Increasing paths in edge-ordered graphs

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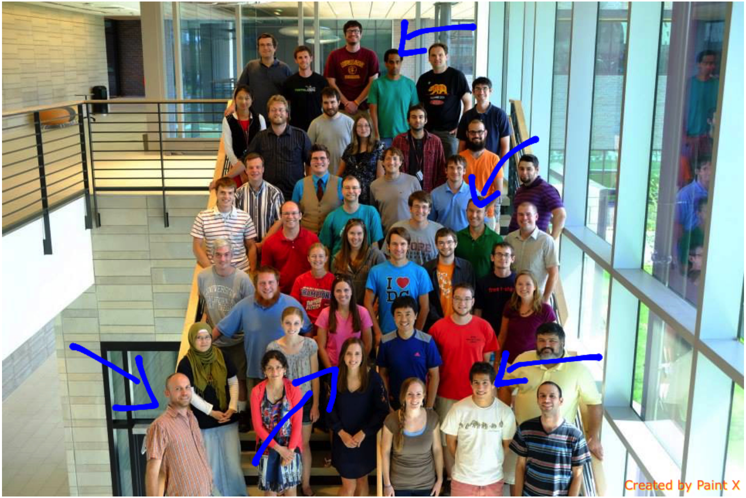
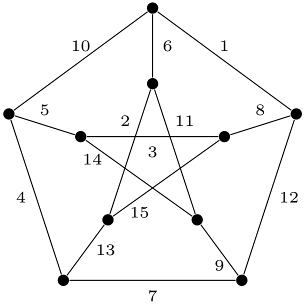


Joint work with Jessica De Silva, Theodore Molla, Florian Pfender, and Troy Retter

A game

Let's play a game



What is the longest increasing path you can find?

Definitions

Definition

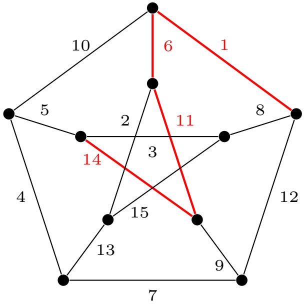
An *edge-ordering* ϕ of a graph G is a bijection $\phi : E(G) \to \{1, \ldots, |E(G)|\}$.

Definition

Given an edge-ordering ϕ, and *increasing path* is a path $e_1 e_2 \cdots e_k$ such that $\phi(e_1) < \phi(e_2) < \cdots < \phi(e_k)$.

Note that a path is a self-avoiding walk, ie no vertex is visited more than once.

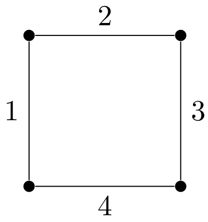
A game

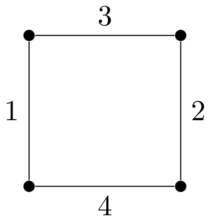


There is an increasing path of length at least 4.

Our opponent

Our goal is to find a long increasing path. Our opponent's goal is to order the edges so that we cannot find a long increasing path.





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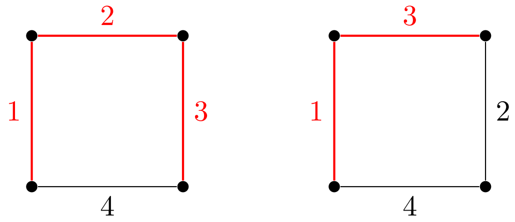


Max-min problem

If both players play optimally, how long will the longest increasing path be? Given a graph G, define $f(G)$ to be this length.

Definition

Fix a graph G. Define

$$f(G) = \min_{\phi} \text{ length of longest increasing path under } \phi$$

where ϕ runs through all edge-orderings.

History

- Chvátal and Komlós ask about $f(K_n)$ in 1971.
- Graham and Kleitman show $f(K_n) \geq \sqrt{n-1}$ in 1973.
- Rödl shows if G has average degree d, then $f(G) \gtrsim \sqrt{d}$ in 1973.
- A series of upper bounds for $f(K_n)$ follow, settling on $f(K_n) < (1/2 + o(1))n$ by Calderbank, Chung, and Sturtevant in 1984.
- Alon and Yuster study graphs of bounded maximum degree in 2001.

