Eight theorems in extremal spectral graph theory

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Questions in *extremal graph theory* ask to optimize a graph invariant over a family of graphs.

- $\mathrm{ex}(n, H)$
- $\chi(G)$ when G has no K_r minor
- Maximum number of triangles in a graph of maximum degree Δ
- Max/Min cut
- Densest subgraph
- Many more... (any graph invariant over any family of graphs)

Spectral graph theory seeks to associate a matrix with a graph and to deduce properties of the graph from the eigenvalues and eigenvectors of the matrix.

- Expander-mixing lemma/Cheeger constant
- Community detection
- Max/Min cut
- Densest subgraph
- Many more... (many graph invariants over many families of graphs)

We are interested in extremal graph theory problems where the graph invariant is spectral.

- Maximize λ_1 over graphs with m edges (Stanley's bound)
- Maximize λ_1 over graphs with no K_{r+1} as a subgraph (Turán's theorem, Zarankiewicz problem)
- Minimize λ_2 over the family of d-regular graphs (Alon-Boppana-Serre theorem)
- Relationship to other graph parameters (Hoffman ratio bound, Wilf bound on chromatic number)

Conjecture 1 (Boots-Royle 1991)

The planar graph on $n \geq 9$ vertices of maximum spectral radius is $P_2 + P_{n-2}$.

Conjecture 2 (Cvetković-Rowlinson 1990)

The outerplanar graph on n vertices of maximum spectral radius is $K_1 + P_{n-1}$.

Conjecture 3 (Cioabă-Gregory 2007)

Let G be a connected graph and let x_{max} and x_{min} be the maximum and minimum entries in the leading eigenvector of the adjacency matrix of G. The *principal ratio* of G is given by $\gamma(G) = \frac{x_{max}}{x_{min}}$. Then the graph on n vertices maximizing $\gamma(G)$ is a kite graph.

Conjecture 4 (Aochiche et al 2008)

The connected graph on n vertices maximizing the quantity

$$\text{(spectral radius minus average degree)}$$

is a pineapple graph.

Conjecture 5 (Aldous and Fill 1994)

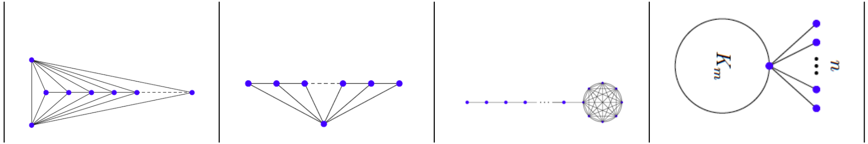
Let τ be the minimum over all connected graphs on n vertices of the second eigenvalue of the normalized Laplacian.

$$\tau \sim \frac{54}{n^3}.$$

Extremal graphs

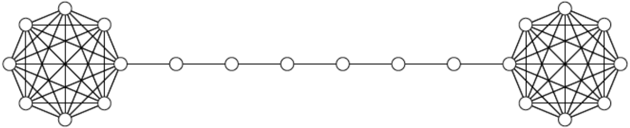
Theorem (Tait-Tobin)

Conjectures 1-4 true for n large enough.



Theorem (Aksoy-Chung-Tait-Tobin)

Conjecture 5 is true.




I ♥ KPBS

The Colin de Verdière parameter

Given a matrix M, define the *corank* of M to be the dimension of its kernel. If G is an n-vertex graph, then the *Colin de Verdière parameter* of G is defined to be the largest corank of any $n \times n$ matrix M such that:

M1 If $i \neq j$ then $M_{ij} < 0$ is $i \sim j$ and $M_{ij} = 0$ if $i \not\sim j$.

M2 M has exactly one negative eigenvalue of multiplicity 1.

M3 There is no nonzero matrix X such that $MX = 0$ and $X_{ij} = 0$ whenever $i = j$ or $M_{ij} \neq 0$.

