21-241 MATRICES AND LINEAR TRANSFORMATIONS SUMMER 1 2012 COURSE NOTES JUNE 25

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Our aim for tomorrow is to prove the following theorem, which I often call *the spectral theorem.*

Theorem. *If A is a Hermitian matrix, then A is diagonalizable by a unitary matrix.*

Today we'll develop the background necessary for the proof of this theorem. It starts with the following definition.

Definition. Let $V_1, \dots, V_k$ be subspaces of $\mathbb{C}^n$, such that $V_i \perp V_j$ for $i \neq j$. Then their *orthogonal sum* is the subspace

$$V_1 \oplus \cdots \oplus V_k = \{v_1 + \cdots + v_k \mid v_1 \in V_1, \dots, v_k \in V_k\}$$

Example 1. Let's take the lines $L = \text{span}\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\}$ and $K = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$ in $\mathbb{R}^3$. Then $L \perp K$. Their orthogonal sum is the plane

$$L \oplus K = \{v + w \mid v \in L \wedge w \in K\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Lemma 1. *Suppose V and W are subspaces of $\mathbb{C}^n$, such that $V \perp W$. Then $\dim(V \oplus W) = \dim(V) + \dim(W)$, and moreover if X is an orthonormal basis for V and Y an orthonormal basis for W, then $X \cup Y$ is an orthonormal basis for $V \oplus W$.*

Corollary 1. *If $V_i \perp V_j$ for $i \neq j$, and X_i is an orthonormal basis for V_i, then $X_1 \cup \cdots \cup X_k$ is an orthonormal basis for $V_1 \oplus \cdots \oplus V_k$.*

Corollary 2. *If V is a subspace of $\mathbb{C}^n$, then $\dim(V) + \dim(V^\perp) = n$.*

Definition. A subspace $V \subseteq \mathbb{C}^n$ is *invariant* for a matrix $A \in M_n(\mathbb{C})$ if

$$\forall v \in V \quad Av \in V$$

Lemma 2. *If V is invariant for A, and A is Hermitian, then so is $V^\perp$.*

Proof. Let $w \in V^\perp$. Then for all $v \in V$,

$$\langle Aw, v \rangle = \langle w, A^H v \rangle = \langle w, Av \rangle = 0$$

since $Av \in V$. Hence $Aw \in V^\perp$. □

Note. Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Suppose V is an invariant subspace for A; thus $V^\perp$ is too by the above lemma. Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_\ell$ be orthonormal bases for V and $V^\perp$ respectively; then $k + \ell = n$, and

$$v_1, \ldots, v_k, w_1, \ldots, w_\ell$$

is an orthonormal basis for $\mathbb{C}^n$. Let U be the matrix whose columns are $v_1, \ldots, v_k, w_1, \ldots, w_\ell$. Then U is a unitary matrix. We'd like to look at $U^H AU$. Its (i, j)-entry is

$$\left\langle U^H AUe_j, e_i \right\rangle = \langle AUe_j, Ue_i \rangle$$

and Ue_j and Ue_i are the jth and ith columns of U, respectively. Say $i \leq k$ and $j > k$. Then Ue_i is one of the v's, and Ue_j is one of the w's; so in particular, $Ue_i \in V$, whereas $Ue_j \in V^\perp$. Since $V^\perp$ is invariant for A, $AUe_j \in V^\perp$ too, so

$$\langle AUe_j, Ue_i \rangle = 0$$

Similarly, if $i > k$ but $j \leq k$, then $AUe_j \in V$ and $Ue_i \in V^\perp$, so

$$\langle AUe_j, Ue_i \rangle = 0$$

This means that A has the block form

$$A = \begin{pmatrix} \hat{A} & 0_{k \times \ell} \\ 0_{\ell \times k} & \check{A} \end{pmatrix}$$

where $\hat{A}$ is some $k \times k$ matrix, and $\check{A}$ is some $\ell \times \ell$ matrix.

Lemma 3. *If A is a Hermitian matrix, $\lambda_1, \ldots, \lambda_k$ are its eigenvalues (without repetition), and*

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$$

then V and $V^\perp$ are invariant subspaces for A.

Proof. Let $v \in V$. Then $v = v_1 + \cdots + v_k$ for some $v_1 \in V_{\lambda_1}, \ldots, v_k \in V_{\lambda_k}$. Now,

$$Av = A(v_1 + \cdots + v_k) = Av_1 + \cdots + Av_k = \lambda_1 v_1 + \cdots + \lambda_k v_k$$

Since $v_i \in V_{\lambda_i}$, so is $\lambda_i v_i$. Hence $Av \in V$. This proves that V is an invariant subspace for A. That $V^\perp$ is also invariant follows from this and the above lemma. □