Our Theorems

Theorem (GRWC 2014)

Let Q_d denote the d-dimensional hypercube. Then for all $d \geq 2$,

$$f(Q_d) \geq \frac{d}{\log d}.$$

Theorem (GRWC 2014)

Let ω be any function tending to infinity, and $p \leq \frac{\log n}{\sqrt{n}}\omega(n)$. Then with probability tending to 1,

$$f(G(n,p)) \geq \frac{(1-o(1))np}{\omega(n)\log n}.$$

Both of these bounds are tight up to the logarithmic factor.

Progress on $f(K_n)$

Our theorem shows that a random graph with expected degree just slightly larger than $\sqrt{n}$ satisfies the same lower bound that Graham and Kleitman showed for K_n.We thought that this was good evidence that the lower bound for $f(K_n)$ was not correct.

Theorem (Milans)

$f(G) = \Omega\left((n/\log n)^{2/3}\right)$.

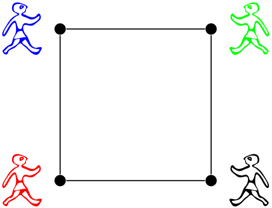
The pedestrian argument

Theorem (Graham-Kleitman 1973, Rödl 1973)

Every edge-ordering of K_n contains an increasing path of length at least $\sqrt{n-1}$. That is

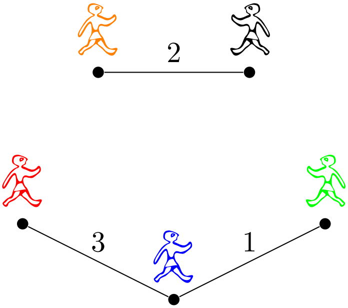
$$f(K_n) \geq \sqrt{n-1}.$$

Place a pedestrian on each vertex.



The pedestrian argument

Call out the edges in order. The two pedestrians switch places unless it would cause one of them to revisit a vertex she has already seen.



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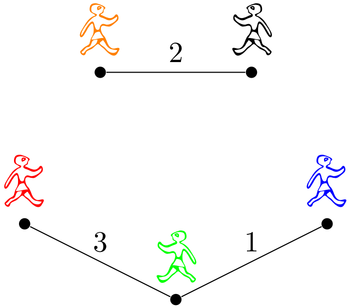

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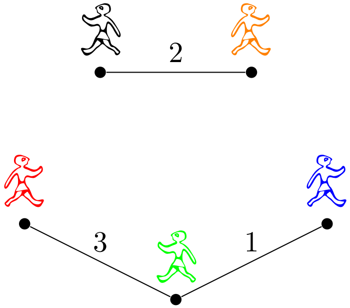

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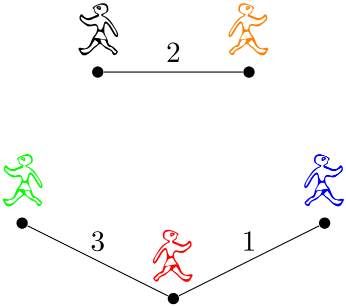

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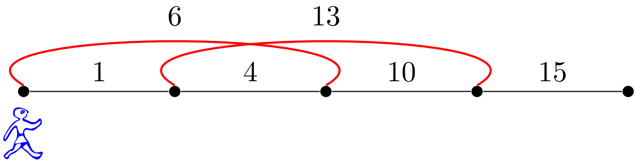

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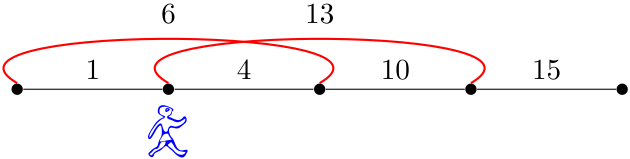

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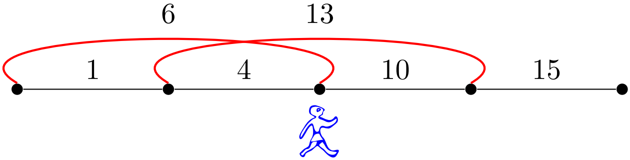

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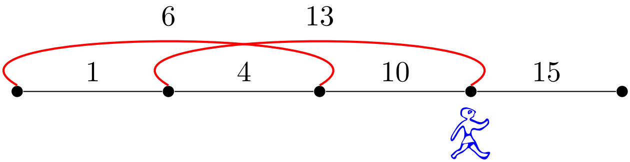
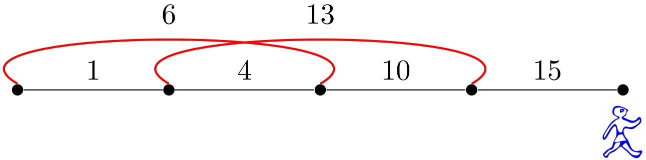


The pedestrian argument



The blue pedestrian has walked an increasing path of length 4 ($1 - 4 - 10 - 15$).

