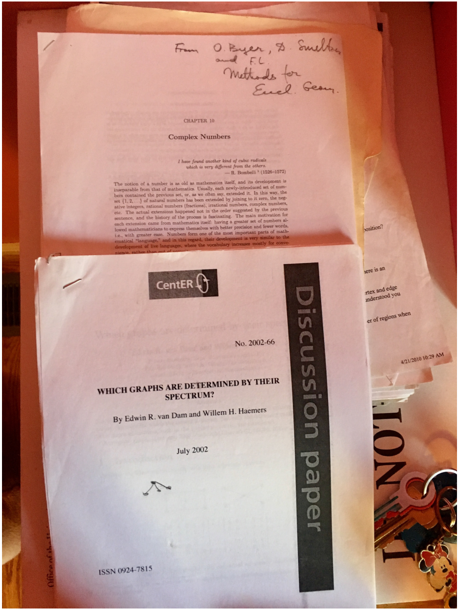
Degree Ramsey numbers for even cycles

Michael Tait

Carnegie Mellon University
mtait@cmu.edu

AEGT: a conference in honor of Willem, Felix, and Andy
University of Delaware

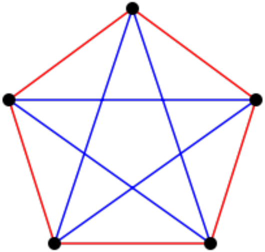
August 8, 2017


From O. Byer, D. Smeltzer and F.L. Methods for Eucl. Geom.
CHAPTER 10
Complex Numbers
I have found another kind of cubic radicals which is very different from the others.
— R. Bombelli 1 (1526–1572)
The notion of a number is as old as mathematics itself, and its development is inseparable from that of mathematics. Usually, each newly-introduced set of numbers contained the previous set, or, as we often say, extended it. In this way, the set {1, 2, ...} of natural numbers has been extended by joining to it zero, the negative integers, rational numbers (fractions), irrational numbers, complex numbers, etc. The actual extensions happened not in the order suggested by the previous sentence, and the history of the process is fascinating. The main motivation for each extension came from mathematics itself: having a greater set of numbers allowed mathematicians to express themselves with better precision and fewer words, i.e., with greater ease. Numbers form one of the most important parts of mathematical "language," and in this regard, their development is very similar to the development of live languages, where the vocabulary increases mostly for convenience, rather than out of necessity.
CentER
Discussion paper
No. 2002-66
WHICH GRAPHS ARE DETERMINED BY THEIR SPECTRUM?
By Edwin R. van Dam and Willem H. Haemers
July 2002
ISSN 0924-7815
4/21/2010 10:29 AM

Ramsey Theory

“Large” implies “structure”

For graphs G and H, $R(G, H)$ denotes the smallest n such that any red/blue coloring of $E(K_n)$ contains a red G or a blue H



$$R(C_n, C_m) = f(n, m)$$
$$R(K_n, T_m) = (m-1)(n-1)+1$$
$$R(K_3, K_t) = \Theta\left(\frac{t^2}{\log t}\right)$$
$$R(K_m, K_n) = ??$$

Parameter Ramsey numbers

We say $H \xrightarrow{s} G$ denote that any s-coloring of $E(H)$ contains a monochromatic G. For any monotone graph parameter ρ, we let

$$R_\rho(G,s) = \min\{\rho(H) : H \xrightarrow{s} G\}.$$

- $R(G,G) = R_{|V(H)|}(G,2)$.
- $\rho = e(H)$: *size Ramsey number*
- $\rho = \chi(H)$: *chromatic Ramsey number*
- $\rho = \Delta(H)$: *degree Ramsey number*

Degree Ramsey numbers

- Trees, cycles (Kinnersley, Milans, West)
- $R_\Delta(C_4, s) = \Omega(s^{14/9})$ (Jiang, Milans, West)
- Closed blowups of trees (Horn, Milans, Rödl)
- $R_\Delta(C_4, s) = \Theta(s^2)$ (Kang, Perarnau)

$R_\Delta(C_4, s) = O(s^2)$: there exists a graph of maximum degree $O(s^2)$ such that any s-coloring of its edges contains a monochromatic C_4.
$R_\Delta(C_4, s) = \Omega(s^2)$: any graph of maximum degree Δ can be partitioned into $O(\Delta^{1/2})$ subgraphs each of which is C_4 free.

