Solutions to Homework Set 3

1) Let R be a commutative ring with unit element. Prove that $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$ is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent elements in R (an element is nilpotent if some power of it is zero).

Solution. First suppose that a_0 is a unit and all other a_i's are nilpotent, say $a_i^{n_i} = 0$ for all $i \geq 1$. Let $g(x) = f(x) - a_0$. Let $r = m \times \max_i n_i$. Consider the polynomial

$$h(x) = a_0^{-1} - g(x)a_0^{-2} + \cdots + (-1)^{r-1}g(x)^{r-1}a_0^{-r}.$$

Then $f(x)h(x) = 1 + (-1)^{r-1}g(x)^r a_0^{-r}$. Now a typical term in $g(x)^r$ is of the form $\prod_{j=1}^r a_{i_j}x^{i_j}$. Since the number of variables is at most m, the coefficient involves some term of the form $a_{i_j}^{r/m}$ and since $r/m \geq n_{i_j}$, this term is 0. Hence $g(x)^r = 0$ and $f(x)h(x) = 1$.

For the other direction, suppose that $f(x)$ is a unit. Let us prove by induction on m that a_0 is a unit and all the other a_i's are nilpotent. Since the constant term is 1, we clearly obtain that a_0 is a unit. Suppose that $f(x)g(x) = 1$, where $g(x) = b_0 + b_1x + \cdots + b_nx^n$. Then clearly $a_mb_n = 0$ and also $a_{m-1}b_n + b_{n-1}a_m = 0$. Multiplying this by a_m yields $b_{n-1}a_m^2 = 0$. The coefficient of x^{m+n-2} gives $a_{m-2}b_n + a_{m-1}b_{n-1} + a_mb_{n-2} = 0$. Multiplying this by a_m^2 gives $b_{n-2}a_m^3 = 0$. Continuing in this way we get $b_0a_m^{n+1} = 0$ which implies that a_m is nilpotent. Now we have $((f(x) - a_mx^m) + a_mx^m)g(x) = 1$ which gives $(f(x) - a_mx_m)g(x) = 1 - a_mx^mg(x)$. Since a_m is nilpotent, every coefficient of $1 - a_mx^mg(x)$ except the constant one is nilpotent. By what we previously showed, this polynomial is therefore a unit. Now by induction on m, all other coefficients of $f(x)$ except a_0 are nilpotent.

2) Let V be a finite dimensional vector space over the reals and $W = \{w_1, \ldots, w_m\}$ be an orthonormal set in V such that

$$\sum_{i=1}^m |\langle w_i, v\rangle|^2 = ||v||^2$$

for every $v \in V$. Prove that W is a basis of V.

Solution. Let $c_i = \langle v, w_i\rangle$. Then

$$||v||^2 = ||v - \sum_i c_iw_i||^2 + ||\sum_i c_iw_i||^2 = ||v - \sum_i c_iw_i||^2 + \sum_i c_i^2,$$

where the first equality holds because of the definition of c_i and the second because W is orthonormal. Now the hypothesis implies that $v - \sum_i c_i w_i = 0$ and so v is in the span of W. Since W is clearly a linearly independent set, it is a basis.

3) Let V be the set of real functions $y = f(x)$ satisfying

$$\frac{d^2y}{dx^2} + 9y = 0.$$

a) Prove that V is a two-dimensional real vector space.
b) In V, define

$$\langle u, v \rangle = \int_0^\pi uv\ dx.$$

Show that this defines an inner product on V and find an orthonormal basis for V.
Solution.
a) Let $z = dy/dx$. Then the equation $dz/dx + 9y = 0$ translates, by the chain rule, to $dz/dy \times dy/dx + 9y = 0$. Substituting z this gives $z(dz/dy) + 9y = 0$. Now we claim that all solutions $z = z(y)$ to this satisfy $z^2 = C - 9y^2$ for some constant C. Indeed, suppose z is a solution, then $d(z^2 + 9y^2)/dy = 2z(dz/dy) + 18y = 0$ and so clearly $z^2 + 9y^2 = C$. This gives us $z = \sqrt{C - 9y^2}$, or $dy/dx = \pm\sqrt{C - 9y^2}$. Solving this gives $\pm 3x + c' = \sin^{-1}(y/\sqrt{C})$ for some constant c' and thus $y = \sqrt{C}\sin(\pm 3x + c) = A \sin 3x + B \cos 3x$ for appropriate constants A, B. The uniqueness of this solution follows by differentiating $\sin^{-1}(y/\sqrt{C}) \pm 3x$ with respect to x and obtaining a constant as before. We have shown that every solution is a linear combination of $\sin 3x$ and $\cos 3x$. Since these two vectors are clearly linearly independent ($\tan 3x$ is not a constant function), V is a two-dimensional real vector space.

b) By properties of integrals, it is an inner product. Since $\int_0^\pi \cos 3x \sin 3x\ dx = 0$, the vectors $\cos 3x$ and $\sin 3x$ are already orthogonal. It suffices to normalize them. Easy computations show that $||\sin 3x||^2 = ||\cos 3x||^2 = \pi/2$, so we must divide each vector by $\sqrt{\pi/2}$ to get an orthonormal basis for V.

4) Let W be a subspace of V and $v \in V$ satisfy $2\langle v, w \rangle \leq \langle w, w \rangle$ for every $w \in W$. Suppose that the inner product is positive definite. Prove that v lies in the orthogonal complement of W.

Solution. Write $v = p + (v - p)$ where p is the projection of v onto W. Now apply the hypothesis with $w = p$. This gives $2\langle p + (v - p), p\rangle \leq \langle p, p\rangle$. Since $v - p$ is orthogonal to p, this simplifies to $2\langle p, p\rangle \leq \langle p, p\rangle$. By positive definiteness, we conclude that $p = 0$ and so v is orgthogonal to W.