SIDON SETS AND 2-CAPS IN $\mathbb{F}_3^n$

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ABSTRACT. For each natural number d, we introduce the concept of a d-cap in $\mathbb{F}_3^n$. A subset of $\mathbb{F}_3^n$ is called a d-cap if, for each $k = 1, 2, \ldots, d$, no $k+2$ of the points lie on a k-dimensional flat. This generalizes the notion of a cap in $\mathbb{F}_3^n$. We prove that the 2-caps in $\mathbb{F}_3^n$ are exactly the Sidon sets and study the problem of determining the size of the largest 2-cap in $\mathbb{F}_3^n$.

1. INTRODUCTION

Throughout, let $\mathbb{F}_q$ denote the field with q elements and let $\mathbb{F}_q^n$ denote n-dimensional affine space over $\mathbb{F}_q$. A *cap* in $\mathbb{F}_3^n$ is a collection of points such that no three are collinear. Although this definition is geometric, there is an equivalent definition that is arithmetic: a set of points C is a cap in $\mathbb{F}_3^n$ if and only if C contains no three-term arithmetic progressions.

Here, we consider natural generalizations of caps in $\mathbb{F}_3^n$. For $d \in \mathbb{N}$, we call a set of points a *d-cap* if, for each $k = 1, 2, \ldots, d$, no $k+2$ of the points lie on a k-dimensional flat. With this definition, a 1-cap corresponds to the usual definition of a cap.

Let $r(1, \mathbb{F}_3^n)$ denote the maximal size of a 1-cap in $\mathbb{F}_3^n$. In general, it is a difficult problem to determine $r(1, \mathbb{F}_3^n)$—in fact, the exact answer is known only when $n \leq 6$. Table 1 lists the best known upper and lower bounds for $r(1, \mathbb{F}_3^n)$ for $n \leq 10$ [10]. It is known that in dimension $n \leq 6$, maximal 1-caps are equivalent up to affine transformation [4, 8, 9].

Dimension	1	2	3	4	5	6	7	8	9	10
Lower bound	2	4	9	20	45	112	236	496	1064	2240
Upper bound	2	4	9	20	45	112	291	771	2070	5619

TABLE 1. The best known bounds for the size of a maximal 1-cap in $\mathbb{F}_3^n$.

The asymptotic bounds on $r(1, \mathbb{F}_3^n)$ are also well-studied. In [5], Edel showed that

$$\limsup_{n\to\infty} \frac{\log_3(r(1, \mathbb{F}_3^n))}{n} \geq 0.724851$$

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and consequently that $r(1,\mathbb{F}_3^n)$ is $\Omega(2.2174^n)$ (using Hardy-Littlewood's Ω notation). In more recent breakthrough work [6], Ellenberg and Gijswijt (adapting a method of Croot, Lev, and Pach [3]) proved that $r(1,\mathbb{F}_3^n)$ is $o(2.756^n)$.

In this paper, we focus on the study of 2-caps in $\mathbb{F}_3^n$. We show that there is an equivalent arithmetic formulation of the definition of a 2-cap. In particular, the 2-caps in $\mathbb{F}_3^n$ are exactly the Sidon sets, which are important objects in combinatorial number theory. Using this definition, we are able to compute the maximal size of a 2-cap in $\mathbb{F}_3^n$ exactly when n is even. We also examine 2-caps in low dimension when n is odd, in particular considering dimension $n = 3$, 5, and 7.

Dimension	1	2	3	4	5	6	7	8	n even	n odd
Lower bound	2	3	5	9	13	27	33	81	$3^{n/2}$	$3^{(n-1)/2}+1$
Upper bound	2	3	5	9	13	27	47	81	$3^{n/2}$	$\lceil 3^{n/2} \rceil$

Table 2. Bounds for the size of a maximal 2-cap in $\mathbb{F}_3^n$.