The pedestrian argument

Theorem (Graham-Kleitman 1973, Rödl 1973)

Every edge-ordering of K_n contains an increasing path of length at least $\sqrt{n-1}$. That is

$$f(K_n) \geq \sqrt{n-1}.$$

Proof:

- Suppose each pedestrian walks $\leq k$ steps during this process.
- Then at most $\frac{kn}{2}$ edges are traversed.
- Each pedestrian declines to walk an edge at most $\binom{k+1}{2} - k$ times.

$$\text{edges walked+edges declined} = \binom{n}{2} \leq \frac{kn}{2} + \binom{k}{2} n = \frac{k^2 n}{2}.$$

The pedestrian argument

Consider the pedestrian algorithm on an arbitrary graph G. Every edge in G is either traversed or is declined by some pedestrian. An edge may only be declined if it is contained in the subgraph induced by the path walked by a pedestrian.

Lemma

Let G be any graph. If $f(G) < k$, there exist sets $V_1, \cdots, V_n \subset V(G)$ such that $|V_i| \leq k$ and every edge of G is contained in a subgraph induced by some V_j.

In particular,

$n \cdot$(# edges in densest subgraph on $f(G)$ vertices) $\geq |E(G)|$.

The hypercube

$n \cdot (\#$ edges in densest subgraph on $f(G)$ vertices$) \geq |E(G)|$

Theorem (GRWC 2014)

$$f(Q_d) \geq \frac{d}{\log d}$$

Proof: **Lemma:** Any subgraph of a hypercube has density less than or equal to a subhypercube of the same size.

The random graph

Theorem (GRWC 2014)

Let $\omega(n)$ be a function tending to infinity arbitrarily slowly. Then for any $p \geq \frac{\log n}{\sqrt{n}}\omega(n)$, with probability tending to 1

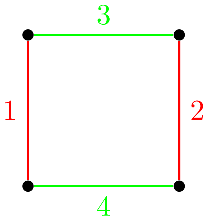
$$f(G(n,p)) \geq \frac{(1-o(1))np}{\omega(n)\log n}$$

Proof: The graphs induced by the pedestrians' paths must cover all of the edges of $G(n,p)$. If $f(G(n,p)) \leq \frac{np}{\omega(n)\log n}$, we get a lower bound on the number of pairs that *cannot* be edges. The probability that this occurs is $o\left(\binom{n}{f(G(n,p))}^n\right)$, i.e. it is so unlikely that even adding up over all possible paths for the pedestrians the probability that it occurs is still $o(1)$.

Upper Bounds

Our opponent wants to label the edges of G so that there is no long increasing path. Constructing an edge-labeling yields an upper bound on $f(G)$.

A first strategy: Consider a proper edge-coloring of a graph G with colors $c_1, \cdots, c_k$. Label the edges with color c_1 with the smallest labels. Label the edges with color c_2 with the next smallest labels. Continue this process. Any increasing path can use at most one edge of each color.



Open problems

Lavrov and Loh studied a variant of this problem. What happens when the edges of K_n are ordered randomly?

Theorem (Lavrov-Loh)

With probability tending to 1, a random edge-ordering of K_n has a monotone path of length at least $.85n$. With probability at least $1/e - o(1)$, a random edge-ordering of K_n has an increasing Hamiltonian path.

Conjecture

With probability tending to 1, a random edge-ordering of K_n contains an increasing Hamiltonian path.

Open problems

- Improve the lower bound $f(K_n) = \Omega\left((n/\log n)^{2/3}\right)$.
- Does $f(Q_d) = d$?
- Are there graphs G with $\Delta(G) = k$ and $f(G) = k + 1$?
- Show a random edge-ordering of K_n contains an increasing Hamiltonian path with probability tending to 1.