Induced Turán numbers

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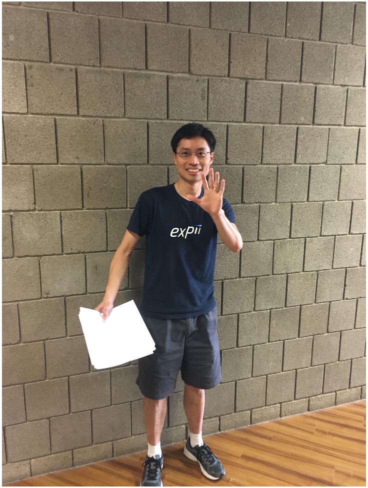
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For $\mathcal{F}$ a family of graphs, $\mathrm{ex}(n, \mathcal{F})$ denotes the maximum number of edges in an n-vertex graph which is F-free for all $F \in \mathcal{F}$.

Theorem (Turán's Theorem)

Let $T(n, r)$ be the complete r-partite graph with parts of size as equal as possible. Then

$$\mathrm{ex}(n, K_{r+1}) = e(T(n, r)).$$

Theorem (Erdős-Stone-Simonovits)

Let $\chi(\mathcal{F})$ denote $\min \chi(F)$ over $F \in \mathcal{F}$, then

$$\mathrm{ex}(n, \mathcal{F}) \sim \left(1 - \frac{1}{\chi(\mathcal{F}) - 1}\right)\binom{n}{2}$$

Determining Turán numbers for bipartite graphs is difficult in general.

Theorem (Kővari-Sós-Turán)

$$\mathrm{ex}(n, K_{s,t}) = O\left(n^{2-1/s}\right).$$

- $\mathrm{ex}(n, K_{2,2}) \sim \frac{1}{2}n^{3/2}$ (Brown, Erdős-Rényi-Sós, Füredi)
- $\mathrm{ex}(n, K_{2,t+1}) \sim \frac{1}{2}\sqrt{t}n^{3/2}$ (Kővari-Sós-Turán, Füredi)
- $\mathrm{ex}(n, K_{3,3}) \sim \frac{1}{2}n^{5/3}$ (Brown, Füredi)
- $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$ for $t > (s-1)!$ (Kővari-Sós-Turán, Alon-Rónyai-Szabó)

Problem

Is it true that $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$?

We begin with a stupid question: Determine

$$\mathrm{ex}(n, K_{s,t}\text{−ind}).$$

i.e. what is the maximum number of edges in an n-vertex graph with no induced copy of $K_{s,t}$?

K_n has no induced $K_{s,t}$ and has $\binom{n}{2}$ edges.

Fix a graph H.

Problem

What is $\mathrm{ex}(n, \{H, K_{s,t}\text{−ind}\})$?

i.e. How many edges may be in an H-free graph with no induced $K_{s,t}$?

Some previous work:

- $\mathrm{ex}(n, \{K_4^{(3)}, e\text{−ind}\})$ (Razborov)
- $\mathrm{ex}(n, \{K_{1,\Delta}, 2K_2\text{−ind}\})$ (Nešetřil-Erdős, Chung-Gyárfás-Trotter-Tuza)
- $\mathrm{ex}(n, \{K_{1,\Delta}, P_n\text{−ind}\})$ (Chung-Jiang-West, Chung-West)
- $\mathrm{ex}(n, \{K_r, C_4\text{−ind}\})$ (Gyárfás-Hubenko-Solymosi, Gyárfás-Sárközy)
- Posets

Definition

The **Ramsey-Turán number** of a graph H, denoted by $\mathbf{RT}(n, H, m)$ is the maximum number of edges in an n-vertex H-free graph which has independence number less than m.

- $\mathbf{RT}(n, H, m) = \text{ex}(n, \{H, I_m\text{−ind}\})$.
- Sós introduced this parameter so that one may not use a Turán graph as the (asymptotically) extremal H-free graph. Our motivation is the same.

Erdős-Hajnal conjecture

For a fixed graph F, there exists a constant $c > 0$ such that any graph with no induced copy of F contains either a clique or an independent set of size n^c.

If $\text{ex}(n, \{K_r, F\text{−ind}\}) = \frac{nd}{2}$, then any graph that is F-induced-free contains either a clique of size r or an independent set of size $\frac{n}{d+1}$.

