in [GrVa1] that for rooted minimal types there are no Vaughtian-pairs.

Definition 4.1. Let $M \in \mathcal{K}$ be a limit model and $p \in \mathrm{ga\text{-}S}(M)$ non-algebraic. Fix $\mu \geq \|M\|$.

(1) A pair of models (N_0, N_1) is said to be a (p, μ)-*Vaughtian pair* provided that N_0, N_1 both have cardinality μ and $M \preceq_\mathcal{K} N_0 \precneqq_\mathcal{K} N_1$ with no $c \in N_1 \backslash N_0$ realizing p.
(2) A (p, μ)-Vaughtian pair (N_0, N_1) is a *true (p, μ)-Vaughtian pair* iff N_0 and N_1 are both limit models.

The ubiquity of the assumption of categoricity in a successor cardinal in the literature concerning Conjecture 1.1 can be explained by the proof of the following central result. The result uses a classical Vaughtian-construction in the spirit of Morley's work, and it appears in [Sh 394] as Claim $(*)_8$ of Theorem 9.7.

Fact 4.2. *Assume that $\mathcal{K}$ is categorical in some $\lambda^+ > \mathrm{LS}(\mathcal{K})^+$. Then for every limit model $M \in \mathcal{K}_{\leq \lambda}$ and every minimal type $p \in \mathrm{ga\text{-}S}(M)$, there are no true (p, λ)-Vaughtian pairs.*

Using the fact that all saturated models are limit models; that the union of an increasing chain of saturated models is saturated (Claim 6.7 of [Sh 394]) and Fact 2.11, the same argument for Fact 4.2 can be carried out to yield the following proposition.

Proposition 4.3. *Assume that $\mathcal{K}$ is categorical in some $\lambda^+ > \mathrm{LS}(\mathcal{K})^+$. Then for every limit model $M \in \mathcal{K}_{\leq \lambda}$ and every quasi-minimal type $p \in \mathrm{ga\text{-}S}(M)$, there are no true (p, λ)-Vaughtian pairs.*

Notice that the previous argument works only when λ is strictly larger than $\mathrm{LS}(\mathcal{K})$. We will come back to this issue in Section 6 and deal with the special case in which $\mathrm{LS}(\mathcal{K}) = \chi = \lambda$ and $\mathcal{K}$ is categorical in both λ and λ^+.

The following Vaughtian-pair transfer theorem is a relative of Theorem 3.3 of [GrVa1]:

Theorem 4.4. *Suppose that $\mathcal{K}$ is categorical in some $\lambda^+ > \mathrm{LS}(\mathcal{K})$. Let p be a deep-rooted quasi-minimal type over a model M of cardinality λ^+. Fix a root $N \prec_{\mathcal{K}} M$ of cardinality λ, with $p \upharpoonright N$ quasi-minimal. If $\mathcal{K}$ has a (p, λ^+)-Vaughtian pair, then there is a true $(p \upharpoonright N, \lambda)$-Vaughtian pair (N_0, N_1).*

Proof. Suppose that (N^0, N^1) form a (p, λ^+)-Vaughtian pair. By categoricity, we know that N^0 and N^1 are both saturated.

Let C denote the set of all realizations of $p \upharpoonright N$ inside N^1. Fix $a \in N^1 \backslash N^0$. We now construct $\langle N_i^0, N_i^1 \in \mathcal{K}_\lambda \mid i < \lambda^+ \rangle$ satisfying the following:

(1) $N_0^0 = N$
(2) $N_i^\ell \precneqq_{\mathcal{K}} N^\ell$ for $\ell = 0, 1$
(3) the sequences $\langle N_i^0 \mid i < \lambda^+ \rangle$ and $\langle N_i^1 \mid i < \lambda^+ \rangle$ are both $\prec_{\mathcal{K}}$-increasing and continuous
(4) N_{i+1}^ℓ is a limit model over N_i^ℓ for $\ell = 0, 1$
(5) $a \in N_i^1 \backslash N_i^0$ and
(6) $C_i := C \bigcap N_i^1 \subseteq N_{i+1}^0$.

The construction follows from the fact that both N^0 and N^1 are saturated and homogeneous and the following:

Claim 4.5. *If $d \in N^1$ realizes $p \upharpoonright N_0^0$, then $d \in N^0$. Thus $C \subseteq N^0$.*

Proof of Claim 4.5. Suppose that $d \in N^1 \backslash N^0$ realizes $p \upharpoonright N_0^0$. Because N^0 is saturated, ga-tp(d/N^0) is not only non-algebraic, it is a big extension of $p \upharpoonright N_0^0$. Since $p \upharpoonright N_0^0$ is quasi-minimal, we have that ga-tp$(d/M) = p$. Since (N^0, N^1) form a (p, λ^+)-Vaughtian pair, it must be the case that $d \in N^0$, contradicting our choice of d.

