Solutions to Homework Set 6

1) A group is simple if it has no nontrivial proper normal subgroups. Let G be a simple group of order 168. How many elements of order 7 are there in G?

Solution: Observe that $168 = 2^3 \cdot 3 \cdot 7$. Every element of order 7 generates a cyclic group of order 7 so let us count the number of such subgroups: By the Sylow theorems, the number of subgroups of order 7 is $\equiv 1 \pmod 7$ and divides 168. The only options are 1 and 8. There cannot be only one Sylow 7-subgroup, since it would be normal, so there are 8 of them. Every two of these subgroups intersects in a subgroup whose order divides 7, so it must be the identity. Every other element in these subgroups must have order 7, so the number of elements of order 7 is $6 \cdot 8 = 48$.

2) Let G be a group of order 231. Prove that the (unique) Sylow 11-subgroup of G is in the center.

Solution: There are several ways to proceed. Here is a sketch of one proof: By arguments similar to the previous problem, there is a unique Sylow 7-subgroup L and a unique Sylow 11-subgroup M, and of course these are both normal. Now if we can produce a normal subgroup N of order $3 \cdot 7 = 21$ then the set MN is actually a group (why?) of order 231 so it must be G. Moreover, M must commute with N, since if $m \in M, n \in n$, then by normality of M, N, we have that $mnm^{-1}n^{-1} \in M \cap N = \{1\}$ so $mn = nm$. It is now easy to show that in fact M must commute with G (since every $g \in G$ can be written as $m'n'$). So it suffices to find a normal subgroup of order 21. To this end, let P be a Sylow 3-subgoup and consider PL. It is easy to show that this is a subgroup of order 21. To show that its normal, argue that its normalizer cannot have order 21 and then it must be all of G.

3) Prove that for every even integer $p \geq 2$, there exists a constant $c(p)$ such that any one-distance set with respect to the L_p-norm in R^n has at most $n^{c(p)}$ points.

Solution: Proceed as in class, by defining $f_i(x) = ||x - v_i||_p^p - \delta^p$, where δ is the distance, and v_i is the incidence vector for set A_i. Then these polynomials are linearly independent and they lie in a space spanned by 1 and x_i^j for $1 \leq i \leq n$ and $1 \leq j \leq p$. The numbe rof these is $1 + pn$.

4) Let λ be a nonzero integer, J be the $m \times m$ all ones matrix, and D be the $m \times m$ diagonal matrix with positive integer diagonal entries $\gamma_1, \ldots, \gamma_m$. Compute the determinant of $A = \lambda J + D$.

Solution: We use a trick called bordering: Extend A to an $(m+1)$ by $(m+1)$ matrix A' by adding a first rwo of all ones and a first column of all ones except the top left entry. Clearly $det(A) = det(A')$. Subtract λ times the first row from each row. Now use the diagonal entries to kill the first column and create an upper triangular matrix. The determinant is now the product of the diagonal. The result is

$$\gamma_1 \cdots \gamma_m \left(1 + \lambda \left(\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_m}\right)\right)$$