HOMEOMORPHISMS OF ČECH-STONE REMAINDERS: THE ZERO-DIMENSIONAL CASE

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Abstract. We prove the result announced in [2, Theorem 4.10.1]: TA and $MA_{\aleph_1}$ together imply that given any two locally compact, zero-dimensional Polish spaces, any homemomorphism between their Čech-Stone remainders is trivial. It follows that two such spaces have homeomorphic remainders if and only if they have cocompact subspaces which are homeomorphic.

1. Introduction

The Čech-Stone remainder $\beta X \setminus X$ of a topological space X is denoted X^*. A continuous map $\varphi : X^* \to Y^*$ is called *trivial* if there is a continuous $e : X \to Y$ such that $\varphi = e^*$, where $e^* = \beta e \setminus e$ and βe is the unique continuous exension of e to βX. It follows that two remainders X^* and Y^* are homeomorphic via a trivial map if and only if there are cocompact subspaces of X and Y which are themselves homeomorphic. In this paper we prove the following, originally announced in [2, Theorem 4.10.1];

Theorem 1. *Assume $TA + MA_{\aleph_1}$. Let X and Y be locally compact, noncompact and zero-dimensional Polish spaces. Then every every homeomorphism between X^* and Y^* is trivial.*

Here TA abbreviates *Todorčević's Axiom* (also widely known as the *Open Coloring Axiom*, see [8]). $MA_{\aleph_1}$ is the usual Martin's Axiom for $\aleph_1$-many dense sets. Both are consequences of the *Proper Forcing Axiom* (PFA); hence the above result proves a special case of the more general conjecture that PFA implies all homeomorphisms between Čech-Stone remainders of locally compact, noncompact Polish spaces are trivial. In comparison, under the Continuum Hypothesis, all Čech-Stone remainders of locally compact, noncompact, zero-dimensional Polish spaces are homeomorphic (a consequence of Parovičenko's theorem). $TA+MA_{\aleph_1}$ thus implies a certain rigidity for such remainders, whereas CH implies the opposite.

Theorem 1 follows a long line of results going back to the late 70's when Shelah proved that, consistently, all autohomeomorphisms of ω^* are trivial ([5]). Shelah and Steprans later showed that the same conclusion holds under PFA ([6]) and Veličković improved their result by reducing the assumption to $TA+MA_{\aleph_1}$. The first author ([2]) extended this by proving Theorem 1 in

The first author was partially supported by NSERC..

the case where both X and Y are countable. All of these results rely heavily on the zero-dimensionality of the spaces X and Y; indeed, all results in this direction in fact deal with isomorphisms between Boolean algebras of the form $\mathscr{C}(X)/\mathscr{K}(X)$, where $\mathscr{C}(X)$ is the algebra of clopen subsets of X, and $\mathscr{K}(X)$ its ideal of compact-open sets. Stone duality provides the connection to X^* in the case where X is zero-dimensional (see e.g. [1]). Our proof does not differ in this regard.

In section 2 we introduce some of the language required to prove Theorem 1. Section 3 treats embeddings of $\mathscr{P}(\omega)/\text{fin}$ into $\mathscr{C}(X)/\mathscr{K}(X)$; we prove in ZFC that such maps are trivial whenever they are "definable" in a certain sense, and then we prove under $TA + MA_{\aleph_1}$ that every such map is trivial. Section 4 completes the proof of Theorem 1 through an analysis of *coherent families of continuous functions.*

2. Notation

Fix a zero-dimensional, locally compact and noncompact Polish space X. We denote by $\mathscr{C}(X)$ the Boolean algebra of clopen subsets of X, and by $\mathscr{K}(X)$ its ideal of compact-open subsets of X. Let $\langle K_n \mid n < \omega\rangle$ be an increasing sequence of compact-open sets in X, such that $X = \bigcup K_n$. Then $\mathscr{K}(X)$ is generated by the sequence $\langle K_n \mid n < \omega\rangle$, ie,

$$K \in \mathscr{K}(X) \iff \exists n\, K \subseteq K_n$$

It is easy to see that $\mathscr{C}(X)$ has size continuum, whereas $\mathscr{K}(X)$ is countable. Let $X_0 = K_0$ and $X_{n+1} = K_{n+1} \setminus K_n$. When $A, B \in \mathscr{C}(X)$ are distinct, we write $\delta(A, B)$ for the least n such that $A \cap X_n \neq B \cap X_n$. If

$$d(A,B) = \begin{cases} 2^{-\delta(A,B)} & A \neq B \\ 0 & A = B \end{cases}$$

then d is a Polish metric on $\mathscr{C}(X)$. In this topology, $\mathscr{K}(X)$ is an F_σ subset of $\mathscr{C}(X)$. We will often identify $\mathscr{C}(X)$ with $\prod_n \mathscr{C}(X_n)$, and $\mathscr{P}(\omega)$ with ${}^\omega 2$. Under these identifications, $\mathscr{K}(X)$ maps to $\bigoplus_n \mathscr{C}(X_n)$ (the set of functions in $\prod_n \mathscr{C}(X_n)$ which are nonempty on only finitely many coordinates) and fin to ${}^{<\omega}2$. If Y and Z are zero-dimensional, locally compact Polish spaces, $\varphi : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(Z)/\mathscr{K}(Z)$ is a homomorphism, and $U \in \mathscr{C}(Y)$, then we write $\varphi \restriction U$ for the restriction $\varphi \restriction \mathscr{C}(U)/\mathscr{K}(U)$.

Finally, we state *Todorčević's Axiom.* Let E be a separable metric space and let $[E]^2 = M_0 \cup M_1$ be a partition of the unordered pairs on E, such that M_0 is open when identified with a symmetric subset of $E \times E$ minus the diagonal. Then one of the following holds.

(1) There is an uncountable set $H \subseteq E$ such that $[H]^2 \subseteq M_0$.
(2) There are sets $H_n \subseteq E$, for $n < \omega$, such that $E = \bigcup H_n$ and for each n, $[H_n]^2 \subseteq M_1$.