The Colin de Verdière parameter

(i) $\mu(G) \leq 1$ if and only if G is the disjoint union of paths.
(ii) $\mu(G) \leq 2$ if and only if G is outerplanar.
(iii) $\mu(G) \leq 3$ if and only if G is planar.
(iv) $\mu(G) \leq 4$ if and only if G is linklessly embeddable.

Theorem (Tait)

For n large enough, the n-vertex graph of maximum spectral radius with Colin de Verdière parameter at most $r+1$ is the join of K_r and a path of length $n-r$.

Theorem (Tait)

For n large enough, the K_{r+2} minor free graph of maximum spectral radius is the join of K_r and an independent set of size $n-r$.

Theorem (Tait)

Let $s \geq r \geq 3$. For n large enough, if G is an n-vertex graph with no $K_{r+1,s+1}$ minor and λ is the spectral radius of its adjacency matrix, then

$$\lambda \leq \frac{r+s+5+\sqrt{(r+s-1)^2+4(r(n-r)-s(r-1))}}{2}$$

with equality if and only if $n \equiv r \pmod{s+1}$ and G is the join of K_r with a disjoint union of copies of K_{s+1}.

Proof Outline

The proofs all have similar structure, though the technical details are different:

- Use the conjectured extremal example to give a lower bound on the invariant in question.
- Use this lower bound to deduce rough structural information about the extremal graph.

- Once rough structure is known, deduce properties of the leading eigenvector.
- Use these properties to turn rough structure into exact structure.

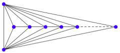
$$\lambda_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

Let $\mathbf{x}$ be the leading eigenvector, and let it be normalized so that it has maximum entry equal to 1. Throughout, let z be the vertex of eigenvector entry 1. Then the eigenvalue equation is, for all $u \in V(G)$

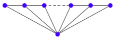
$$\lambda_1 \mathbf{x}_u = \sum_{v \sim u} \mathbf{x}_v.$$

$$\lambda_1 = \sum_{u \sim z} \mathbf{x}_u \leq d_z.$$

$$\lambda_1^2 = \sum_{u \sim z} \lambda_1 \mathbf{x}_u = \sum_{u \sim z} \sum_{v \sim u} \mathbf{x}_v = \sum_{u \sim z} \sum_{\substack{v \sim u \\ v \in N(z)}} \mathbf{x}_v + \sum_{u \sim z} \sum_{\substack{v \sim u \\ v \notin N(z)}} \mathbf{x}_v$$



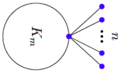
$\lambda_1 > \sqrt{r(n-r)}$



$\lambda_1 > \sqrt{n-1}$

Vertex of large degree

$\gamma(G) \geq n^{(1-\varepsilon)n}$



$\lambda_1 - \overline{d} \gtrsim \frac{n}{4}$

$\tau_1 \gtrsim \frac{54}{n^3}$

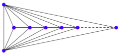
Lemma

For any $u \in V(G)$,

$$d_u > n \cdot \mathbf{x}_u - 11\sqrt{n}.$$

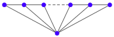
Proof:

$$\begin{aligned}
\lambda_1^2 \mathbf{x}_u &= \sum_{y\sim u}\sum_{v\sim y} \mathbf{x}_v && \text{(count 2-walks from } u\text{)} \\
&\leq d_u + \sum_{y\sim u}\sum_{\substack{v\sim y \\ v\neq u}} \mathbf{x}_v && (\mathbf{x}_u \leq 1) \\
&\leq d_u + 2\sum_{y\neq u} \mathbf{x}_y && (\text{no } K_{2,3}) \\
&\leq d_u + \frac{2}{\lambda_1}\sum_{u\not\sim v} d_v && \text{(eigenvalue equation)} \\
&\leq d_u + \frac{4(2n-3)}{\lambda_1} && (e(G) \leq 2n-3).
\end{aligned}$$