$\dashv$

The construction is enough: Define

$$
E := \left\{ \delta < \lambda^+ \;\middle|\; \begin{array}{l} \delta \text{ is a limit ordinal,} \\ \text{for all } i < \delta \text{ and } x \in N_i^1, \\ \text{if there exists } j < \lambda^+ \text{ such that } x \in C_j, \\ \text{then there exists } j < \delta, \text{ such that } x \in C_j \end{array} \right\}.
$$

Notice that E is a club. (We only use the fact that E is non-empty.) Fix $\delta \in E$.

Claim 4.6. *For every $c \in N_\delta^1 \cap C$, we have $c \in N_\delta^0$.*

Proof of Claim 4.6. Since $\langle N_i^1 \mid i < \lambda^+ \rangle$ is continuous, there is $i < \delta$ such that $c \in N_i^1$. Thus by the definition of E, there is a $j < \delta$ with $c \in C_j$. By condition (6) of the construction, we would have put $c \in N_{j+1}^0 \prec_{\mathcal{K}} N_\delta^0$.

$\dashv$

Notice that $N_\delta^1 \neq N_\delta^0$ since $a \in N_\delta^1 \backslash N_\delta^0$. Thus Claim 4.6 allows us the conclude that we have constructed a $(p \restriction N, \lambda)$-Vaughtian pair. We complete the proof by observing that condition (4) of the construction and our choice of a limit ordinal δ imply that both N_δ^0 and N_δ^1 are limit models.

$\dashv$

Remark 4.7. The same proof of Theorem 4.4 works with minimal in place of quasi-minimal. Thus for p a minimal type, notice that for (N_0, N_1) in the conclusion of Theorem 4.4 we have that (N_0, N_1) is a true $(p \restriction N_0, \lambda)$-Vaughtian pair and $p \restriction N_0$ is minimal. Furthermore N_0 is a limit model over N. This extra information will be used in Section 6.

Corollary 4.8. *Let $\lambda > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ^+ and p is a deep-rooted quasi-minimal type over a model of cardinality λ^+, then there are no (p, λ^+)-Vaughtian pairs.*

Proof. Suppose that (N^0, N^1) is a (p, λ^+)-Vaughtian pair and that N is both a root of p (with $p \restriction N$ quasi-minimal) and a limit model of size λ. Then by Theorem 4.4, there is a true $(p \restriction N, \lambda)$-Vaughtian pair. This contradicts Proposition 4.3.

$\dashv$

Corollary 4.9. *Let $\lambda > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ^+, then every deep-rooted quasi-minimal type over a model N of cardinality λ^+ is realized λ^{++} times in every model of cardinality λ^{++} extending N.*

Proof. Suppose $M \in \mathcal{K}_{\lambda^{++}}$ realizes p only $\alpha < \lambda^+$ times.

Let $A := \{a_i \mid i < \alpha\}$ be an enumeration of the realizations of p in M. We can find $N_0 \in \mathcal{K}_{\lambda^+}$ such that $N \bigcup A \subseteq N_0 \prec_\mathcal{K} M$. Since M has cardinality λ^{++}, we can find $N_1 \in \mathcal{K}_{\lambda^+}$ such that $N_0 \subsetneq_\mathcal{K} N_1 \prec_\mathcal{K} M$. Then (N_0, N_1) forms a (p, λ^+)-Vaughtian pair contradicting Corollary 4.8.

$\dashv$

5. The Main Result

Now that we have established the existence of deep-rooted quasi-minimal types with no Vaughtian pairs, we proceed as in [GrVa1] to transfer categoricity upwards using the following result which is a variation of Theorem 4.1 of [GrVa1].

Theorem 5.1. *Let $\lambda \geq \chi$. Suppose $M_0 \in \mathcal{K}_\lambda$ and $r \in \mathrm{ga\text{-}S}(M_0)$ is a quasi-minimal type such that $\mathcal{K}$ has no (r, λ)-Vaughtian pairs.*

Let α be an ordinal $< \lambda^+$ such that $\alpha = \lambda \cdot \alpha$. Suppose $M \in \mathcal{K}_\lambda$ has a resolution $\langle M_i \in \mathcal{K}_\lambda \mid i < \alpha \rangle$ such that for every $i < \alpha$, there is $c_i \in M_{i+1} \backslash M_i$ realizing r. Then M is saturated over M_0. Moreover if $\mathcal{K}$ is Galois-stable in λ, then M is a (λ, α)-limit model over M_0.