Table 2 lists the bounds we obtain for the size of a maximal 2-cap in $\mathbb{F}_3^n$. The values in dimension 3, 5, and 7 are given by Theorems 3.9 and 3.10, and Proposition 3.12, respectively. The bounds for even dimension follow from Theorem 3.4. The upper bound in odd dimension n follows from Proposition 3.3 and the lower bound is given by adding one affinely independent point to the construction in dimension $n-1$. Knowing the exact value in even dimension also allows us to conclude that asymptotically, the maximal size of a 2-cap in $\mathbb{F}_3^n$ is $\Theta(3^{n/2})$.

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2. Preliminaries

In this section, we establish basic notation, definitions, and background. The set of natural numbers is denoted $\mathbb{N} = \{1, 2, 3, \dots\}$. Throughout, d and n will always denote natural numbers. An element $\mathbf{a} \in \mathbb{F}_3^n$ will be written as a row vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ with each $a_i \in \{0, 1, 2\}$. We will sometimes order the vectors of $\mathbb{F}_3^n$ lexicographically—i.e., by regarding them as ternary strings. We use the notation $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ to denote the n standard basis vectors in an n-dimensional vector space.

A k-dimensional affine subspace of a vector space is called a k-dimensional *flat*. In particular, a 1-dimensional flat is also called a *line*. In the affine space $\mathbb{F}_3^n$, every line consists of the points

$\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{a}+2\mathbf{b}\}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{F}_3^n$ where $\mathbf{b} \neq \mathbf{0}$. Hence, the lines in $\mathbb{F}_3^n$ correspond to three-term arithmetic progressions. It is easy to see that three distinct points in $\mathbb{F}_3^n$ are collinear if and only if they sum to $\mathbf{0}$. Likewise, a 2-dimensional flat is called a *plane*. Any three non-collinear points determine a unique plane. For $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{F}_3^k$ with $k < n$. The subset of $\mathbb{F}_3^n$ whose first k entries are $a_1, a_2, \ldots, a_k$ is an $(n-k)$-dimensional flat which we call *the* $\mathbf{a}$*-affine subspace* of $\mathbb{F}_3^n$.

Two subsets C and D of a vector space are called *affinely equivalent* if there exists an invertible affine transformation T such that $T(C) = D$. It is clear that affine equivalence determines an equivalence relation on the power set of a vector space. Given a set of points X in a vector space, its affine span is given by the set of all affine combinations of points of X. A set X is called *affinely independent* if no proper subset of X has the same affine span as X. Equivalently, $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is affinely independent if and only if $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0\}$ is linearly independent.

Definition 2.1. A subset C of $\mathbb{F}_3^n$ is called a *d-cap* if, for each $k = 1, 2, \ldots, d$, no $k+2$ points of C lie on a k-dimensional flat. Equivalently, C is a d-cap if and only if any subset of C of size at most $d+2$ is affinely independent. A d-cap is called *complete* if it is not a proper subset of another d-cap and is called *maximal* if it is of the largest possible cardinality.

We will denote the size of a maximal d-cap in $\mathbb{F}_3^n$ by $r(d, \mathbb{F}_3^n)$. As mentioned in the introduction, a 1-cap is a classical cap. We note that $C \subseteq \mathbb{F}_3^{d+1}$ is a d-cap if and only if the points of C are in general position. Since invertible affine transformations preserve affine independence, the image of a d-cap under an invertible affine transformation is again a d-cap. As a warm-up, we prove some basic facts about maximal d-caps in $\mathbb{F}_3^n$.

Lemma 2.2. *We have that $r(d, \mathbb{F}_3^n) \geq n+1$ with equality if $n \leq d$.*

Proof. The set $\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an affinely independent subset of $\mathbb{F}_3^n$ of size $n+1$ and hence is a d-cap for any $d \in \mathbb{N}$. Therefore, $r(d, \mathbb{F}_3^n) \geq n+1$.