3. Embeddings of $\mathscr{P}(\omega)/\operatorname{fin}$ into $\mathscr{C}(X)/\mathscr{K}(X)$

Let $e : X \to \omega$ be a continuous map. If $e^{-1}(n)$ is compact for every n, then we say e is *compact-to-one*. If e is compact-to-one, then the map $a \mapsto e^{-1}(a)$, from $\mathscr{P}(\omega)$ to $\mathscr{C}(X)$, induces a homomorphism $\varphi_e : \mathscr{P}(\omega)/\operatorname{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$. Moreover, φ_e is injective if and only if e is finite on compact sets. We call a homomorphism $\varphi : \mathscr{P}(\omega)/\operatorname{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$ *trivial* if it is of the form φ_e for some compact-to-one, continuous e.

Lemma 3.1. *Suppose $Y \in \mathscr{C}(X)$ and $e, f : Y \to \omega$ are continuous, compact-to-one maps, such that $e^{-1}(a)\Delta f^{-1}(a)$ is compact for every $a \subseteq \omega$. Then $\{x \in Y \mid e(x) \neq f(x)\}$ is compact.*

Proof. Suppose not; then for some infinite set $I \subseteq \omega$ and all $n \in I$, there is a point $x_n \in Y \cap X_n$ such that $e(x_n) \neq f(x_n)$. Since e and f are compact-to-one, we may assume also that $m \neq n$ implies $e(x_m) \neq e(x_n)$ and $f(x_m) \neq f(x_n)$. Now define a coloring $F : [I]^2 \to 3$ by

$$F(\{m < n\}) = \begin{cases} 0 & e(x_m) \neq f(x_n) \wedge f(x_m) \neq e(x_n) \\ 1 & e(x_m) = f(x_n) \wedge f(x_m) \neq e(x_n) \\ 2 & e(x_m) \neq f(x_n) \wedge f(x_m) = e(x_n) \end{cases}$$

By Ramsey's theorem, there is an infinite set $a \subseteq I$ which is homogeneous for this coloring. Suppose first that a is 1-homogeneous, and let $m < n < k$ be members of a. Then

$$e(x_m) = f(x_n) \quad \text{and} \quad e(x_m) = f(x_k) \quad \text{and} \quad e(x_n) = f(x_k)$$

which implies $e(x_n) = f(x_n)$, a contradiction. Similarly, a cannot be 2-homogeneous.

Now suppose a is 0-homogeneous. Let $a = a_0 \cup a_1$ be a partition of a into two infinite sets, and put $Z_i = \{x_n \mid n \in a_i\}$ and $Z = \{x_n \mid n \in a\} = Z_0 \cup Z_1$. From the homogeneity of a, it follows that $e''Z \cap f''Z = \emptyset$, and hence (as e and f are injective on Z)

$$Z \cap e^{-1}((e''Z_0) \cup (f''Z_1)) = Z_0 \quad \text{and} \quad Z \cap f^{-1}((e''Z_0) \cup (f''Z_1)) = Z_1$$

So, if $b = e''Z_0 \cup f''Z_1$, we have $Z \subseteq e^{-1}(b)\Delta f^{-1}(b)$. But Z is not compact, so this is a contradiction. □

3.1. **Definable embeddings.**

Lemma 3.2. *Suppose $\varphi : \mathscr{P}(\omega)/\operatorname{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$ is an embedding with a continuous lift $F : G \to \mathscr{C}(X)$ on some comeager set $G \subseteq \mathscr{P}(\omega)$. Then φ is trivial.*

Proof. First we work with the case $G = \mathscr{P}(\omega)$. For $s \in {}^{<\omega}2$ and $S \in \bigoplus \mathscr{C}(X_n)$ let

$$N_s = \{a \in 2^\omega \mid s \subseteq a\} \quad \text{and} \quad N_S = \left\{A \in \prod \mathscr{C}(X_n) \;\middle|\; A \text{ extends } S\right\}$$

So N_s and N_S are basic clopen sets for ${}^\omega 2$ and $\mathscr{C}(X)$ respectively. If $F''N_s \subseteq N_S$ we say that s *forces* S.

First we build, by induction on i, an increasing sequence n_i of integers, and a sequence of functions $t_i : [n_i, n_{i+1}) \to 2$, such that

(1) for all i, if $s \in {}^{n_i}2$, then $s \cup t_i$ forces some S of length n_i,
(2) for all i, $s, s' \in {}^{n_i}2$, $k > n_{i+1}$, and $u : [n_{i+1}, k) \to 2$, if $s \cup t_i \cup u$ and $s' \cup t_i \cup u$ force S and S' respectively, then $S \Delta S' \subseteq K_{n_{i+1}}$.

Assume n_i and t_{i-1} are defined. First, let s_j, $j < 2^{n_i}$ enumerate ${}^{n_i}2$, and construct functions t_i^j for $j \leq 2^{n_i}$, such that t_i^j has domain $[n_i, k)$ for some k, $t_i^j \subseteq t_i^{j+1}$ for all j, and $s_j \cup t_i^j$ forces some S of length n_i (this last is possible by continuity of F). Now any t which extends $t_i^{2^{n_i}}$ and has domain disjoint from n_i will satisfy (1) in place of t_i.

As for condition (2), we first claim that for any $s, s' \in {}^{n_i}2$ and any $u : [n_i, n) \to 2$, where $n > n_i$, there is some $v : [n_i, k) \to 2$ extending u, such that whenever $s \cup v \cup w$ and $s' \cup v \cup w'$ force S and S' respectively, then $S \Delta S' \subseteq K_n$. Assume otherwise. Then we can construct increasing sequences $u_j : [n_i, j) \to 2$, $S_j, S_j' \in \mathscr{K}(X)$, and $\ell_j < \omega$, such that $s \cup u_j$ and $s' \cup u_j$ force S_j and S_j' respectively, but $S_j \cap X_{\ell_j} \neq S_j' \cap X_{\ell_j}$. Put $x = s \cup \bigcup_j u_j$ and $x' = s' \cup \bigcup_j u_j$. Then $x =^* x'$ but $F(x) \cap X_{\ell_j} \neq F(x') \cap X_{\ell_j}$ for all j. Now to ensure condition (2), we apply the claim for each pair $s, s' \in 2^{n_i}$ in some order, starting with $u = t_i^{2^{n_i}}$ from above and repeatedly extending u via the v as in the claim. We end with a function $t_i : [n_i, n_{i+1}) \to 2$ as required.