Proof. At the referee's request we have included a proof of this result. Let r, M_0, M and $\langle M_i \mid i < \alpha \rangle$ be as in the statement of the theorem. Let $p \in \text{ga-S}(M_0)$ be given. We will show that M realizes p.

First, fix M' an extension of M_0 of cardinality λ realizing p. It is enough to construct an isomorphism between M and some extension of M'. We build such an extension and isomorphism by inductively defining increasing and continuous sequences $\langle M'_i \mid i < \alpha \rangle$ and $\langle h_i \mid i < \alpha \rangle$ so that $h_i : M_i \to M'_i$. During this construction we also fix $\langle a_{\lambda \cdot i + j} \mid j < \lambda \rangle$ an enumeration (possibly repeating) of the realizations of r inside M_i. After stage $\beta = \lambda \cdot i + j$ of the construction, we require that $a_\beta \in h_{\beta+1}(M_{\beta+1})$.

To see that such a construction is possible, let us examine the successor case. The base and limit stages of the construction are routine to carry out. Suppose that we have defined M'_i and h_i and that we have fixed an enumeration $\langle a_{\lambda \cdot k + j} \mid j < \lambda \rangle$ of all realizations of r in M'_k for each $k \leq i$. By properties of ordinal arithmetic, there is exactly one pair j, k with $k \leq i$ for which $i + 1 = \lambda \cdot k + 1$. If a_i is already in $h_i(M_i)$ there is nothing to do but extend h_i to include M_{i+1} in its domain and choose an appropriate M'_{i+1} containing $h_{i+1}(M_{i+1})$. When $a_i \notin h_i(M_i)$, more care is needed. The important thing to notice here is that in this case, $\text{ga-tp}(a_i/h_i(M_i))$ is a non-algebraic extension of r. By the quasi-minimality of r, we know that regardless of which extension $\check{h}$ of h_i to an automorphism of $\mathfrak{C}$ that one would consider, we have $\text{ga-tp}(\check{h}(c_i)/h_i(M_i)) = \text{ga-tp}(a_i/h_i(M_i))$. Thus we can choose $\check{h}$ to be an automorphism of $\mathfrak{C}$ extending h_i so that $\check{h}(c_i) = a_i$. Now define $h_{i+1} := \check{h} \restriction M_{i+1}$ and choose an appropriate extension M'_{i+1} of M'_i containing the image of M_{i+1} under h_{i+1}.

Once we have completed the construction outlined above, the issue of whether or not $h := \bigcup_{i < \alpha} h_i$ is an isomorphism between M and $\bigcup_{i < \alpha} M'_i$ remains to be addressed. First notice that by our assumption that $\alpha = \lambda \cdot \alpha$, if $a \in M'$ realizes r, then at some stage in the construction, we would have put a into the range of h. Therefore, if h were not an isomorphism, $h(M)$ and M' would form a (r, λ)-Vaughtian pair contradicting our hypothesis on r.

If in addition to the hypothesis given, we assume that $\mathcal{K}$ is Galois-stable in λ, we could conclude that M is a (λ, α)-limit model by altering the construction. At stage i of the construction we choose M'_{i+1} as above, only now require that M'_{i+1} to be universal over M'_i. ⊣

Using Theorem 5.1, we are able to transfer categoricity from λ to λ^+ by showing that every model of cardinality λ^+ is saturated:

Theorem 5.2. *Suppose that $\mathcal{K}$ has arbitrarily large models, is χ-tame and satisfies the amalgamation and joint embedding properties. Let λ be such that $\lambda > \text{LS}(\mathcal{K})$ and $\lambda \geq \chi$. If $\mathcal{K}$ is categorical in λ^+ then $\mathcal{K}$ is categorical in all $\mu \geq \lambda^+$.*

Proof. First we prove that $\mathcal{K}$ is categorical in λ^{++} by establishing that every model N of cardinality λ^{++} is saturated. Let $M \prec_{\mathcal{K}} N$ have cardinality λ^+. We will show that N realizes every type over M. First notice that Proposition 3.5 and categoricity in λ^+ guarantees that there exists a deep-rooted quasi-minimal $r \in$ ga-S(M). By Corollary 4.9, we know that N realizes r λ^{++}-times.

