Homework Set 2

1) Let R be the ring of 2×2 matrices with rational entries. Prove that the only ideals of R are (0) and R.

2) Let R be the ring of all real valued continuous functions on $[0,1]$. Let M be a maximal ideal of R. Prove that there is a real number $\gamma \in [0,1]$ such that $M = \{f(x) \in R : f(\gamma) = 0\}$. Hint: Proceed by contradiction. Use the fact that $[0,1]$ is compact, so every open cover of it has a finite subcover.

3) Let R be a Euclidean ring and $a, b \in R$. The least common multiple c of a and b is an element of R such that $a|c$ and $b|c$ and such that whenever $a|x$ and $b|x$ for $x \in R$, then $c|x$. Prove that c exists and that $c \times (a,b) = ab$, where (a,b) is the gcd of a and b.

4) Define the derivative $f'(x)$ of the polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ as $f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$. Prove that if $f(x) \in F[x]$, where F is the field of rational numbers, then $f(x)$ is divisible by the square of a polynomial if and only if $f(x)$ and $f'(x)$ have a gcd $d(x)$ of positive degree.