Put $a^\epsilon = \bigcup \{[n_i, n_{i+1}) \mid i \equiv \epsilon \pmod 3\}$ and $x^\epsilon = \bigcup \{t_i \mid i \equiv \epsilon \pmod 3\}$, for $\epsilon = 0, 1, 2$. For every $x \subseteq a^0$, let

$$F^0(x) = F(x \cup x^1 \cup x^2) \setminus F(x^1 \cup x^2)$$

Then $F^0(x) \Delta F(x)$ is compact, for every $x \subseteq a^0$. Moreover, by the properties of the sequence t_i, there are functions $h_i^0 : \mathscr{P}([n_{3i}, n_{3i+1})) \to \mathscr{C}(K_{n_{3i+2}} \setminus K_{n_{3i-1}})$ such that for all $x \subseteq a^0$,

$$F^0(x) = \bigcup_i h_i^0(x \cap [n_{3i}, n_{3i+1}))$$

Now we claim that for almost all i, h_i^0 is a (Boolean algebra) homomorphism. To see this, suppose for instance that for infinitely many i, there are $u_i, v_i \subseteq [n_{3i}, n_{3i+1})$ such that $h_i^0(u_i \cup v_i) \neq h_i^0(u_i) \cup h_i^0(v_i)$. Put $u = \bigcup u_i$ and $v = \bigcup v_i$; then $F^0(u \cup v) \Delta (F^0(u) \cup F^0(v))$ has nonempty intersection with infinitely many X_n, and hence is not compact, contradicting the fact that φ is a homomorphism. Let $A_i^0 = h_i^0([n_{3i}, n_{3i+1}))$; then there is a continuous map $e_i^0 : A_i^0 \to [n_{3i}, n_{3i+1})$ such that $h_i^0(x) = (e_i^0)^{-1}(x)$ for all x in the domain of h_i^0. Notice that the sets A_i^0 are pairwise disjoint; so if $A^0 = \bigcup_i A_i^0$ and $e^0 = \bigcup_i e_i^0$, then $e^0 : A^0 \to a^0$, and

$$\forall x \in \mathscr{P}(a_0) \quad F^0(x) = (e^0)^{-1}(x)$$

Similarly, we may define F^ϵ, A^ϵ, and e^ϵ for $\epsilon = 1, 2$. Notice that since $A^\epsilon \Delta F(a^\epsilon)$ is compact, and the sets a^ϵ form a partition of ω, it follows that $A^\delta \cap A^\epsilon$ is compact for $\delta \neq \epsilon$, and $A^0 \cup A^1 \cup A^2$ is cocompact. Hence by putting the functions e^ϵ together on a cocompact set, we obtain a function $e : X \to \omega$ such that for all $x \subseteq \omega$, $F(x)\Delta e^{-1}(x)$ is compact.

This completes the proof when $G = \mathscr{P}(\omega)$. For the general case, recall (by a Theorem of Talagrand and Jalali-Naini, see [7, 3]) that there are a partition $\omega = a_0 \cup a_1$, and sets $s_i \subseteq a_i$, such that for all $x \subseteq a_i$, $x \cup s_{1-i} \in G$. Hence the function $F_i(x) = F(x \cup s_{1-i}) \setminus F(s_{1-i})$ induces φ on $\mathscr{P}(a_i)$. By the special case, then, we get functions $e_0 : Y_0 \to a_0$ and $e_1 : Y_1 \to a_1$ (where $Y_i = F(a_i)$) inducing φ on a_0 and a_1 respectively. Then $Y_0 \cap Y_1$ and $(Y_0 \cup Y_1)\Delta X$ are compact, and hence we may put together e_0 and e_1 on a cocompact set to get a function $e : X \to \omega$ satisfying our requirements. $\square$

Lemma 3.3. *Let $\varphi : \mathscr{P}(\omega)/\mathrm{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$ be an embedding with a lift $F : \mathscr{P}(\omega) \to \mathscr{C}(X)$. Suppose that there are Borel functions $F_n : \mathscr{P}(\omega) \to \mathscr{C}(X)$, for $n < \omega$, such that for all $a \subseteq \omega$ there is n with $F(a)\Delta F_n(a) \in \mathscr{K}(X)$. Then φ is trivial.*

Proof. Define
$$\mathscr{I} = \{a \subseteq \omega \mid \varphi \restriction a \text{ is trivial}\}$$
Then $\mathscr{I}$ is an ideal containing the ideal of finite sets. For each $a \in \mathscr{I}$, we fix a continuous, compact-to-one map $e_a : F(a) \to a$ which induces $\varphi \restriction a$. We also define, for such a, the function $f_a : a \to \mathscr{C}(F(a))$ given by
$$f_a(n) = e_a^{-1}(\{n\})$$
Clearly, e_a is uniquely determined by f_a.

Claim 3.1. $\mathscr{I}$ is not a maximal nonprincipal ideal.

Proof. Assume otherwise. Fix a dense G_δ subset W of $\mathscr{P}(\omega)$, such that each F_n is continuous on W. As usual, we may find a partition $\omega = a_0 \cup a_1$ into infinite sets, along with sets $t_0 \subseteq a_0$ and $t_1 \subseteq a_1$, such that for all $x \subseteq a_i$, $x \cup t_{1-i} \in W$. By the assumption, one of a_0 or a_1 is not in $\mathscr{I}$; without loss of generality, say it's a_0. Now, the function
$$G_n(x) = F_n(x \cup t_1) \cap F_n(a_0)$$
is continuous on $\mathscr{P}(a_0)$, and moreover for every $x \subseteq a_0$ there is some n such that $G_n(x)\Delta F(x)$ is compact. Let $\mathscr{J} = \mathscr{I} \cap \mathscr{P}(a_0)$. Fix $a \in \mathscr{J}$ and for each $n, m < \omega$ let
$$D^a_{n,m} = \left\{x \subseteq a \;\middle|\; e_a^{-1}(x) \setminus K_m = G_n(x) \setminus K_m\right\}$$
Then each $D^a_{n,m}$ is closed, and $\mathscr{P}(a) = \bigcup_{n,m} D^a_{n,m}$. By the Baire category theorem, it follows that there are some $n, m < \omega$ and a nonempty clopen subset U of $\mathscr{P}(a)$ such that $U \subseteq D^a_{n,m}$. Let H_n, $n < \omega$ enumerate all functions from $\mathscr{P}(a_0)$ to $\mathscr{C}(X)$ of the form
$$x \mapsto (G_\ell((x \setminus j) \cup t) \setminus K_m) \cup s''(x \cap k)$$
where $j, k, \ell, m < \omega$, $t \subseteq j$, and $s : k \to \mathscr{C}(K_m)$. Then it follows that each H_n is continuous, and for every $a \in \mathscr{J}$ there is some n such that
$$\forall x \subseteq a \quad H_n(x) = e_a^{-1}(x)$$

Let $\mathscr{J}_n$ be the ideal of all $a \in \mathscr{J}$ for which the above holds. Then for any $a, b \in \mathscr{J}_n$, we have $f_a \restriction a \cap b = f_b \restriction a \cap b$, and so the function

$$f_n = \bigcup_{a \in \mathscr{J}_n} f_a$$

induces φ on every $a \in \mathscr{J}_n$. If $\mathscr{J}_n$ is cofinal in $\mathscr{J}$ with respect to $\subseteq^*$, then it follows that f_n induces φ on a_0, contradicting $a_0 \notin \mathscr{I}$.