Let $\alpha < \lambda^+$ be such that $\alpha = \lambda^+ \cdot \alpha$. By the Downward-Löwenheim Skolem Axiom of AECs, we can construct a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $\langle M_i \prec_{\mathcal{K}} N \mid i < \alpha \rangle$ such that $M = M_0$ for every $i < \alpha$, we can fix $a_i \in M_{i+1} \backslash M_i$ realizing r. This construction is possible since there are λ^{++}-many realizations of r from which to choose. By Fact 5.1, $\bigcup_{i<\alpha} M_i$ realizes every type over M.

We have explained that categoricity in λ^+ implies categoricity in λ^+ and λ^{++}. Now, an application of Fact 1.4 provides categoricity in all larger cardinalities.

$\dashv$

A combination of our upward result and Shelah's downward result from [Sh 394] yields

Theorem 5.3. *Let $\mathcal{K}$ be a χ-tame abstract elementary class satisfying the amalgamation and joint embedding properties. If $\mathcal{K}$ is categorical in λ^+ for some $\lambda > \max\{LS(\mathcal{K}), \chi\}$, then $\mathcal{K}$ is categorical in all $\mu \geq \min\{\lambda^+, \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}\}$.*

It remains open whether or not categoricity in λ^+ implies categoricity in λ^{++} for the special case where $\aleph_0 < LS(\mathcal{K}) = \chi = \lambda$. For this case, a substitute for Fact 4.2 is missing. We will provide some partial results concerning this problem in the following section.

6. Categoricity in $LS(\mathcal{K})$ and $LS(\mathcal{K})^+$

In this section, we examine an abstract elementary class which is categorical in both λ and λ^+ and $\lambda = LS(\mathcal{K}) = \chi$. We assume the class has no maximal models and satisfies the amalgamation and joint embedding properties. This is motivated by questions of John Baldwin and Olivier Lessmann concerning perceived limitations of [GrVa1]. From these assumptions, we derive categoricity in all $\mu \geq LS(\mathcal{K})$. The difficulty in working with a class that is categorical in $LS(\mathcal{K})^+$ is that there are no saturated models of cardinality $LS(K)$. However, from stability we do have limit models of cardinality $LS(\mathcal{K})$, and in this section we have an extra categoricity assumption which tells us that all models of cardinality $LS(\mathcal{K})$, while not saturated, are limit models. This allows us to use minimal types instead of quasi-minimal types.

We begin with a replacement for Fact 4.2.

Theorem 6.1. *Assume that $\mathcal{K}$ is categorical in λ and λ^+ with $\lambda = LS(\mathcal{K}) = \chi$. Then for every limit model $M \in \mathcal{K}_\lambda$ there is a minimal type $p \in$ ga-S(M),*

such that there are no true (p, λ)-Vaughtian pairs of the form (N_0, N_1) with $M = N_0$.

Proof. Suppose every minimal type over a limit model had a true Vaughtian pair. Let M be a limit model of cardinality μ and fix $p \in$ ga-S(M) minimal with true Vaughtian pair (M, N_1) where $N_1 \in \mathcal{K}_\lambda$. We can construct a $\prec_{\mathcal{K}}$-increasing and continuous chain $\langle N_i \mid i < \lambda^+ \rangle$ of limit models such that for each $i < \lambda^+$

  (1) $N_0 = M$
  (2) $N_i \in \mathcal{K}_\lambda$
  (3) N_i is a limit model and
  (4) no $a \in N_{i+1} \backslash N_i$ realizes p.

Suppose i is a limit ordinal and that we have defined N_j for all $j < i$. Let $N_i := \bigcup_{j<i} N_j$. By categoricity in λ we know that N_i must be a limit model (but it may not be a limit model over M).

For the successor step of the construction, suppose that N_i has been defined. Since M is a limit model, we can find $p_i \in$ ga-S(N_i) a unique non-algebraic extension of p (by Remark 3.4). Since N_i is a limit model and p_i is a minimal type, by our assumption it must be the case that there is N_{i+1} a limit model extending N_i which together with N_i forms a (p_i, λ)-Vaughtian pair. Since no $a \in N_{i+1} \backslash N_i$ realizes p_i, we can conclude by the minimality of p that condition (4) holds.

To see why the construction is enough to get a contradiction, let $N_{\lambda^+} := \bigcup_{i<\lambda^+} N_i$. From condition (4) of the construction, we find that N_{λ^+} does not realize p. Thus N_{λ^+} is not saturated, which contradicts categoricity in λ^+. ⊣

We now prove a slight variation of Corollary 4.8.

