21-241 MATRICES AND LINEAR TRANSFORMATIONS
SUMMER 1 2012
COURSE NOTES
DAY 4

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Definition 1. A matrix A is in *reduced row echelon form (rref)* if

(1) Every zero row in A is below every nonzero row.
(2) The leading term in any nonzero row in A is strictly to the right of all the leading terms in the rows above it.
(3) The leading term in a nonzero row in A is a 1, and is the only nonzero entry in its column.

A system of linear equations $Ax = b$ is in reduced row echelon form if and only if A is.

Definition 2. Let A be an $m \times n$ matrix.

(1) An $n \times m$ matrix B such that $BA = I_n$ is called a *left inverse* for A.
(2) An $n \times m$ matrix C such that $AC = I_m$ is called a *right inverse* for A.
(3) An $n \times m$ matrix D which is both a left inverse and a right inverse for A is called an *inverse* for A.

If A has a left inverse or right inverse, A is called *left-invertible* or *right-invertible* respectively. A is called *invertible* if it is both left-invertible and right-invertible.

Notice that, on the face of it, a matrix may be invertible but have no inverse. The following result should correct our doubt about that.

Fact 1. Suppose A is invertible, and B and C are a left inverse and a right inverse for A, respectively. Then $B = C$, and hence A has an inverse.

Proof. We have

$$B = BI = B(AC) = (BA)C = IC = C$$

□

We also get a uniqueness theorem out of this, which will allow us to make the following definition.

Corollary 1. *Suppose A is invertible. Then there is exactly one inverse for A.*

Proof. If B and C are inverses for A, then in particular, B is a left inverse and C a right inverse. Then $B = C$ by the above fact. ☐

Definition 3. If A is an invertible matrix, then A^{-1} is the unique inverse of A.

Theorem 1. *Let A be an $m \times n$ matrix. Then the following are equivalent.*

(1) *For any choice of constants $b_1, \ldots, b_m$, the system of linear equations*

$$\begin{aligned} a_{11}x_1 + \cdots + \ a_{1n}x_n &= \ b_1 \\ a_{21}x_1 + \cdots + \ a_{2n}x_n &= \ b_2 \\ &\ \ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

has at least one solution.

(2) *For every $b \in \mathbb{R}^m$, there is some $s \in \mathbb{R}^n$ such that $As = b$.*

(3) *$T_A : \mathbb{R}^n \to \mathbb{R}^m$ is surjective.*

(4) *A is right-invertible.*

Proof. The equivalence of (1), (2), and (3) is just a recapitulation of the definitions in play. The real work is in proving they're equivalent to (4).

I'll prove (2) is equivalent to (4). First, assume (2); we'll prove (4). Consider the standard basis vectors in $\mathbb{R}^m$;

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots e_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

By our assumption, there are vectors $s_1, \ldots, s_m \in \mathbb{R}^n$ such that $As_1 = e_1, \ldots, As_m = e_m$. Now let B be the matrix with columns $s_1, \ldots, s_m$, in that order. Then B is $n \times m$, and for any i, j, $B_{ij} = (s_j)_i$, the ith entry of column vector s_j. Then for all $i, j \leq m$, we have

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ik}(s_j)_k = (As_j)_i = (e_j)_i = I_{ij}$$

so $AB = I$.

Now assume (4), and let B be a right inverse for A. Let $b \in \mathbb{R}^m$ be given. Then

$$A(Bb) = (AB)b = Ib = b$$

and so there is some $s \in \mathbb{R}^n$ such that $As = b$, namely $s = Bb$. ☐