If no $\mathscr{J}_n$ is cofinal in $\mathscr{J}$, then $\mathscr{J}$ can't be countably directed, and it follows that there is a partition $a_0 = \bigcup_n b_n$ such that $b_n \in \mathscr{J}$ for all n, but there is no $b \in \mathscr{J}$ such that $b_n \subseteq^* b$ for all n. Let $\mathscr{U}$ be the set of all $b \subseteq a_0$ such that $b \cap b_n =^* \emptyset$ for all n; then $\mathscr{U}$ is a countably-directed subideal of $\mathscr{J}$. If $\mathscr{U}_n = \mathscr{J}_n \cap \mathscr{U}$, then there is some n for which $\mathscr{U}_n$ is cofinal in $\mathscr{U}$. As above, we let

$$f = \bigcup_{a \in \mathscr{U}_n} f_a$$

and

$$e(x) = k \iff x \in f(k)$$

and it follows that e induces φ on every $a \in \mathscr{U}$. Now consider the set

$$T = \{m < \omega \mid e \restriction F(b_m) \text{ does not induce } \varphi \restriction b_m\}$$

Suppose T is infinite. Then for each $m \in T$ we may choose some infinite $c_m \subseteq b_m$ such that $e^{-1}(c_m) \cap F(c_m)$ is compact; moreover, by shrinking c_m we may ensure that $e^{-1}(c_m) \cap F(c_k)$ is compact for every $m, k \in T$. We can then find some D such that $F(c_m) \setminus D$ and $e^{-1}(c_m) \cap D$ are compact for all m. Choose some c such that $F(c) \Delta D$ is compact. Then $c_m \subseteq^* c$ for every m, since $F(c_m) \setminus D$ is compact for all m. So we may choose some $i_m \in c_m \cap c$ such that $e(i_m) \notin D$. Then the set

$$b = \{i_m \mid m \in T\}$$

is in $\mathscr{U}$. Hence $e^{-1}(b) \Delta F(b)$ is compact, and since $b \subseteq c$, $e^{-1}(b) \setminus F(c)$ must be compact as well. But $e^{-1}(b) \cap D = \emptyset$, a contradiction.

Suppose now that T is finite. Then e induces φ on every a in the ideal generated by $\mathscr{U}$ and $\{b_m \mid m \notin T\}$. This ideal is dense in the powerset of $\bigcup_{m \notin T} b_m$ and it follows that e induces φ on this set. But this means $a_0 \in \mathscr{I}$, another contradiction.

□

Now by induction we build subsets a_n and x_n of ω, for $n < \omega$, such that

1. (1) $a_n \cap a_m = \emptyset$ for $n \neq m$,
2. (2) $x_n \subseteq a_n$,
3. (3) φ is nontrivial on $\omega \setminus \bigcup_{i<n} a_i$,
4. (4) for every $x \subseteq \omega \setminus \bigcup_{i<n} a_i$,

$$\left(F_n\left(\bigcup_{i<n} x_i \cup x\right) \cap F(a_n)\right) \Delta F(x_n) \notin \mathscr{K}(X)$$

The existence of such a sequence clearly contradicts our assumption, since if $x = \bigcup_i x_i$, then it follows that for every n, $(F_n(x) \cap F(a_n))\Delta F(x_n)$ is not compact.

Suppose a_i and x_i, for $i < n$, have been constructed so as to satisfy the conditions above. Let $c_n = \omega \setminus \bigcup_{i<n} a_i$, and $z_n = \bigcup_{i<n} x_i$. Since φ is nontrivial on c_n, by Claim 3.1 it follows that there are infinite sets d_n and $\bar{d}_n$ which partition c_n and on both of which φ is nontrivial. For each $y \subseteq d_n$, put

$$H_n(y) = \left\{x \subseteq \bar{d}_n \;\middle|\; (F_n(z_n \cup y \cup x) \cap F(d_n))\Delta F(y) \in \mathscr{K}(X)\right\}$$

Then $H_n(y)$ is a Borel set.

Claim 3.2. There is some $y \subseteq d_n$ such that $H_n(y)$ is not comeager.

Proof. Suppose otherwise. Then for all $(y, Y) \in \mathscr{P}(d_n) \times \mathscr{C}(X)$, $\varphi[y] = [Y]$ if and only if the set

$$\left\{x \subseteq \bar{d}_n \;\middle|\; (F_n(z_n \cup y \cup x) \cap F(d_n))\Delta Y \in \mathscr{K}(X)\right\}$$

is comeager. Then $\mathrm{Gr}(\varphi \restriction d_n)$ is analytic. By the Jankov-von Neumann theorem ([4]), there is a uniformization of $\mathrm{Gr}(\varphi \restriction d_n)$ which is C-measurable, and hence continuous on a comeager set. By Lemma 3.2, φ is trivial on d_n, a contradiction. □

Fix $y \subseteq d_n$ so that $H_n(y)$ is not comeager. Since $H_n(y)$ is Borel, there is a basic clopen set N_s in $\mathscr{P}(\bar{d}_n)$ such that $H_n(y)$ is meager in N_s. Let $u \subseteq \bar{d}_n$ be the domain of s. Then there is a partition $\bar{d}_n \setminus u = \bar{d}_n^0 \cup \bar{d}_n^1$ along with sets $t_i \subseteq \bar{d}_n^i$ such that for any $x \subseteq \bar{d}_n^i$, $s \cup x \cup t_{1-i}$ is not in $H_n(y)$. By Claim 3.1, φ must be nontrivial on one of $\bar{d}_n^0$ or $\bar{d}_n^1$; say it's $\bar{d}_n^i$. Set

$$a_n = d_n \cup u \cup \bar{d}_n^{1-i} \qquad x_n = y \cup s \cup t_i$$

This completes the induction, and hence the proof of the theorem. □

3.2. **Embeddings under $TA + MA_{\aleph_1}$.**

Theorem 2. *Assume $TA + MA_{\aleph_1}$, and suppose*

$$\varphi : \mathscr{P}(\omega)/\operatorname{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$$

is an embedding. Then φ is trivial.