Corollary 6.2. *Let λ be as in Theorem 6.1. For every $M \in \mathcal{K}_{\lambda^+}$, there is $q \in$ ga-S(M), a deep-rooted minimal type with no (q, λ^+)-Vaughtian pairs.*

Proof. Let $M \in \mathcal{K}_{\lambda^+}$ be given. Fix $N \prec_{\mathcal{K}} M$ a limit model of cardinality λ. By Theorem 6.1, we can choose a minimal $p \in$ ga-S(N) such that there are no true (p, λ)-Vaughtian pairs. By Proposition 3.5, we can extend p to a deep-rooted minimal type $q \in$ ga-S(M). Suppose that N^0, N^1 form a (q, λ^+)-Vaughtian pair. Then Theorem 4.4 and Remark 4.7 tell us that there are limit models N_0, N_1 with $N \prec_{\mathcal{K}} N_0 \prec_{\mathcal{K}} N^0$ with (N_0, N_1) a (p, λ)-Vaughtian pair and N_0 a limit model over N. Furthermore, we have that (N_0, N_1) form a $(q \restriction N_0, \lambda)$-Vaughtian pair.

We will now show that by our choice of p such $(q \restriction N_0, \lambda)$-Vaughtian pairs cannot exist. Since N is a limit model, we can find a resolution $\langle N_i^+ \mid i < \omega \rangle$ of N such that N_{i+1}^+ is universal over N_i^+. By Fact 2.9, there is $i < \omega$ such that p does not λ-split over N_i^+. Observe that N is a limit model over N_i^+. Additionally, since N_0 is a limit model over N it is also a limit model over N_i^+. Then, N and N_0 are isomorphic over N_i^+. Let $f : N \cong N_0$ with

$f \upharpoonright N_i^+ = \mathrm{id}_{N_i^+}$. Since there are no (p, λ)-Vaughtian pairs with N as the first model in the pair, there are no $(f(p), \lambda)$-Vaughtian pairs with N_0 as the first model in the pair. By invariance and our choice of N_i^+, we have that $f(p)$ does not μ-spit over N_i^+. This implies that $f(p) \geq p$, otherwise f^{-1} would witness that $f(p)$ λ-splits over N_i^+. Now we have that $f(p)$ and $q \upharpoonright N_0$ are both non-algebraic extensions of p to N_0. By minimality of p, $f(p) = q \upharpoonright N_0$ and we can conclude that there are no $(q \upharpoonright N_0)$-Vaughtian pairs with N_0 as the first model of the pair. This gives us a contradiction and completes the proof. $\dashv$

Corollary 6.2 is enough to carry out the argument of Corollary 4.9 and the remaining arguments in Section 5. This allows us to conclude the second theorem in the abstract, restated here:

Theorem 6.3. *Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-tame abstract elementary class satisfying the amalgamation and joint embedding properties with arbitrarily large models. If $\mathcal{K}$ is categorical in both $\mathrm{LS}(K)$ and $\mathrm{LS}(\mathcal{K})^+$, then $\mathcal{K}$ is categorical in all $\mu \geq \mathrm{LS}(\mathcal{K})$.*

References

[Ba1] John Baldwin. **Categoricity**. Monograph, in preparation. Available at `http://www2.math.uic.edu/~jbaldwin/model.html`

[BaK] J. Baldwin and A. Kolesnikov. Categoricity, amalgamation, and tameness, preprint.

[Ch] C. C. Chang. Some remarks on the model theory of infinitary languages, **The Syntax and Semantics of Infinitary Languages**, Springer-Verlag Lecture Notes in Mathematics **72**, edited by J. Barwise (1968) pages 36–63.

[Gra] George Grätzer. **Universal algebra**. Van Nostrand, 1968. ix+368pp.

[Gr1] Rami Grossberg. Classification theory for non-elementary classes. **Logic and Algebra**, ed. Yi Zhang, *Contemporary Mathematics*, **302**, (2002) AMS, pp. 165–204.

[Gr2] Rami Grossberg. **A Course in Model Theory.** Book in Preparation.

[GrKv] R. Grossberg and A. Kolesnikov, Superior Abstract Elementary Classes are Tame. (26 pages). Submitted to *JSL*. Preprint available at `http://www.math.cmu.edu/~rami/`

[GrVa1] Rami Grossberg and Monica VanDieren. Shelah's Categoricity Conjecture from a successor for Tame Abstract Elementary Classes, *Journal of Symbolic Logic.* **71**, (2006) 2, 553–568.