Kang-Perarnau: If one can partition K_Δ into C_4 free graphs efficiently, then one can partition any graph of maximum degree Δ into C_4 free graphs efficiently.

Theorem (Tait)

$$R_\Delta(C_6, s) = \Theta(s^{3/2})$$
$$R_\Delta(C_{10}, s) = \Theta(s^{5/4})$$

If one can partition K_Δ into C_6 or C_{10} free graphs efficiently then one can partition any graph of maximum degree Δ into C_6 or C_{10} free graphs efficiently (this is not quite true).

Assume H is a graph of maximum degree Δ. We would like to partition it into "few" graphs all of which are C_{2k}-free.

Lemma

There is a subgraph spanning H' of H and a proper coloring of $V(H')$ with $O(\Delta)$ colors such that

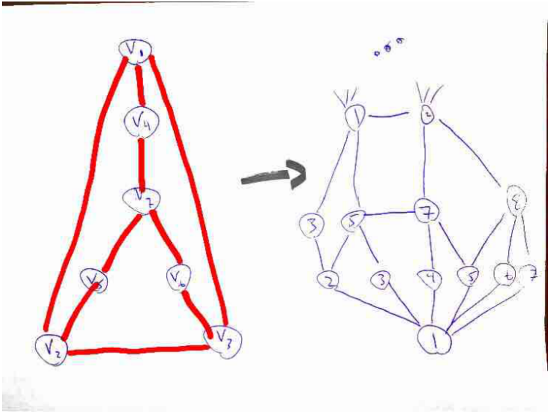
1. *For all v, $d_{H'}(v) \geq \delta d_H(v)$*
2. *Every vertex in H' sees a rainbow*

Note: by iterating, it suffices to show that H' can be partitioned into "few" graphs all of which are C_{2k}-free.

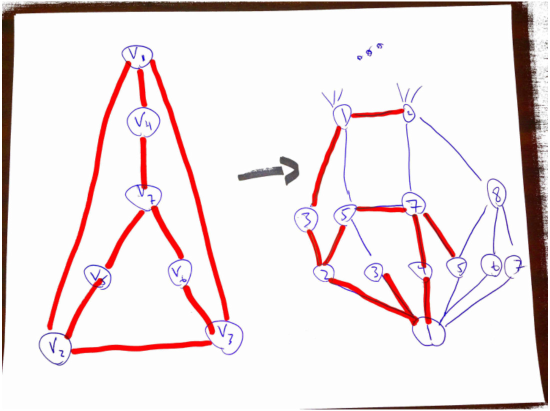
Assume H' is properly colored with 100Δ colors so that each neighborhood is a rainbow. Also assume that we have partitioned $K_{100\Delta}$ into subgraphs $G_1, \cdots, G_m$ none of which contain C_4.

We show that we can also partition H' into m subgraphs none of which contain a C_4.

Map from K_Δ to H'



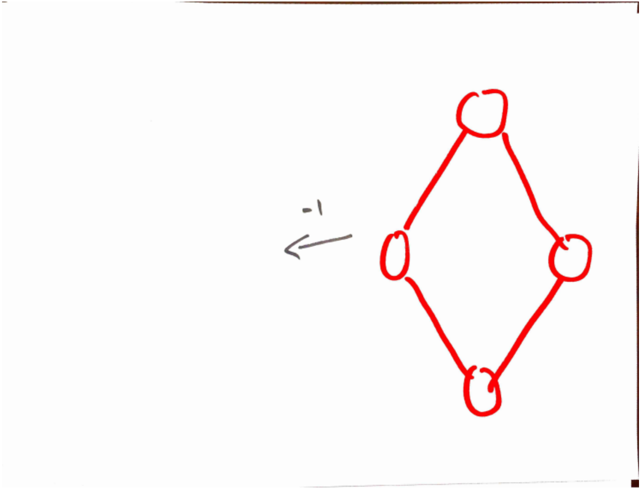
Map from K_Δ to H'



Since $G_1, \cdots, G_m$ partition $K_{100\Delta}$, these graphs partition H'.

