21-241 MATRICES AND LINEAR TRANSFORMATIONS
SUMMER I 2012
HOMEWORK 5

Definition. If V is a subspace of $\mathbb{C}^n$, we define the *orthogonal subspace of* V to be

$$V^\perp = \{x \in \mathbb{C}^n \mid \forall v \in V \; x \perp v\}$$

We say that subspaces V and W of $\mathbb{C}^n$ are *orthogonal*, and write $V \perp W$, if

$$\forall v \in V \; \forall w \in W \quad v \perp w$$

Definition. A square, *complex* matrix P is called a *projection matrix* if $P^2 = P = P^H$. (Note the difference with the earlier definition of projection.)

(1) Find an orthonormal basis for each of the following subspaces. [10 each]

$$(a) \quad \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 5 \end{pmatrix} \right\}$$

$$(b) \quad \operatorname{col}\begin{pmatrix} 0 & -2 & 1 \\ 3-2i & 0 & i \\ -1 & -i & 0 \\ 0 & 1 & 2 \\ 1+i & 1-i & 0 \end{pmatrix}$$

(2) Find the eigenvalues of the following matrices. For each eigenvalue, find an orthonormal basis for its associated eigenspace. (Feel free, in finding the roots of the characteristic polynomial, to use a calculator.)

$$(a) \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad [5]$$

(b) $$\begin{pmatrix} 0 & i & -1 \\ -i & 0 & i \\ -1 & -i & 0 \end{pmatrix}$$ [10]

(c) $$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$ [10]

(d) The $n \times n$ shift matrix. [10]

(3) Let P and Q be $n \times n$ projections. Show that the following are equivalent; [20]
 (i) $PQ = 0$.
 (ii) $\operatorname{ran}(P) \perp \operatorname{ran}(Q)$.
 (iii) $P + Q$ is a projection.

(4) Prove or disprove: $\operatorname{rank}(AB) = \operatorname{rank}(BA)$ for all square matrices A and B of the same size. [10]

(5) Prove that for all $x, y \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$,
 (a) If $x \perp y$ then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. [5]
 (b) $\|\lambda x\| = |\lambda| \, \|x\|$. [5]
 (c) $\|x + y\| \leq \|x\| + \|y\|$. (Hint: use the Cauchy-Schwartz inequality.) [10]

(6) Let S and T be subspaces of $\mathbb{C}^n$, and suppose $S \perp T$. Prove that $S \cap T = \{0\}$. [5]

(7) Let A be a matrix satisfying $A = A^H$. Prove that $\operatorname{spec}(A) \subseteq \mathbb{R}$. (Hint: look at the number $\langle Av, v \rangle$, where v is an eigenvector of A.) [10]