21-241 MATRICES AND LINEAR TRANSFORMATIONS
SUMMER I 2012
HOMEWORK 6

Definition. A square, *complex* matrix P is called a *projection matrix* if $P^2 = P = P^H$. (Note the difference with the earlier definition of projection.)

Definition. A matrix $A \in M_n(\mathbb{C})$ is *diagonalizable* if there are some diagonal D and invertible S such that $A = SDS^{-1}$. In this case we say S *diagonalizes* A.

Definition. A matrix $U \in M_n(\mathbb{C})$ such that $U^H U = I$ and $UU^H = I$ (ie, U is invertible and $U^{-1} = U^H$) is called *unitary*. If $U \in M_n(\mathbb{R})$ and U is unitary, we sometimes say that U is *orthogonal*.

The following theorem may be useful in the problems below.

Theorem. *If A is a Hermitian matrix, then there is some unitary U such that UAU^H is diagonal, and moreover the diagonal values of UAU^H are exactly the eigenvalues of A.*

You may also use the following characterization of the determinant (often called the Laplace expansion).

Fact. Let A be an $n \times n$ matrix with entries a_{ij}. Then for any i or j,

$$\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(M_{ik}) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(M_{kj})$$

where M_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing its ith row and jth column entirely. (This is the so-called (i,j)-*minor* of A.)

(1) Given each matrix A below, check whether A is diagonalizable; if it is, diagonalize it. Otherwise, demonstrate an eigenvalue of A whose geometric multiplicity is different from its algebraic multiplicity. [10 each]

$$(a)\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 1 & i & -1 & -i \\ -i & 1 & i & -1 \\ -1 & -i & 1 & i \\ i & -1 & -i & 1 \end{pmatrix}$$

$$(c)\quad \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \qquad (d)\quad \begin{pmatrix} 1 & i & 0 \\ -i & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

$$(e)\quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (f)\quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(2) Use your answer from 1(f) to prove that

$$f_n = \frac{1}{\sqrt{5}}(\phi_+^n - \phi_-^n)$$

where f_n is the nth Fibonacci number, and

$$\phi_\pm = \frac{1 \pm \sqrt{5}}{2}$$

(Recall from previous homework that if $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then $A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$.) [10]

(3) Show that there is an $A \in M_2(\mathbb{R})$ such that $\det(A) \geq 0$, but there is no $B \in M_2(\mathbb{R})$ such that $A = B^2$. [10]

(4) Show that if $A \in M_n(\mathbb{C})$ is Hermitian and $\operatorname{spec}(A) \subseteq [0, \infty)$, then there is some $B \in M_n(\mathbb{C})$ such that $A = B^2$. [10]

(5) Let P be a projection matrix. Show that $\operatorname{tr}(P) = \operatorname{rank}(P)$. [10]

(6) We call a matrix $U \in M_n(\mathbb{C})$ *distance-preserving* if $\|Ux - Uy\| = \|x - y\|$ for all $x, y \in \mathbb{C}^n$. Prove that the following are equivalent; [20]
 (a) U is distance-preserving.
 (b) For all $x \in \mathbb{C}^n$, $\|Ux\| = \|x\|$.
 (c) For all $x, y \in \mathbb{C}^n$, $\langle Ux, Uy \rangle = \langle x, y \rangle$.
 (d) U is unitary.

(Hint: for the (6b) $\implies$ (6c) implication, consider the quantities $\|Ux - Uy\|^2$ and $\|Ux - U(iy)\|^2$.)

(7) Prove that if U is an orthogonal matrix then every *real* eigenvalue of U is either $+1$ or -1, and either $\det(U) = +1$ or $\det(U) = -1$. [10]

(8) Let U be a 2×2 orthogonal matrix. Prove that U is either a rotation matrix or a reflection matrix. [10]

(9) Let n be an odd natural number. Prove that every $A \in M_n(\mathbb{R})$ has at least one real eigenvalue. [10]

(10) Let U be a 3×3 orthogonal matrix.
  (a) Prove that there is some unit vector $v \in \mathbb{R}^3$ such that either $Uv = v$ or $Uv = -v$. [10]
  (b) Prove that there are vectors $t, u \in \mathbb{R}^3$ such that $\{t, u, v\}$ is an orthonormal basis of $\mathbb{R}^3$. [5]
  (c) Let $\hat{U}$ be the matrix of T_U with respect to the (ordered) basis (t, u, v). Show that $\hat{U}$ has one of the following forms; [15]

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & 0 \\ \sin(2\theta) & -\cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & 0 \\ \sin(2\theta) & -\cos(2\theta) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

  (d) Describe the linear transformations T_U and $T_{\hat{U}}$, in geometric terms, in each of the cases above. (You may write down a single description of what's going on, so long as it covers all the cases, and isn't ambiguous.) [10]

(11) Let S be a subspace of $\mathbb{C}^n$ and let $x \in \mathbb{C}^n$. Prove that the distance between $\mathbb{P}_S(x)$ and x is the smallest possible distance between x and any vector in S. Stated more formally, prove that

$$\|x - \mathbb{P}_S(x)\| \leq \|x - s\|$$

for all $s \in S$. (Hint; note that $x - s = (x - \mathbb{P}_S(x)) + (\mathbb{P}_S(x) - s)$.) [10]

(12) Let S be the $n \times n$ shift matrix. Find a polynomial in S and $S^\top$ which evaluates to [10]

$$\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & n-1 \end{pmatrix}$$