First Observation

$$\mathrm{ex}(n, \{H, F\text{−ind}\}) \geq \mathrm{ex}(n, \{H, F\}).$$

Second Observation

$$\mathrm{ex}(n, \{C_3, C_4\text{−ind}\}) = \mathrm{ex}(n, \{C_3, C_4\}).$$

Theorem (Bollobás-Györi)

$$\mathrm{ex}(n, \{C_5, C_4\text{−ind}\}) \geq \left(\frac{2}{3\sqrt{3}} - o(1)\right) n^{3/2}.$$

Theorem (Erdős-Simonovits)

$$\mathrm{ex}(n, \{C_5, C_4\}) \leq \frac{1}{2\sqrt{2}} n^{3/2} + 4\left(\frac{n}{2}\right)^{1/2}.$$

Theorem (Loh-Tait-Timmons)

Let r, s, t be fixed, and G be a K_r-free graph on n vertices with no induced copy of $K_{s,t}$. Then

$$e(G) \leq C_{r,s,t} n^{2-1/s}.$$

Our main lemma is a clique-counting theorem:

Theorem (Loh-Tait-Timmons)

Let G be an n-vertex, K_r-free graph with no copy of $K_{s,t}$ as an induced subgraph. If $t_m(G)$ is the number of cliques of size m in G, then

$$m \cdot t_m(G) \leq 2(t+r)^{tm/s}(r+s)^s n^{m-\frac{m-1}{s}} + (r+s)^s n^{m-1}.$$

Recently Alon and Shikhelman the quantity $\mathrm{ex}(n, T, H)$, denoting the maximum number of copies of T in an H-free graph.

$$\mathrm{ex}(n, K_2, H) = \mathrm{ex}(n, H).$$

- $\mathrm{ex}(n, K_t, K_r)$ (Erdős)
- $\mathrm{ex}(C_5, K_3)$ (Hatami-Hladký-Král-Norine-Razborov, Grzesik)
- $\mathrm{ex}(n, K_3, C_{2k+1})$ (Bollobás-Győri, Győri-Li)

Theorem (Alon-Shikhelman)

Let $m \geq 2$ and $t \geq s \geq m-1$, then

$$\mathrm{ex}(n, K_m, K_{s,t}) = O\left(n^{m-\frac{m(m-1)}{2s}}\right).$$

Our theorem says $\mathrm{ex}(n, K_m, \{K_r, K_{s,t}\text{−ind}\}) = O\left(n^{m-\frac{m-1}{s}}\right)$.

Proof sketch that $e(G) = O(n^{2-1/s})$

Lemma

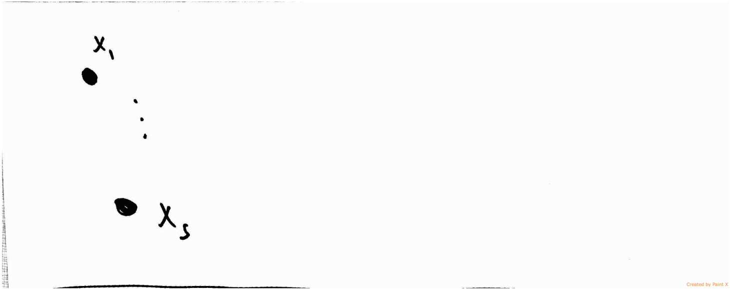
If F is a graph on n vertices and $n > 2 \cdot 4^s$, then

$$t_s(F) + t_s(\overline{F}) \geq \epsilon_s n^s.$$

Any set of $4^s > R(s,s)$ vertices contains a clique of size s in either F or $\overline{F}$. Each set of size s is contained in $\binom{n-s}{4^s-s}$ sets of size 4^s. Therefore

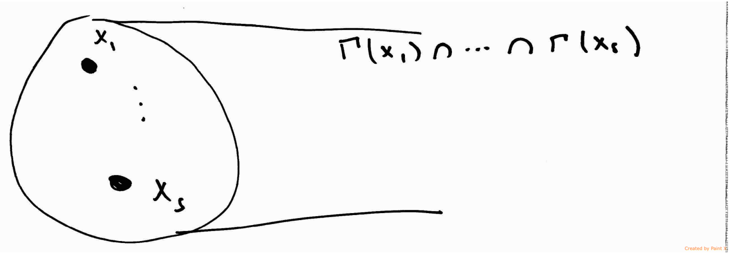
$$t_s(F) + t_s(\overline{F}) \geq \frac{\binom{n}{4^s}}{\binom{n-s}{4^s-s}} > \frac{n^s}{2^s 4^{s^2}}.$$

Proof sketch that $e(G) = O(n^{2-1/s})$



Let $\mathcal{I}_s$ denote the set of independent sets of size s.