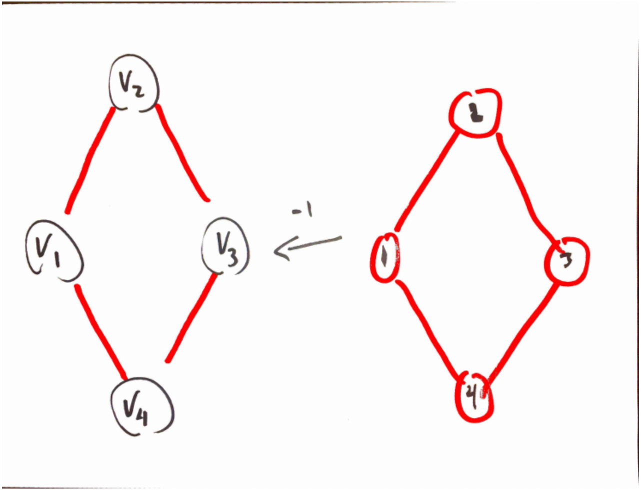
C_4 free?

What is the preimage of a C_4?



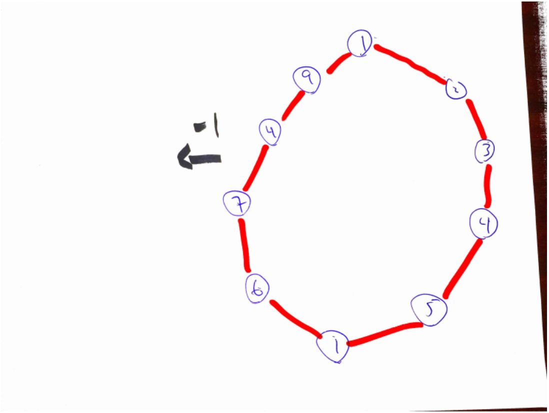
C_4 free?

What is the preimage of a C_4?



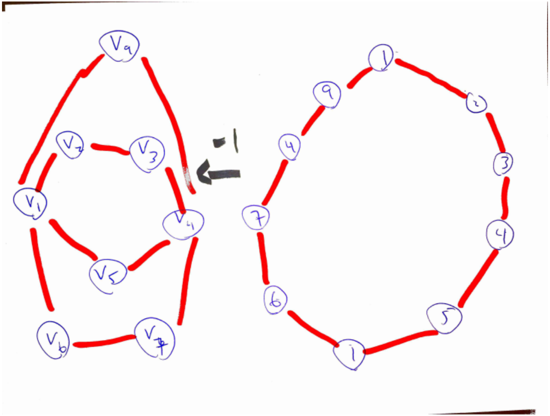
C_{2k} free?

What is the preimage of a C_{2k}?



C_{2k} free?

What is the preimage of a C_{2k}?



The preimage is a nonbacktracking closed walk of length $2k$. It contains a cycle of length at most $2k$.

If we can partition $K_{C\Delta}$ into "few" graphs with girth greater than $2k$, then we can partition H' into the same number of graphs with no C_{2k}.

Theorem (Chung-Graham 1975, Lazebnik-Woldar 2000)

K_n can be partitioned into m C_4 free subgraphs where $m \sim n^{1/2}$.

Theorem (Li-Lih 2009)

K_n can be partitioned into $O(n^{2/3})$ C_6-free graphs or $O(n^{4/5})$ C_{10}-free graphs.

Theorem

K_n can be partitioned in $O(n^{2/3})$ subgraphs of girth 8 or $O(n^{4/5})$ subgraphs of girth 12.

Kinnersley, Milans, and West showed that $R_\Delta(C_{2k}, s) \geq 2s$.

Theorem (Lazebnik, Ustimenko, Woldar 1995)

Let k be fixed and $\delta = 0$ if k is odd and 1 if k is even. The graphs $CD(k,q)$ are graphs on n vertices with $\Omega\left(n^{1+\frac{2}{3k-3+\delta}}\right)$ edges and girth at least $2k+2$.

Corollary

Let $k \geq 2$ and $\delta = 0$ if k is odd and $\delta = 1$ if k is even. Then

$$R_\Delta(C_{2k}, s) = \Omega\left(\left(\frac{s}{\log s}\right)^{1+\frac{2}{3k-5+\delta}}\right).$$

Open Problem 1

Get rid of the log in the previous theorem.

Open Problem 2

Can any graph with spectral radius λ be decomposed into $O(\lambda^{2/3})$ subgraphs each of which are C_4-free?

Open Problem 3

Is there a function $f(\Delta, s)$ such that $R_\Delta(G, s) \leq f(\Delta(G), s)$?