Towards the proof of Theorem 2, we fix an embedding $\varphi : \mathscr{P}(\omega)/\operatorname{fin} \to \mathscr{C}(X)/\mathscr{K}(X)$, and an arbitrary lift $F : \mathscr{P}(\omega) \to \mathscr{C}(X)$ of φ. Again we consider the ideal

$$\mathscr{I} = \{a \subseteq \omega \mid \varphi \restriction a \text{ is trivial}\}$$

A family $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is called *almost disjoint* if for all distinct $a, b \in \mathscr{A}$, $a \cap b =^* \emptyset$. Such a family $\mathscr{A}$ is called *treelike* if there is some tree T on ω and a bijection $t : \omega \to {}^{<\omega}\omega$ under which each $a \in \mathscr{A}$ corresponds to a branch through T, and vice-versa.

Lemma 3.4. *Assume TA. Let $\mathscr{A}$ be an uncountable, tree-like, almost-disjoint family of subsets of ω. Then $\mathscr{I} \setminus \mathscr{A}$ is countable.*

Proof. Put

$$E = \{(a,b) \mid \exists c \in \mathscr{A}\ b \subseteq a \subseteq c\}$$

and for each $(a,b) \in E$, let $\sigma(a)$ be the unique element of $\mathscr{A}$ such that $a \subseteq \sigma(a)$. Since $\mathscr{A}$ is treelike, σ is a continuous map. Define a coloring $[E]^2 = M_0 \cup M_1$ by placing $\{(a,b),(\bar{a},\bar{b})\}$ in M_0 if and only if

(i) $\sigma(a) \neq \sigma(\bar{a})$,
(ii) $a \cap \bar{b} = \bar{a} \cap b$, and
(iii) $F(a) \cap F(\bar{b}) \neq F(\bar{a}) \cap F(b)$.

Then M_0 is open in the topology on E obtained by identifying (a,b) with $(a,b,F(a),F(b))$.

Claim 3.3. There is no uncountable, M_0-homogeneous subset H of E.

Proof. Assume H is such. Put

$$d = \bigcup_{(a,b)\in H} b$$

Then for all $(a,b) \in H$, $d \cap a = b$, and hence $(F(d) \cap F(a)) \Delta F(b)$ is compact. Then we may find an n, and sets $A, B \in \mathscr{C}(K_n)$, so that for uncountably many $(a,b) \in H$ we have

- $(F(d) \cap F(a)) \Delta F(b) \subseteq K_n$,
- $F(a) \cap K_n = A$, and
- $F(b) \cap K_n = B$.

Then, for any such (a,b) and $(\bar{a},\bar{b})$ in H,

$$F(a) \cap F(\bar{b}) \cap K_n = A \cap B = F(\bar{a}) \cap F(b) \cap K_n$$

and moreover,

$$(F(a) \cap F(\bar{b})) \setminus K_n = (F(a) \cap F(d) \cap F(\bar{a})) \setminus K_n = (F(b) \cap F(\bar{a})) \setminus K_n$$

Hence there is an uncountable M_1-homogeneous subset of H, a contradiction. □

By TA, there is a sequence E_n, $n < \omega$, of M_1-homogeneous sets which cover E. Let D_n be a countable, dense subset of E_n, in the topology on E described above. Fix $c \in \mathscr{A}$ so that c is not equal to $\sigma(a)$ for any $(a,b) \in D_n$, for any n. We'll show that $\varphi \restriction c$ is trivial.

Claim 3.4. There is a partition $c = c_0 \cup c_1$, so that for all n and $(a,b) \in E_n$, if $a \subseteq c_i$ for some $i < 2$ then for every m there is $(\bar{a},\bar{b}) \in D_n$ with

(1) $a \cap m = \bar{a} \cap m$ and $b \cap m = \bar{b} \cap m$,
(2) $F(a) \cap K_m = F(\bar{a}) \cap K_m$ and $F(b) \cap K_m = F(\bar{b}) \cap K_m$, and
(3) $a \cap \bar{b} = \bar{a} \cap b$.

Proof. For each i, fix an enumeration $\left\langle A_i^j \mid j < \omega \right\rangle$ of $\mathscr{C}(K_i)$. Fix also an enumeration $p \mapsto ((p)_0,(p)_1,(p)_2)$ of the triples in ω, so that $(p)_i \leq p$ for each p and $i < 3$.

Notice that if $(a,b) \in E_n$ then for any m there is some $(\bar{a},\bar{b}) \in D_n$ satisfying conditions (1) and (2) above, simply by density of D_n. Moreover, if $a \subseteq c$, then a and $\bar{a}$ must be almost disjoint (by choice of c), and hence $a \cap \bar{b}$ and $\bar{a} \cap b$ are both finite. This motivates the following definition.

Given m, we choose m^+ large enough that for all $n, p < m$ and $s, t \subseteq m$, if there is some $(a,b) \in E_n$ with

- $a \cap m = s$, $b \cap m = t$,
- $F(a) \cap K_{(p)_0} = A^{(p)_1}_{(p)_0}$, and $F(b) \cap K_{(p)_0} = A^{(p)_2}_{(p)_0}$,

then there is $(\bar{a},\bar{b}) \in D_n$ with the same properties, which moreover satisfies $\bar{a} \cap c \subseteq m^+$. Put $m_0 = 0$ and $m_{k+1} = m_k^+$ for each k, and set

$$c_0 = \bigcup_k c \cap [m_{2k}, m_{2k+1}) \qquad c_1 = \bigcup_k c \cap [m_{2k+1}, m_{2k+2})$$

