21-241 MATRICES AND LINEAR TRANSFORMATIONS
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COURSE NOTES
DAY 3

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Definition 1. A *linear transformation* is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

(1) $T(x + y) = T(x) + T(y)$, and
(2) $T(\lambda x) = \lambda T(x)$.

Fact 1. (a) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, $x_1, \ldots, x_k \in \mathbb{R}^n$, and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, then

$$T(\lambda_1 x_1 + \cdots + \lambda_k x_k) = \lambda_1 T(x_1) + \cdots + \lambda_k T(x_k)$$

(b) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(0) = 0$.
(c) If $\lambda \in \mathbb{R}$ and $S, T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, then so are the functions $S + T$ and λT defined by

$$(S + T)(x) = S(x) + T(x) \qquad (\lambda T)(x) = \lambda(T(x)) \qquad x \in \mathbb{R}^n$$

Example. The following are all linear transformations.

(1) The *identity map* $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ which takes $x \in \mathbb{R}^n$ to itself.
(2) The map which rotates a point x in the plane $\mathbb{R}^2$ around the origin by a fixed angle θ.
(3) The map which takes a given $x \in \mathbb{R}^n$ to $Ax \in \mathbb{R}^m$, where A is a fixed $m \times n$ matrix.

It is an important fact that the last example is completely representative of all linear transformations.

Theorem 1. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is some $m \times n$ matrix A such that for all $x \in \mathbb{R}^n$,*

$$T(x) = Ax$$

Moreover such an A is unique; ie, there is exactly one *$m \times n$ matrix A such that the above holds for all $x \in \mathbb{R}^n$.*

Proof. We will make use of the following vectors in $\mathbb{R}^n$, which you will get to know very well over the course of the semester;

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \qquad \cdots \qquad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

These are called the *standard basis vectors for* $\mathbb{R}^n$. Now let $a_i = T(e_i)$ for each $i \leq n$. Then each a_i is a column vector in $\mathbb{R}^m$. Let A be the $m \times n$ matrix whose ith column (from left to right) is a_i. We'll show that this A works, ie,

$$\forall x \in \mathbb{R}^n \quad T(x) = Ax$$

To see this, let $x \in \mathbb{R}^n$ be given. Then

$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \end{aligned}$$

Hence, by Fact 1,

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n) = x_1 a_1 + \cdots + x_n a_n = Ax$$

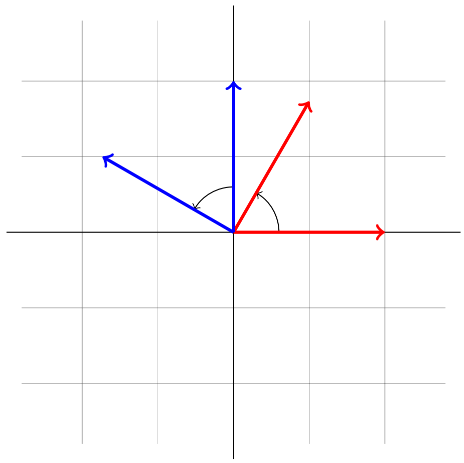
Since $x \in \mathbb{R}^n$ was given, this proves that there is a matrix A which implements T. To show uniqueness, ie that there is *at most one* such A, suppose both A and B are $m \times n$ matrices which implement T. Then,

$$\forall x \in \mathbb{R}^n \quad Ax = T(x) = Bx$$

Now we look again at the standard basis vectors; we have $Ae_i = T(e_i) = Be_i$ for each $i \leq n$. Notice that Ae_i and Be_i are the ith columns of A and B respectively. Hence the ith column of A is equal to the ith column of B, for each $i \leq n$; it follows then that $A = B$. □

Definition 2. The *standard matrix* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the unique $m \times n$ matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$.

Example. Let's work out the standard matrix of the linear transformation R_θ which rotates a point $x \in \mathbb{R}^2$ around the origin (counterclockwise) by a fixed angle θ. By the proof of the above theorem, to find the columns of this matrix we need only find $R_\theta(e_1)$ and $R_\theta(e_2)$, where $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. In this case e_1 is the unit vector in the x direction; so by trigonometry, rotation by the angle θ brings us to the point $\left(\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right)$.



Some geometric pondering also shows that rotating e_2, the unit vector in the y direction, by the angle θ brings us to $\left(\begin{smallmatrix} -\sin\theta \\ \cos\theta \end{smallmatrix}\right)$. Hence the standard matrix for T is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$