21-241 MATRICES AND LINEAR TRANSFORMATIONS
SUMMER I 2012
HOMEWORK 3

Definition. The *transpose* $A^\top$ of an $m \times n$ matrix A is the $n \times m$ matrix with entries $(A^\top)_{ij} = A_{ji}$. A square matrix P is called a *projection* if $P^2 = P$ and $P^\top = P$.

(1) Show that $(AB)^\top = B^\top A^\top$ for all matrices A and B for which AB is defined. [10]
(2) Prove that a square matrix A is invertible if and only if $A^\top$ is invertible. [10]
(3) Prove the following theorem from class. [15]

Theorem. *Let $a_1, \dots, a_n \in \mathbb{R}^m$ and let A be the $m \times n$ matrix whose columns are $a_1, \dots, a_n$. Then the following are equivalent;*
(a) $\operatorname{span}\{a_1, \dots, a_n\} = \mathbb{R}^m$,
(b) A is right-invertible.

(4) Let P be an $n \times n$ diagonal matrix whose diagonal values are all either zero or one. Prove that P is a projection, and find bases for $\operatorname{ran}(P)$ and $\operatorname{null}(P)$. [10]
(5) Show that there are 2×2 projections P and Q such that $PQ \neq QP$, and $\operatorname{tr}(P) = \operatorname{tr}(Q)$. [10]
(6) Find bases for $\operatorname{null}(A)$ and $\operatorname{ran}(A)$, where A is the following matrix; [10]

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 2 & 1 & 0 & 1 \\ 0 & 4 & 5 & 2 \end{pmatrix}$$

(7) Find a basis for the subspace of $\mathbb{R}^3$ spanned by the following set of vectors; [10]

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 5 \\ 8 \end{pmatrix} \right\}$$

(8) Let A be an $n \times n$ matrix. Show that the sequence of integers

$$d_t = \dim \operatorname{null}(A^t)$$

is nondecreasing and eventually constant. Show that if d is the eventual value of this sequence, then $d_k = d$ for all $k \geq n$. (That is, the sequence reaches this eventual value at $t = n$ at the latest.) [20]