Now suppose $(a,b) \in E_n$ and $a \subseteq c_0$, and let $m > n$ be given. Choose p so that $(p)_0 = m$, and

$$F(a) \cap K_m = A^{(p)_1}_m \qquad F(b) \cap K_m = A^{(p)_2}_m$$

Find k so that $p < m_{2k+1}$. Then there is $(\bar{a},\bar{b}) \in D_n$ such that

$$\bar{a} \cap m_{2k+1} = a \cap m_{2k+1} \qquad \bar{b} \cap m_{2k+1} = b \cap m_{2k+1}$$

and

$$F(\bar{a}) \cap K_m = F(a) \cap K_m \qquad F(\bar{b}) \cap K_m = F(b) \cap K_m$$

and, moreover, $\bar{a} \cap c \subseteq m_{2k+2}$. But then $\bar{a} \cap c_0 \subseteq m_{2k+1}$, and hence $a \cap \bar{b} = \bar{a} \cap b$ as required. □

Now define

$$F_n(b) = \bigcup \left\{ F(c_0) \cap F(\bar{b}) \;\middle|\; (\bar{a},\bar{b}) \in D_n \text{ and } c_0 \cap \bar{b} = \bar{a} \cap b \right\}$$

Then F_n is Borel. We claim moreover that, if $(c_0, b) \in E_n$, then $F_n(b) \Delta F(b) \in \mathscr{K}(X)$. To see this, first suppose $(\bar{a},\bar{b}) \in D_n$ and

$$c_0 \cap \bar{b} = \bar{a} \cap b$$

Then by M_1-homogeneity of E_n (since $\sigma(c_0) \neq \sigma(\bar{a})$), it follows that

$$F(c_0) \cap F(\bar{b}) = F(\bar{a}) \cap F(b)$$

Hence $F_n(b) \subseteq F(b)$. The claim above also implies that for any m there is such a pair $(\bar{a},\bar{b}) \in D_n$ with $F(b) \cap K_m = F(\bar{b}) \cap K_m$; hence

$$F_n(b) = F(c_0) \cap F(b)$$

Since $b \subseteq c_0$, we have $F(b) \setminus F(c_0) \in \mathscr{K}(X)$, so $F_n(b) \Delta F(b) \in \mathscr{K}(X)$. By Lemma 3.3 it follows that $c_0 \in \mathscr{I}$, and the same argument shows that $c_1 \in \mathscr{I}$. Then $c \in \mathscr{I}$, as required. □

Lemma 3.5. *Assume $TA + MA_{\aleph_1}$. If $\mathscr{I}$ is a dense P-ideal then φ is trivial.*

Proof. For each $a \in \mathscr{I}$, we fix some continuous, compact-to-one map $e_a : F(a) \to a$ which induces φ on a. We also define $f_a : \omega \to \mathscr{C}(X)$ by

$$f_a(n) = e_a^{-1}(\{n\})$$

Define a partition $[\mathscr{I}]^2 = M_0 \cup M_1$ by placing $\{a, b\} \in M_0$ if and only if there is some $n \in a \cap b$ such that $f_a(n) \neq f_b(n)$. Then M_0 is open when $a \in \mathscr{I}$ is identified with $f_a \in {}^\omega\mathscr{C}(X)$.

Claim 3.5. There is no uncountable, M_0-homogeneous subset H of $\mathscr{I}$.

Proof. Assume H is such a set, and that $|H| = \aleph_1$. Since $\mathscr{I}$ is a P-ideal, there is a set $\bar{H} \subseteq \mathscr{I}$ such that for every $a \in H$ there is some $b \in \bar{H}$ with $a \subseteq^* b$, and moreover $\bar{H}$ is a chain of order-type ω_1 with respect to $\subseteq^*$. By (a weakening of) TA, there is an uncountable subset of $\bar{H}$ which is homogeneous for one of the two colors M_0 and M_1; hence, by passing to this subset, we may assume $\bar{H}$ is either M_0 or M_1 homogeneous.

Say $\bar{H}$ is M_1-homogeneous. Put $\bar{a} = \bigcup \bar{H}$, and $\bar{f} = \bigcup_{a \in \bar{H}} f_a$. Then $\bar{f} : \bar{a} \to \mathscr{C}(X)$, and for all $a \in H$ we have $a \subseteq^* \bar{a}$ and $f_{\bar{a}} \restriction (a \cap \bar{a}) =^* f_a \restriction (a \cap \bar{a})$. Choose n so that for uncountably many $a \in H$, we have $a \setminus n \subseteq \bar{a}$, and $f_{\bar{a}} \restriction a \setminus n = f_a \restriction a \setminus n$. Then if $a, b \in H$ are such, and $f_a \restriction n = f_b \restriction n$, we have $\{a, b\} \in M_1$, a contradiction.

So $\bar{H}$ is M_0-homogeneous. Define a poset $\mathbb{P}$ as follows. Put $p \in \mathbb{P}$ if and only if $p = (A_p, m_p, H_p)$ where $m_p < \omega$, $A_p \in \mathscr{C}(K_{m_p})$, and $H_p \in [\bar{H}]^{<\omega}$, and for all distinct $a, b \in H_p$, there is an $n \in a \cap b$ such that

$$\neg(f_a(n) \cap A_p = \emptyset \iff f_b(n) \cap A_p = \emptyset)$$

That is, one of $f_a(n)$, $f_b(n)$ is disjoint from A_p, and the other isn't. Put $p \leq q$ if and only if $m_p \geq m_q$, $A_p \cap K_{m_q} = A_q$, and $H_p \supseteq H_q$.

First we must show that $\mathbb{P}$ is ccc. Suppose $\mathcal{X}$ is an uncountable subset of $\mathbb{P}$. We may assume without loss of generality that for some fixed m and $A \in \mathscr{C}(K_m)$, and for all $p \in \mathcal{X}$, $m_p = m$ and $A_p = A$, and moreover that H_p is the same size for all $p \in \mathcal{X}$. Let a_p be the minimal element of H_p under $\subseteq^*$, for each $p \in \mathcal{X}$. Find n_p so that for all $a \in H_p$,

$$f_{a_p} \restriction (a_p \setminus n_p) \subseteq f_a \qquad e''_{a_p} K_m \subseteq n_p$$

We may assume that for some fixed n, we have $n_p = n$ for all $p \in \mathcal{X}$. Find $p, q \in \mathcal{X}$ with $f_{a_p} \restriction n = f_{a_q} \restriction n$. Since $\{a_p, a_q\} \in M_0$, there is some $k \in a_p \cap a_q$ such that $f_{a_p}(k) \neq f_{a_q}(k)$. Then $k \geq n$, and so $f_{a_p}(k) \cap K_m = f_{a_q}(k) \cap K_m = \emptyset$. At least one of $f_{a_p}(k) \setminus f_{a_q}(k)$ and $f_{a_q}(k) \setminus f_{a_p}(k)$ must be nonempty; whichever one it is, call it B. Put $A_r = A \cup B$ and $H_r = H_p \cup H_q$, and choose m_r large enough that $A_r \subseteq K_{m_r}$. Then $r = (A_r, m_r, H_r) \in \mathbb{P}$, and $r \leq p, q$.

