21-241 MATRICES AND LINEAR TRANSFORMATIONS
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COURSE NOTES
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1. Invertibility Revisited

Recall the corollary to rank-nullity that we established;

Corollary 1. *Let A be an $m \times n$ matrix. Then* $\text{rank}(A) + \text{nullity}(A) = n$.

[We also proved via rank-nullity that the column rank and row rank are the same, hence my usage of the generic term "rank" above.]

Theorem 1. *Let A be an $n \times n$ matrix. Then the following are equivalent.*

(1) A is invertible.
(2) The columns of A form a basis of $\mathbb{R}^n$.
(3) For every $b \in \mathbb{R}^n$, there is exactly one solution to the system $Ax = b$.
(4) For every $b \in \mathbb{R}^n$, there is at most one solution to the system $Ax = b$.
(5) For every $b \in \mathbb{R}^n$, there is at least one solution to the system $Ax = b$.
(6) The only solution to $Ax = 0$ is 0.
(7) The columns of A are linearly independent.
(8) The columns of A span $\mathbb{R}^n$.
(9) $\text{nullity}(A) = 0$.
(10) $\text{rank}(A) = n$.
(11) A is left-invertible.
(12) A is right-invertible.
(13) T_A is injective.
(14) T_A is surjective.
(15) $A^\top$ is invertible.

Proof. We've seen that

$$(4) \iff (6) \iff (7) \iff (9) \iff (11) \iff (13)$$

and

$$(5) \iff (8) \iff (10) \iff (12) \iff (14)$$

We've also seen (1) $\iff$ (3) $\iff$ (2) and clearly (2) $\implies$ (7), (8). The rank-nullity theorem fills in the gap by proving (9) $\iff$ (10); for if $\text{null}(A) = 0$, then $\text{rank}(A) = n - 0 = n$ by rank-nullity, and similarly if $\text{rank}(A) = n$ then $\text{null}(A) = n - n = 0$.

Finally, you've shown on your homework that (1) $\iff$ (15). $\square$

2. Example Problems

Example. Calculate the rank and nullity of the matrix A below. Find bases for $\text{row}(A)$, $\text{null}(A)$ and $\text{col}(A)$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 8 \\ 5 & 8 & 13 \end{pmatrix}$$

What about $A^\top$?

Note that (132134) is in $\text{row}(A)$ and $(1, 1, 1)$ is in $\text{null}(A)$. Find the unique linear combinations of your bases which produce these vectors.

Example. Let $d_1, \ldots, d_n \in \mathbb{R}$ be given. Calculate the rank and nullity of $D = \text{diag}(d_1, \ldots, d_n)$, and bases for $\text{row}(D)$, $\text{null}(D)$ and $\text{col}(D)$.

We've been working exclusively with real numbers up to this point. Soon we'll have to use complex numbers. If I haven't said it before, let me say it now; everything we've done up to now works for complex numbers too. Let's do some computations with those. Recall the following fact about complex numbers;

Fact 1. Let $z = a + bi$ be a nonzero complex number (so at least one of a and b is nonzero). Then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a - bi}{a^2 + b^2}$$

In particular if z lies on the unit circle (so $z = \cos\theta + i\sin\theta$ for some θ) then

$$\frac{1}{z} = \bar{z}$$

Example. Find the rank and nullity of the matrix A below. Find bases for $\text{row}(A)$, $\text{null}(A)$ and $\text{col}(A)$.

$$\begin{pmatrix} 1 + 2i & -1 & 0 \\ 1 - 2i & i & 3 \\ 0 & 2 & 1 \end{pmatrix}$$

Example.
1. Let p and q be points in $\mathbb{R}^2$ (ie, the plane). Show that $\{p, q\}$ is linearly independent if and only if the points 0, p, and q are not colinear.
2. Let $p, q, r \in \mathbb{R}^3$. Show that $\{p, q, r\}$ is linearly independent if and only if the points 0, p, q, and r are not coplanar.