$\lambda_1 > \sqrt{r(n-r)}$

r vxs of large degree



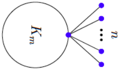
$\lambda_1 > \sqrt{n-1}$

Vertex of large degree

Other entries small

$\gamma(G) \geq n^{(1-\varepsilon)n}$

Diameter large



$\lambda_1 - \overline{d} \gtrsim \frac{n}{4}$

$\lambda_1 \sim \frac{n}{2}$, $\overline{d} \sim \frac{n}{4}$

$\tau_1 \gtrsim \frac{54}{n^3}$

Diameter large

Lemma

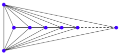
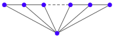
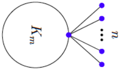
For all other $u \in V(G)$,

$$\mathbf{x}_u = O\left(\frac{1}{\sqrt{n}}\right).$$

Proof:

- The vertex of eigenvector entry 1 has $n - C\sqrt{n}$ neighbors.
- G has no $K_{2,3}$, so every other vertex has degree at most $C\sqrt{n} + 2$.

$$d_u > n \cdot \mathbf{x}_u - O\left(\sqrt{n}\right).$$

				
$\lambda_1 > \sqrt{r(n-r)}$	$\lambda_1 > \sqrt{n-1}$	$\gamma(G) \geq n^{(1-\varepsilon)n}$	$\lambda_1 - \overline{d} \gtrsim \frac{n}{4}$	$\tau_1 \gtrsim \frac{54}{n^3}$
r vxs of large degree	Vertex of large degree	Diameter large	$\lambda_1 \sim \frac{n}{2}$, $\overline{d} \sim \frac{n}{4}$	Diameter large
Other entries small	Other entries small	Pendant path	$\sim K_{n/2}$	Vector bimodal
	Vertex of degree $n-1$			

Lemma

The vertex of large degree has degree $n-1$.

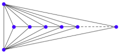
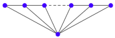
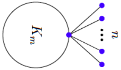
Proof: Let $B = V(G) \setminus (N(z) \cup \{z\})$. Assume $y \in B$ and show a contradiction.

Claim

$\sum_{u \in B} \mathbf{x}_u < \frac{C}{\sqrt{n}} < 1$.

$$V(G^+) = V(G)$$
$$E(G^+) = E(G) \cup \{zy\} \setminus \{\{vy\} : vy \in E(G)\}.$$

$$\lambda_1(A^+) - \lambda_1(A) \geq \frac{\mathbf{x}^T(A^+ - A)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{2\mathbf{x}_y}{\mathbf{x}^t\mathbf{x}}\left(1 - \sum_{v \sim y} \mathbf{x}_v\right).$$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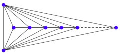
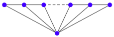
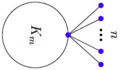
Other entries small	Other entries small	Pendant path	$\sim K_{n/2}$	Vector bimodal
r vxs of degree $n-1$	Vertex of degree $n-1$	very dense piece	$\sim \frac{n}{2}$ leaves	“+/−” cliques
	$G = P_1 + P_{n-1}$			

Theorem

The outerplanar graph maximizing λ_1 *is* $P_1 + P_{n-1}$.

Proof:

- Any other vertex may have degree at most 3 (including z), otherwise G contains $K_{2,3}$.
- The neighborhood of the vertex of degree $n-1$ may not contain a cycle.
- G is a subgraph of $P_1 + P_{n-1}$. By Perron-Frobenius and Rayleigh quotient, $G = P_1 + P_{n-1}$.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$G = K_r + H$	$G = P_1 + P_{n-1}$	$G = P_m \wedge K_{n-m}$	$G = PA\left(\lceil \frac{n}{2} \rceil + 1\right)$	?

Question

Characterize the extremal graph minimizing τ_1.

Question

What is the regular graph minimizing τ_1?

Question

Which graph minimizes the principal eigenvector's 1-norm?

Question

Which graph maximizes $\lambda_1 - \lambda_n$?

Question

Maximize the spectral radius of a planar subgraph of a random graph (say $p = 1/2$).


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