By $MA_{\aleph_1}$, there is a set $A \in \mathscr{C}(X)$ and an uncountable $H^* \subseteq \bar{H}$ such that for all distinct $a, b \in H^*$,

$$\exists n \in a \cap b \quad \neg(f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset)$$

Fix $x \subseteq \omega$ such that $F(x) = A$. Then for all $a \in H^*$, $e_a^{-1}(x \cap a)\Delta(A \cap F(a))$ is compact; hence there are k_a and m_a such that

$$e_a^{-1}(x \cap a \setminus k_a) = (A \cap F(a)) \setminus K_{m_a} \qquad \text{and} \qquad e_a^{-1}(a \setminus k_a) = F(a) \setminus K_{m_a}$$

Then, for all $n \in a \setminus k_a$, $n \in x$ implies $f_a(n) \subseteq A$, and $n \notin x$ implies $f_a(n) \cap A = \emptyset$. Fix distinct $a, b \in H^*$ with $k_a = k_b = k$, and $f_a \restriction k = f_b \restriction k$. Then,

$$\forall n \in a \cap b\ (f_a(n) \cap A = \emptyset \iff f_b(n) \cap A = \emptyset)$$

This contradicts the choice of A. □

Now by TA, there is a cover of $\mathscr{I}$ by countably many sets $\mathscr{I}_n$, each of which is M_1-homogeneous. Since $\mathscr{I}$ is a P-ideal, at least one of the $\mathscr{I}_n$'s must be cofinal in $\mathscr{I}$ with respect to $\subseteq^*$. Choose such an $\mathscr{I}_n$, and let $f = \bigcup\{f_a \mid a \in \mathscr{I}_n\}$. Then f is a function from some subset of ω to $\mathscr{C}(X)$. Setting $e(x) = n$ if and only if $x \in f(n)$, we get a function $e : X \to \omega$, and since $\mathscr{I}$ is dense and $\mathscr{I}_n$ cofinal in $\mathscr{I}$, $a \mapsto e^{-1}(a)$ is a lift for φ. □

4. Coherent families of continuous functions

Theorem 3. *Assume $TA + MA_{\aleph_1}$. Let X and Y be zero-dimensional, locally compact Polish spaces, and let $\varphi : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$ be an isomorphism. Then there are compact-open $K \subseteq X$ and $L \subseteq Y$, and a homeomorphism $e : X \setminus K \to Y \setminus L$, such that the map $A \mapsto e^{-1}(A)$ is a lift of φ.*

By Stone duality, a homeomorphism $\varphi : X^* \to Y^*$ induces an isomorphism $\hat{\varphi} : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$, and any map e as in the conclusion to Theorem 3 will in this case be a witness to the triviality of φ. Hence Theorem 3 implies Theorem 1.

Corollary 4.1. *Suppose $\varphi : \mathscr{C}(Y)/\mathscr{K}(Y) \to \mathscr{C}(X)/\mathscr{K}(X)$ is an isomorphism, with a lift $F : \mathscr{C}(Y) \to \mathscr{C}(X)$ which is Borel measurable. Then φ is trivial.*

Proof of Corollary 4.1. The assertion that φ is trivial is

$$\exists e \in C(X, Y)\ \forall U \in \mathscr{C}(Y)\ F(U)\Delta e^{-1}(U) \in \mathscr{K}(X)$$

which is visibly $\mathbf{\Sigma^1_2}$, and hence absolute between the ground model and any forcing extension. Since there is a forcing extension satisfying $TA + MA_{\aleph_1}$ ([8]), the result is proven. □

Before the proof of Theorem 3 we set down some more notation. Fix X, Y and φ as in the statement of the theorem. Let L_n be an increasing sequence of compact subsets of Y, with union Y, and let $Y_{n+1} = L_{n+1} \setminus L_n$ and $Y_0 = L_0$. Let $\mathscr{B}$ be a countable base for Y consisting of compact-open sets, such that

- for all $U \in \mathscr{B}$, the set of $V \in \mathscr{B}$ with $V \supseteq U$ is finite and linearly ordered by $\subseteq$, and

- for all $U \in \mathscr{B}$ and all $n < \omega$, either $U \subseteq Y_n$ or $U \cap Y_n = \emptyset$.

It follows that for all $U, V \in \mathscr{B}$, either $U \cap V = \emptyset$, $U \subseteq V$, or $V \subseteq U$. Let $\mathbb{P}$ be the poset of all partitions of Y into elements of $\mathscr{B}$, ordered by refinement;

$$P \prec Q \iff \forall U \in P \, \exists V \in Q \quad U \subseteq V$$

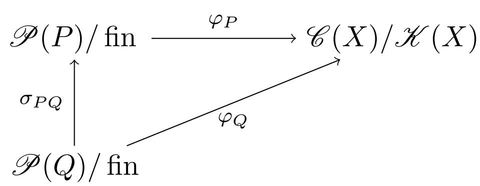
We also use $\prec^*$ to denote *eventual refinement*;

$$P \prec^* Q \iff \forall^\infty U \in P \, \exists V \in Q \quad U \subseteq V$$

When $P \prec^* Q$ we let $\Gamma(P, Q)$ be the least n such that every $U \in P$ disjoint from L_n is contained in some element of Q.

For a given $P \in \mathbb{P}$, let $s_P : Y \to P$ be the unique function satisfying $x \in s_P(x)$ for all $x \in Y$; similarly, when $P, Q \in \mathbb{P}$ and $P \prec Q$ we let $s_{PQ} : P \to Q$ be the unique function satisfying $U \subseteq s_{PQ}(U)$ for all $U \in P$. These maps induce embeddings $\sigma_P : \mathscr{P}(P)/\operatorname{fin} \to \mathscr{C}(Y)/\mathscr{K}(Y)$ and $\sigma_{PQ} : \mathscr{P}(Q)/\operatorname{fin} \to \mathscr{P}(P)/\operatorname{fin}$ in the usual way.