Proof sketch that $e(G) = O(n^{2-1/s})$



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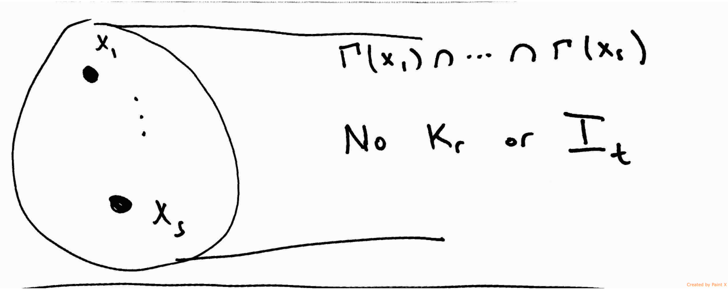


$$d(x_1, \cdots, x_s) \le R(r,t)$$

$$\sum_{\{x_1,\cdots,x_s\}\in \mathcal{I}_s} d(x_1,\cdots,x_s) \le \binom{n}{s} R(r,t)$$

Proof sketch that $e(G) = O(n^{2-1/s})$

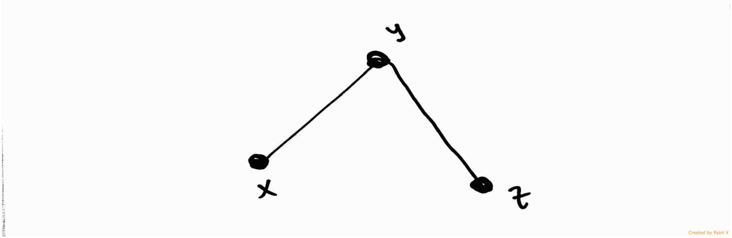
Now we double count and use the lemma and convexity.

$$
\begin{aligned}
(r+t)^t n^s &\geq \sum_{(x_1,\cdots,x_s)\in\mathcal{I}_s} d(x_1,\cdots,x_s) = \sum_{v\in V(G)} t_s(\overline{\Gamma(v)}) \\
&\geq \sum_{v\in V(G)} \epsilon_s (d(v))^s - t_s(\Gamma(v)) \\
&\geq \epsilon_s n \left(\frac{1}{n}\sum_{v\in V(G)} d(v)\right)^s - \sum_{v\in V(G)} t_s(\Gamma(v)) \\
&= \frac{\epsilon_s 2^s (e(G))^s}{n^{s-1}} - (s+1)t_{s+1}(G).
\end{aligned}
$$

Our clique counting theorem gives $t_{s+1}(G) = O(n^s)$.

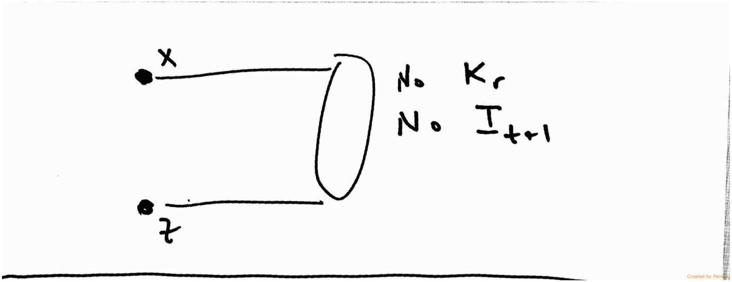
Sketch of proof that $t_{s+1}(G) = O(n^s)$

(Sketch of proof that G K_r-free with no induced $K_{2,t+1}$ implies $O(n^2)$ triangles)



Sketch of proof that $t_{s+1}(G) = O(n^s)$

(Sketch of proof that G K_r-free with no induced $K_{2,t+1}$ implies $O(n^2)$ triangles)



$x \not\sim z$ implies $d(x, z) < R(r, t+1)$

$$\#\{x, y, z\} \leq \sum_{x \not\sim z} d(x, z) < R(r, t+1)n^2$$

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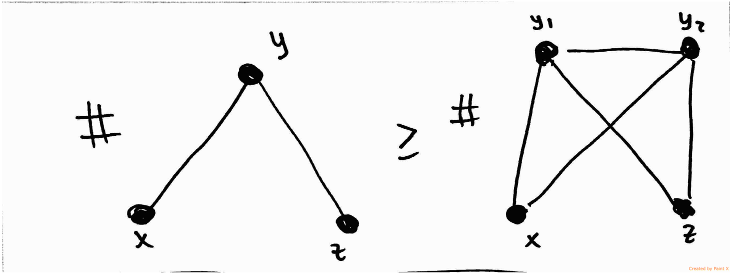

