21-241 MATRICES AND LINEAR TRANSFORMATIONS SUMMER 1 2012 COURSE NOTES JUNE 7

PAUL MCKENNEY

Definition. $M_n(\mathbb{C})$ is the set of $n \times n$ matrices with entries from $\mathbb{C}$. $M_n(\mathbb{C})$ is the set of $n \times n$ matrices with entries from $\mathbb{C}$. For today we will be identifying a matrix $A \in M_n(\mathbb{C})$ with its sequence of rows, $\rho_1, \dots, \rho_n \in \mathbb{C}^n$.

A *multilinear map* is a function $T : M_n(\mathbb{C}) \to \mathbb{C}$ such that

(1) For all $\rho_1, \dots, \rho_n \in \mathbb{C}^n$ and $\sigma \in \mathbb{C}^n$,
$$T(\rho_1, \dots, \rho_i + \sigma, \dots, \rho_n) = T(\rho_1, \dots, \rho_i, \dots, \rho_n) + T(\rho_1, \dots, \sigma, \dots, \rho_n)$$
(2) For all $\rho_1, \dots, \rho_n \in \mathbb{C}^n$ and $t \in \mathbb{C}$,
$$T(\rho_1, \dots, t\rho_i, \dots, \rho_n) = tT(\rho_1, \dots, \rho_i, \dots, \rho_n)$$

In other words, T is a linear map on its ith argument when all others are fixed.

A multilinear map T is *alternating* if in addition we have
$$T(\rho_1, \dots, \rho_j, \dots, \rho_i, \dots, \rho_n) = -T(\rho_1, \dots, \rho_i, \dots, \rho_j, \dots, \rho_n)$$
whenever $\rho_1, \dots, \rho_n \in \mathbb{C}^n$ and $i < j$.

Lemma 1. *Suppose $T : M_n(\mathbb{C}) \to \mathbb{C}$ is an alternating multilinear map, and $A \in M_n(\mathbb{C})$ is some matrix with two rows which are the same, or a row of all zeroes. Then $T(A) = 0$.*

Proof. Suppose A has two rows which are the same, ie $\rho_i = \rho_j$ for some $i \neq j$. Then
$$T(\rho_1, \dots, \rho_i, \dots, \rho_j, \dots, \rho_n) = -T(\rho_1, \dots, \rho_j, \dots, \rho_i, \dots, \rho_n) = -T(\rho_1, \dots, \rho_i, \dots, \rho_j, \dots, \rho_n)$$
The only real (or complex) number t satisfying $t = -t$ is $t = 0$.

Now suppose A has a zero row in the ith place. Then by linearity,
$$T(\rho_1, \dots, 0_{1\times n}, \dots, \rho_n) = T(\rho_1, \dots, 0 \cdot 0_{1\times n}, \dots, \rho_n) = 0 \cdot T(\rho_1, \dots, 0_{1\times n}, \dots, \rho_n) = 0$$
□

Theorem 1. *Suppose $T : M_n(\mathbb{C}) \to \mathbb{C}$ is an alternating multilinear map, and A and B are $n \times n$ matrices such that B is the result of applying a single row operation to A.*

Then $T(A)$ and $T(B)$ are related in the following way depending on the row operation in question;

$$\begin{array}{ccc} i \neq j & \rho_i \leftrightarrow \rho_j & T(B) = -T(A) \\ i \neq j & \rho_i \to \rho_i + \lambda\rho_j & T(B) = T(A) \\ & \rho_i \to \lambda\rho_i & T(B) = \lambda T(A) \end{array}$$

Proof. The swap and scaling cases are simply part of the definition of an alternating multilinear map. The row-combination case follows from the lemma above;

$$\begin{aligned} T(\rho_1, \dots, \rho_i + \lambda\rho_j, \dots, \rho_j, \dots, \rho_n) &= T(\rho_1, \dots, \rho_i, \dots, \rho_j, \dots, \rho_n) + \lambda T(\rho_1, \dots, \rho_j, \dots, \rho_j, \dots, \rho_n) \\ &= T(\rho_1, \dots, \rho_i, \dots, \rho_j, \dots, \rho_n) \end{aligned}$$

□

Theorem 2. *If T is an alternating multilinear map, and A is not invertible, then $T(A) = 0$.*

Proof. Let R be the reduced row-echelon form of A. By the above theorem, $T(A) = 0$ if and only if $T(R) = 0$. Since A is not invertible, R cannot be I; but then R has some zero row. So $T(R) = 0$, and hence $T(A) = 0$. □

Theorem 3. *If $S, T : M_n(\mathbb{C}) \to \mathbb{C}$ are alternating multilinear maps, and $S(I) = T(I)$, then $S = T$.*

Proof. We've already seen that if A is not invertible then S and T both send A to zero. If A is invertible, then it is row-reducible to I. Theorem 1 (along with a routine induction) shows that

$$T(I) = (-1)^k t_1 \cdots t_\ell T(A) \qquad S(I) = (-1)^k t_1 \cdots t_\ell S(A)$$

where k is the number of swaps used, ℓ is the number of scaling operations used, and $t_1, \dots, t_\ell$ are the scaling factors. Hence if $S(I) = T(I)$, then

$$S(A) = (-1)^k \frac{1}{t_1} \cdots \frac{1}{t_\ell} S(I) = (-1)^k \frac{1}{t_1} \cdots \frac{1}{t_\ell} T(I) = T(A)$$

□

Definition. The *determinant* $\det : M_n(\mathbb{C}) \to \mathbb{C}$ is the unique alternating multilinear map such that $\det(I) = 1$.

Theorem 4. *Let A be a square matrix. Then the following are equivalent.*

(1) A is invertible.
(2) $\det(A) \neq 0$.

Example. Let's calculate the determinants of the following matrices.

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 10 \\ 6 & 11 & 17 \end{pmatrix} \qquad \begin{pmatrix} -2 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix}$$