Proof of Theorem 3. For each $P \in \mathbb{P}$, let $\varphi_P = \varphi \circ \sigma_P$. Then φ_P is an embedding of $\mathscr{P}(P)/\operatorname{fin}$ into $\mathscr{C}(X)/\mathscr{K}(X)$. By Theorem 3, there is a continuous map $e_P : X \to P$ such that $a \mapsto e_P^{-1}(a)$ lifts φ_P. Note that if $P, Q \in \mathbb{P}$ and $P \prec^* Q$, then the following diagram commutes;



So by Lemma 3.1, the set $\{x \in X \mid s_{PQ}(e_P(x)) \neq e_Q(x)\}$ is compact.

Now let $[\mathbb{P}]^2 = M_0 \cup M_1$ be the partition defined by

$$\{P, Q\} \in M_0 \iff \exists x \in X \quad s_{P, P \vee Q}(e_P(x)) \neq s_{Q, P \vee Q}(e_Q(x))$$

Here $P \vee Q$ is the finest partition coarser than both P and Q. If we define $f_P : \mathscr{B} \to \mathscr{C}(X)$ by

$$f_P(U) = \{x \in X \mid e_P(x) \subseteq U\}$$

then we have

$$\{P, Q\} \in M_0 \iff \exists U \in \mathscr{B} \quad f_P(U) \neq f_Q(U)$$

and it follows that M_0 is open in the topology on $\mathbb{P}$ obtained by identifying P with f_P.

Claim 4.1. There is no uncountable, M_0-homogeneous subset of $\mathbb{P}$.

Proof. Suppose H is such, and has size $\aleph_1$. Using $MA_{\aleph_1}$ with a simple modification of Hechler forcing, we see that there is some $\bar{P} \in \mathbb{P}$ such that $P \succ^* \bar{P}$ for all $P \in H$. By thinning out H and refining a finite subset of $\bar{P}$, we may assume that $P \succ \bar{P}$ for all $P \in H$, and moreover that there is an $\bar{n}$ such that for all $P \in H$,

$$\left\{x \in X \;\middle|\; s_{\bar{P},P}(e_{\bar{P}}(x)) \neq e_P(x)\right\} \subseteq K_{\bar{n}}$$

Now fix $P, Q \in H$ such that $e_P \restriction K_{\bar{n}} = e_Q \restriction K_{\bar{n}}$. Then $s_{P,P\vee Q} \circ e_P = s_{Q,P\vee Q} \circ e_Q$, contradicting the fact that $\{P, Q\} \in M_0$. □

By TA, there is a countable cover of $\mathbb{P}$ by M_1-homogeneous sets; since $\mathbb{P}$ is countably directed under $\succ^*$, it follows that one of them, say $\mathbb{Q}$, is cofinal in $\mathbb{P}$. It follows moreover that for some n, we have

$$\forall P \in \mathbb{P}\; \exists Q \in \mathbb{Q} \quad \Gamma(Q, P) \leq n$$

That is, $\mathbb{Q}$ is cofinal in $\mathbb{P}$ under $\succ^n$ defined by

$$P \prec^n Q \iff \forall U \in P \;\; (U \cap L_n = \emptyset \implies \exists V \in Q \; U \subseteq V)$$

Claim 4.2. There is a compact set $K \subseteq X$ and a unique continuous map $e : X \setminus K \to Y$ satisfying

$$\forall x \in X \setminus K \quad e(x) \in \bigcap_{P \in \mathcal{Q}} e_P(x)$$

Proof. Fix $x \in X$. If $P, Q \in \mathbb{Q}$, then by M_1-homogeneity of $\mathbb{Q}$ we have

$$s_{P,P\vee Q}(e_P(x)) = s_{Q,P\vee Q}(e_Q(x))$$

Then, the unique member of $P \vee Q$ containing $e_P(x)$ is the same as the unique member of $P \vee Q$ containing $e_Q(x)$. It follows that $e_P(x) \cap e_Q(x) \neq \emptyset$, and so either $e_P(x) \subseteq e_Q(x)$ or vice-versa. Then the collection $\{e_P(x) \mid P \in \mathbb{Q}\}$ is a chain, and hence by compactness has nonempty intersection.

Now let

$$K = \{x \in X \mid \forall P \in \mathbb{Q}\; e_P(x) \subseteq L_n\} \subseteq \bigcap_{P \in \mathbb{Q}} e_P^{-1}(P \cap \mathscr{C}(L_n))$$

Then K is contained in a compact set. If $x \in X \setminus K$ and $P \in \mathbb{Q}$, then $e_P(x)$ is disjoint from L_n. Then for any $x \in X \setminus K$ and $\epsilon > 0$, there is some $P \in \mathbb{Q}$ such that $e_P(x)$ has diameter less than ϵ (since $\mathbb{Q}$ is cofinal in $\mathbb{P}$ under $\succ^n$). Thus e, as defined above, is unique.

To see that e is continuous, note that for any open $U \subseteq X$,

$$x \in e^{-1}(U) \iff \exists P \in \mathbb{Q} \quad e_P(x) \subseteq U$$

□

Claim 4.3. The map $U \mapsto e^{-1}(U)$ lifts φ.

Proof. Fix $P \in \mathbb{Q}$, and let $U \in P$. Then clearly, for all $x \in X \setminus K$, $e_P(x) = U$ if and only if $e(x) \in U$. Since there are only finitely many $U \in P$ such that one of $e_P^{-1}(\{U\})$ or $e^{-1}(U)$ meets K, it follows that

$$\forall^\infty U \in P \; e_P^{-1}(\{U\}) = e^{-1}(U)$$

Then $U \mapsto e^{-1}(U)$ lifts φ_P.

Now fix $A \in \mathscr{C}(Y)$. Then there is some $P \in \mathbb{P}$ such that A can be written as a union of a subset of P. Find $Q \in \mathbb{Q}$ with $Q \prec^* P$; then, up to a compact set, A can be written as a union of some subset a of Q. Hence,

$$\varphi[A] = \varphi_Q[a] = [e^{-1}(A)]$$

□

□

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