[GrVa2] Rami Grossberg and Monica VanDieren. Galois-stability for Tame Abstract Elementary Classes, *Journal of Mathematical Logic.* **6**, No. 1 (2006) 25–49.

[GrVaVi] Rami Grossberg, Monica VanDieren and Andres Villaveces. Uniqueness of limit models in stable classes with amalgamation. (18 pages) submitted.

[HtSh] B. Hart and S. Shelah, Categoricity over P for first order T or categoricity for $\varphi \in L_{\omega_1 \omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$, *Israel J Math* **70** (1990) 219-235.

[KoSh] Oren Kolman and Saharon Shelah. Categoricity of Theories in $L_{\kappa,\omega}$ when κ is a measurable cardinal. Part I. *Fundamentae Mathematicae*, **151**, 209–240, 1996.

[Le] Olivier Lessmann. Upward Categoricity from a Successor for Tame Abstract Elementary Classes. Preprint.

[MaSh] Michael Makkai and Saharon Shelah. Categoricity of theories $L_{\kappa\omega}$ with κ a compact cardinal. *Annals of Pure and Applied Logic*, **47**, 41-97, 1990.

[Sh a] Saharon Shelah. **Classification Theory and the Number of Nonisomorphic Models**, North-Holland, 1978, Amsterdam.

[Sh 48] Saharon Shelah. Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$, *Israel J Math* **20** (1975) 127-148.

[Sh 87a] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A. *Israel J. Math.*, **46**:212–240, 1983.

[Sh 87b] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B. *Israel J. Math.*, **46**:241–273, 1983.

[Sh 88] Saharon Shelah. Classification of nonelementary classes, II. Abstract Elementary Classes. In **Classification Theory (Chicago IL 1985)**, volume 1292 of *Lecture Notes in Mathematics*, pages 419–497. Springer, Berlin, 1987. Proceedings of the USA-Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[Sh 394] Saharon Shelah. Categoricity of abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 1999.

[Sh 472] Saharon Shelah. Categoricity of Theories in $L_{\kappa^*,\omega}$ when κ^* is a measurable cardinal. Part II. Dedicated to the memory of Jerzy Łoś. *Fundamenta Mathematica*. **170** (2001), no.1-2, 165–196.

[Sh 576] S. Shelah, Categoricity of an abstract elementary class in two successive cardinals, *Israel J. of Math.*, **126**, pages 29–128. 2001.

[Sh 600] Saharon Shelah. Categoricity in abstract elementary classes: going up inductive step. Preprint available at `http://shelah.logic.at/short600.html`.

[Sh 705] Saharon Shelah. Toward classification theory of good λ frames and abstract elementary classes Preprint available at `http://shelah.logic.at/short700.html`.

[ShVi] Saharon Shelah and Andrés Villaveces. Categoricity in abstract elementary classes with no maximal models. *Annals of Pure and Applied Logic*, **97**, 1999, no.1–3, 1–25.

[Va] Monica VanDieren. Categoricity in Abstract Elementary Classes with No Maximal Models, *Annals of Pure and Applied Logic*, **141**, 2006, pages 108–147.

[ViZa] Andrés Villaveces and Pedro Zambrano. Hrushovski constructions and tame abstract elementary classes. in preparation.

[Zi] Boris Zilber. A categoricity theorem for quasi-minimal excellent classes, Logic and its applications, *Contemp. Math.*, **380**, Amer. Math. Soc., 2005, pages 297–306.

[Zi2] B.I. Zilber. Model theory, geometry and arithmetic of universal covers of a semi-abelian variety. In Model Theory and Applications, Quaterna di matematica, pages 427–458, 2003.

[Zi3] B.I. Zilber. Analytic and pseudo-analytic structures. Logic Colloquium 2000, Lect. Notes Log., 19, Assoc. Symbol. Logic, Urbana, IL, 2005. pages 392–408.

E-mail address, Rami Grossberg: `rami@andrew.cmu.edu`

DEPARTMENT OF MATHEMATICS, CARNEGIE MELLON UNIVERSITY, PITTSBURGH PA 15213

E-mail address, Monica VanDieren: `mvd@umich.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR MI 48109-1109

CURRENT: DEPARTMENT OF MATHEMATICS, ROBERT MORRIS UNIVERSITY, MOON TOWNSHIP, PA 15108