21-241 MATRICES AND LINEAR TRANSFORMATIONS
SUMMER I 2012
HOMEWORK 4

Definition. The $n \times n$ *shift* matrix is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

(Often I'll call it S, but I won't make that a convention.)

Definition. The *Hermitian* of an $m \times n$ complex matrix A is the $n \times m$ matrix A^H with entries $(A^H)_{ij} = \overline{A_{ji}}$. In other words A^H is the *conjugate transpose* of A.

Definition. Suppose $m_1, \ldots, m_k$ and $n_1, \ldots, n_\ell$ are natural numbers greater than zero. Let $m = m_1 + \cdots + m_k$ and $n = n_1 + \cdots + n_\ell$. Let A^{ij} be an m_i by n_j matrix, for each $i \leq k$ and $j \leq \ell$. Then *the block matrix with blocks* A^{ij} is the $m \times n$ matrix which looks like this;

$$\begin{pmatrix} A^{11} & A^{12} & \cdots & A^{1\ell} \\ A^{21} & A^{22} & \cdots & A^{2\ell} \\ \vdots & \vdots & & \vdots \\ A^{k1} & A^{k2} & \cdots & A^{k\ell} \end{pmatrix}$$

Definition. Let $\mathscr{A}$ be some set of $n \times n$ matrices. (In other words, $\mathscr{A} \subseteq M_n(\mathbb{C})$.) The *commutant* of $\mathscr{A}$ is

$$\mathscr{A}' = \{B \in M_n(\mathbb{C}) \mid \forall A \in \mathscr{A}\ AB = BA\}$$

Definition. An $n \times n$ matrix U is *upper triangular* if $U_{ij} = 0$ whenever $i > j$. In other words, U is upper triangular if it looks like this;

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & U_{nn} \end{pmatrix}$$

(1) Find bases for the following subspaces. Here S denotes the $n \times n$ shift matrix. Of course, you must show your work in each case. [5 each]

(a) The span of

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ -3 \\ -6 \end{pmatrix} \right\}$$

(b) col(A), where

$$A = \begin{pmatrix} 1 & i & -i & 0 \\ -i & 2 & 1 & 2-i \\ i & 1 & 3 & -1 \\ 0 & 2+i & -1 & 4 \end{pmatrix}$$

(c) null($S^\top S$). (You should work out what $S^\top S$ is.)

(d) null($SS^\top$). (You should work out what $SS^\top$ is.)

(2) Given each set X and vector v below, (i) find a subset of X which is a basis for span(X), (ii) decide whether v is in span(X), and (iii) find a linear combination of your basis vectors which produces v, if it is. [10 each]

(a) $X = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\} \quad v = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

(b) $X = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 5 \\ 0 \end{pmatrix} \right\} \quad v = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$

(3) Find the determinant of each matrix below. You may use your work from other problems if it's relevant. [5 each]

(a) $\begin{pmatrix} 1 & i & -i & 0 \\ -i & 2 & 1 & 2-i \\ i & 1 & 3 & -1 \\ 0 & 2+i & -1 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & i & -1 \\ i & -1 & -i \\ -1 & -i & 1 \end{pmatrix}$

(4) Let A be an $n \times n$ complex matrix such that $A = -A^\top$. Prove that if n is odd, then $\det(A) = 0$. [5]

(5) Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $M' = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ be block matrices, where each of A, B, C, D and A', B', C', D' has size $n \times n$. Prove the following formula for the product of M and M'. [10]

$$MM' = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix}$$

(6) Let A be the block matrix $\left(\begin{smallmatrix} I_n \\ 0_{k\times n} \end{smallmatrix}\right)$. Show that [10]

$$A^\top A = I_n \qquad \text{and} \qquad AA^\top = \begin{pmatrix} I_n & 0_{n\times k} \\ 0_{k\times n} & 0_{k\times k} \end{pmatrix}$$

(7) Let $n \geq 1$ be a natural number. Find the commutant of each of the following sets of matrices.

(a) $\mathscr{A} = \{S\}$, where S is the $n \times n$ shift matrix. [10]

(b) $\mathscr{B} = \{A \in M_n(\mathbb{C}) \mid A \text{ is invertible}\}$. [10]

(c) The set [10]

$$\mathscr{C} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in M_n(\mathbb{C}) \right\}$$

(So $\mathscr{C} \subseteq M_{2n}(\mathbb{C})$.)

(8) Let A be an $m \times m$ matrix and B an $n \times n$ matrix. Prove that [15]

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A)\det(B)$$

(9) Let U be an $n \times n$ upper triangular matrix. Prove that $\det(U) = U_{11} \cdots U_{nn}$. [15]