**Solutions to Homework Set 4**

1) Suppose that Eventown has fewer than $2^{\lfloor n/2 \rfloor}$ clubs. Prove that there is room for a new club without violating the Eventown rules.

**Solution:** It suffices to show that every maximal totally isotropic subspace of $F_2^n$ has dimension $\lfloor n/2 \rfloor$. Let $U$ be a totally isotropic subspace of dimension $\leq (n-2)/2$. So $dim(U^\perp) \geq 2 + dim(U)$. This means that we can find two vectors $u, v \in U^\perp$ such that no nontrivial linear combination of them belongs to $U$. If either $u$ or $v$ is isotropic, then adding them to $U$ contradicts maximality of $U$. Otherwise, $(u+v)\cdot(u+v) = u \cdot u + v \cdot v = 0$ and we can add $u+v$ to $U$.

2) Show that if $n$ is even, then there exist at least $2^{n(n+2)/8}/(n!)^2$ nonisomorphic solutions to the Oddtown problem of size $n$. Prove that for large $n$ this is greater than $2^{n^2/9}$.

**Solution:** Let $n = 2k$. Let $A$ be any symmetric $k$ by $k$ matrix with 0-1 entries, and let

$$B = \begin{pmatrix} A + I_k & A \\ A & A + I_k \end{pmatrix}.$$

Then it is easy to see that $BB^T = I_n$ (over the field of 2 elements), so $B$ is an oddtown incidence matrix. The number of of such $A$ is $2^{1+\cdots+k} = 2^{k(k+1)/2} = 2^{n(n+2)/8}$, since the entries in the lower half including the diagonal determine $A$. We must divide this by $(n!)^2$ to obtain a lower bound on the number of pairwise nonisomorphic matrices, since any permutation of the rows and columns does not change the isomorphism class. For large $n$, $2^{n(n+2)/8}/(n!)^2 > 2^{n^2/9}$, since

$$\left(\frac{n(n+2)}{8} - \frac{n^2}{9}\right)\ln 2 = \left(\frac{n^2}{72} + \frac{n}{4}\right)\ln 2 > (2 - o(1)) n \ln n.$$

(here you need the estimate $\log(n!) \sim n \log n$).

3) Let $V$ be a vector space of dimension $n$ over $K$. Let $V^{**}$ be the dual space of $V^*$. Give an explicit isomorphism between $V$ and $V^{**}$.

**Solution:** To each element $w \in V$, assign the element $g_v \in V^{**}$, defined by $g_v(f) = f(v)$, for all $f \in V^*$. It is standard to check that this is linear and 1-1. So $V$ is isomorphic to some subspace of $V^{**}$. But we know that $V$ and

$V^{**}$ have the same dimension, so they must be isomorphic. (I dont think this uses the existence of a scalar product on $V$)

4) Let $V$ be finite dimensional over $R$ with positive definite scalar product. Let $A$ be an operator on $V$. Show that the image of $A^T$ is the orthogonal space to the kernel of $A$.

**Solution:** For every $v \in V$ and $w \in ker(A)$ we have

$$\langle A^T v, w \rangle = \langle v, Aw \rangle = \langle v, 0 \rangle = 0$$

so $A^T(V) \subset ker(A)^{\perp}$. On the other hand, if $v \in A^T(V)^{\perp}$ and $x \in V$, then $\langle Av, x \rangle = \langle v, A^T x \rangle = 0$ so $Av = 0$ and $v \in ker(A)$. Thus $A^T(V)^{\perp} \subset ker(A)$. Taking the orthogonal complement of both sides yields the opposite containment.