Now suppose $n \leq d$. Since, by definition, a d-cap must be an n-cap, we have that $r(d, \mathbb{F}_3^n) \leq r(n, \mathbb{F}_3^n)$. A maximal affinely independent set in $\mathbb{F}_3^n$ has size $n+1$ so $r(n, \mathbb{F}_3^n) \leq n+1$. and so $r(d, \mathbb{F}_3^n) = n+1$. □

Corollary 2.3. *When $n \leq d$, all maximal d-caps in $\mathbb{F}_3^n$ are affinely equivalent.*

Proof. By Lemma 2.2, when $n \leq d$, a maximal d-cap in $\mathbb{F}_3^n$ is a maximal affinely independent set, i.e., an affine basis of $\mathbb{F}_3^n$. All affine bases in an affine space are equivalent up to affine transformation. □

Lemma 2.4. *For fixed d, $r(d, \mathbb{F}_3^n)$ is a non-decreasing function of n and for fixed n, $r(d, \mathbb{F}_3^n)$ is a non-increasing function of d.*

Proof. Since $\mathbb{F}_3^{n-1}$ is an affine subspace of $\mathbb{F}_3^n$, a d-cap in $\mathbb{F}_3^{n-1}$ naturally embeds as a d-cap in $\mathbb{F}_3^n$. Hence $r(d, \mathbb{F}_3^{n-1}) \leq r(d, \mathbb{F}_3^n)$ so the first statement follows. The second statement follows since, by definition, a d-cap in $\mathbb{F}_3^n$ must be a $(d-1)$-cap. Hence, $r(d-1, \mathbb{F}_3^n) \geq r(d, \mathbb{F}_3^n)$. □

3. 2-caps in $\mathbb{F}_3^n$

We now restrict our attention to the study of 2-caps in $\mathbb{F}_3^n$. Our first observation is that in $\mathbb{F}_3^n$, the definition of a 2-cap is equivalent to the definition of a Sidon set.

Definition 3.1. Let G be an abelian group. A subset $A \subseteq G$ is called a *Sidon set* if, whenever $a + b = c + d$ with $a, b, c, d \in A$, the pair (a, b) is a permutation of the pair (c, d).

Theorem 3.2. *A subset C of $\mathbb{F}_3^n$ is a 2-cap if and only if it is a Sidon set.*

Proof. First suppose that C is not a 2-cap. Then C contains three points which are collinear or C contains four points which are coplanar. Every line in $\mathbb{F}_3^n$ is of the form $\mathbf{a}, \mathbf{b}, -\mathbf{a} - \mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathbb{F}_3^n$. But $\mathbf{b} + \mathbf{b} = \mathbf{a} + (-\mathbf{a} - \mathbf{b})$, so if C contains three collinear points, then C is not a Sidon set.

Suppose therefore that no three points in C are collinear. Then C contains four coplanar points. Every 2-dimensional flat F in $\mathbb{F}_3^n$ is determined by any three non-collinear points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in F$. Namely, we have that

$F =$

$\mathbf{a}$	$\mathbf{b}$	$-\mathbf{a} - \mathbf{b}$
$\mathbf{c}$	$-\mathbf{a} + \mathbf{b} + \mathbf{c}$	$\mathbf{a} - \mathbf{b} + \mathbf{c}$
$-\mathbf{a} - \mathbf{c}$	$\mathbf{a} + \mathbf{b} - \mathbf{c}$	$-\mathbf{b} - \mathbf{c}$

It is easy to check that given any four points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in F$, no three of which are collinear, there is a choice of labeling so that $\mathbf{x} + \mathbf{y} = \mathbf{z} + \mathbf{w}$. Hence, if C is not a 2-cap, then C is not a Sidon set.

Conversely, suppose that C is not a Sidon set. Then either C contains three distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} + \mathbf{a} = \mathbf{b} + \mathbf{c}$ or C contains four distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ such that $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$. In the first case, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ so C contains a line. In the second case, $\mathbf{d} = \mathbf{a} + \mathbf{b} - \mathbf{c}$, so $\mathbf{d}$ lies in the plane determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and hence the four points are coplanar. In either case, C is not a 2-cap. □

Since, in $\mathbb{F}_3^n$, 2-caps correspond to Sidon sets, we will use the terms interchangeably throughout. We obtain an upper bound on $r(2, \mathbb{F}_3^n)$ by an easy counting argument (c.f. [2, Corollary 2.2]).