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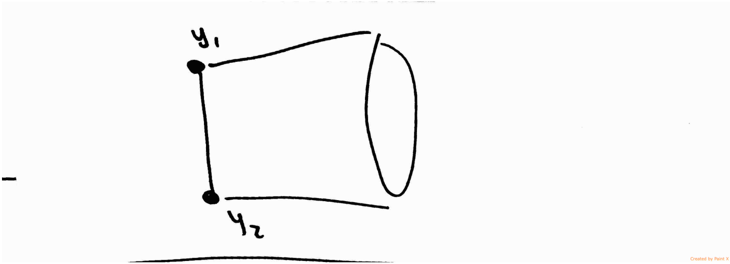

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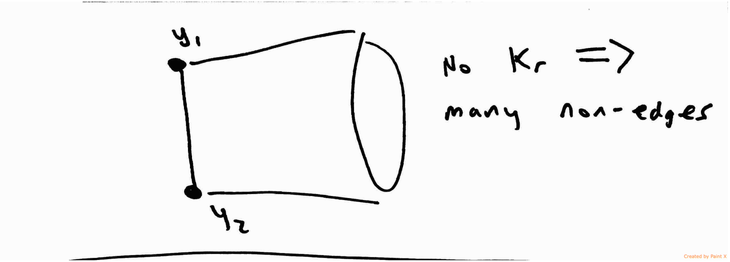

- $\Gamma(y_1) \cap \Gamma(y_2)$ contains at least $\epsilon(d(y_1, y_2))^2$ nonadjacent pairs
- Each nonadjacent pair is counted at most $\binom{R(r,t+1)}{2}$ times this way

Sketch of proof that $t_{s+1}(G) = O(n^s)$

(Sketch of proof that G K_r-free with no induced $K_{2,t+1}$ implies $O(n^2)$ triangles)

$$\begin{aligned} O(n^2) = \#\{x, y, z\} &\geq \frac{1}{\binom{R(r,t+1)}{2}} \sum_{y_1 \sim y_2} \epsilon(d(y_1, y_2))^2 \\ &\geq \frac{\epsilon_1}{n^2} \left(\sum_{y_1 \sim y_2} d(y_1, y_2) \right)^2 \\ &= \frac{\epsilon_1}{n^2} (3t_3(G))^2 \end{aligned}$$

$$\mathrm{ex}(n, \{K_r, K_{s,t}\text{−ind}\}) = O\left(n^{2-1/s}\right).$$

If the average degree of a graph with no induced $K_{s,t}$ is much larger than $n^{1-1/s}$, then the graph has a large clique.

Theorem (Gyárfás-Hubenko-Solymosi 2002, Gyárfás-Sárközy 2015)

If G has average degree d and no induced C_4, then

$$\omega(G) = \Omega\left(\frac{d^2}{n}\right)$$

Corollary (Loh-Tait-Timmons)

If G has average degree d and no induced copy of $K_{s,t}$, then

$$\omega(G) \geq \Omega\left(\left(\frac{d^s}{n^{s-1}}\right)^{\frac{s}{t(s+1)+s^2}}\right).$$

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Theorem (Loh-Tait-Timmons)

If G has average degree d and no induced $K_{2,t+1}$, then

$$\omega(G) = \Omega\left(\left(\frac{d^2}{nt}\right)^{1/t}\right).$$

When $s = 2$ or when H has a specific structure which can be exploited, we can obtain better estimates.

Theorem (Loh-Tait-Timmons)

Let H be a graph on v_H vertices. Then

$$\left(\frac{1}{4}\sqrt{t} - o(1)\right) n^{3/2} \leq \mathrm{ex}(n, \{H, K_{2,t+1}\text{−ind}\}) \leq (\sqrt{2t} + o(1)) v_H^{t/2} n^{3/2}.$$

Theorem (Loh-Tait-Timmons)

$$\mathrm{ex}(n, \{C_{2k+1}, K_{2,t+1}\text{−ind}\}) \leq (\sqrt{2kt} + o(1)) n^{3/2}.$$

Problem

Can one find nontrivial s, t and a sequence of graphs H and determine $\mathrm{ex}(n, \{H, K_{s,t}\text{−ind}\})$ up to a constant factor?

Problem

We showed if G is H-free and $K_{s,t}$ induced free, then the number of cliques of order m is at most

$$O\left(n^{m-\frac{m-1}{s}} + n^{m-1}\right).$$

Can this be improved? Alon and Shikhelmen showed if G is $K_{s,t}$-free, then the number of cliques of order m is at most

$$O\left(n^{m-\frac{m(m-1)}{2s}}\right).$$

Problem

Let G be an n-vertex K_r-free graph with no induced copy of $K_{s,t}$. Let λ be the spectral radius of its adjacency matrix. Show that

$$\lambda = O\left(n^{1-1/s}\right).$$