Proposition 3.3. *For any $n \in \mathbb{N}$, $r(2, \mathbb{F}_3^n) \cdot (r(2, \mathbb{F}_3^n) - 1) \leq 3^n - 1$.*

Proof. Suppose $C \subset \mathbb{F}_3^n$ is a 2-cap and hence, by Theorem 3.2, a Sidon set. For $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in C$, if $\mathbf{a} - \mathbf{b} = \mathbf{c} - \mathbf{d}$ then $\{\mathbf{a}, \mathbf{d}\} = \{\mathbf{c}, \mathbf{b}\}$ and so we have either $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} = \mathbf{c}$ and $\mathbf{b} = \mathbf{d}$. Therefore the set $\{\mathbf{a} - \mathbf{b} : \mathbf{a}, \mathbf{b} \in C, \mathbf{a} \neq \mathbf{b}\}$ has size $|C|(|C| - 1)$. Since these differences are nonzero, we have

$$|C|(|C| - 1) \leq 3^n - 1.$$

□

3.1. **Even dimension.**

Theorem 3.4. *If n is even, then $r(2, \mathbb{F}_3^n) = 3^{n/2}$.*

Proof. First we will show the lower bound, $r(2, \mathbb{F}_3^n) \geq 3^{n/2}$. Since $\mathbb{F}_3^n$ is additively isomorphic to $\mathbb{F}_3^{n/2} \times \mathbb{F}_3^{n/2}$, it suffices to construct a Sidon set of size $3^{n/2}$ in $\mathbb{F}_3^{n/2} \times \mathbb{F}_3^{n/2}$. As vector spaces over $\mathbb{F}_3$, $\mathbb{F}_3^{n/2}$ is isomorphic to $\mathbb{F}_{3^{n/2}}$, the finite field with $3^{n/2}$ elements. Hence, it suffices to construct a Sidon set of size $3^{n/2}$ in $\mathbb{F}_{3^{n/2}} \times \mathbb{F}_{3^{n/2}}$ This follows easily from the following claim, which was first proved in [1]. We include a short proof for completeness.

Claim 1. *Let q be an odd prime power and $\mathbb{F}_q$ be the finite field of order q. Then the set $\{(x, x^2) : x \in \mathbb{F}_q\}$ is a Sidon set in $\mathbb{F}_q \times \mathbb{F}_q$.*

Proof. Assume that

$$(a, a^2) + (b, b^2) = (c, c^2) + (d, d^2),$$

for some $a, b, c, d \in \mathbb{F}_q$. We must show that $\{a, b\} = \{c, d\}$. Without loss of generality, assume that $a \neq c$. Then we have

$$a - c = d - b$$

$$a^2 - c^2 = d^2 - b^2.$$

Since $a \neq c$, we may divide the second equation by the first to see that $a + c = b + d$. But this combined with the first equation gives $2a = 2d$. Since q is odd we have $a = d$ and therefore $b = c$. □

It is clear that the set $\{(x, x^2) : x \in \mathbb{F}_{3^{n/2}}\}$ has size $3^{n/2}$ and so we have $r(2, \mathbb{F}_3^n) \geq 3^{n/2}$. For the upper bound, let $C \subset \mathbb{F}_3^n$ be a 2-cap. Since n is even, $3^{n/2}$ is an integer, and if $|C| \geq 3^{n/2} + 1$, this contradicts Proposition 3.3. Therefore, $r(2, \mathbb{F}_3^n) \leq 3^{n/2}$. □

Corollary 3.5. *As $n \to \infty$, $r(2, \mathbb{F}_3^n)$ is $\Theta(3^{n/2})$.*

The construction above can be leveraged into the following partitioning theorem.

Theorem 3.6. *When n is even, there is a partition of $\mathbb{F}_3^n$ into maximal caps.*

This serves as an analogue to similar results for 1-caps in $\mathbb{F}_3^n$. It is well-known that $\mathbb{F}_3^3$ can be partitioned into three maximal 1-caps of size 9. It is possible to partition $\mathbb{F}_3^2$ into a single point and two disjoint maximal 1-caps of size 4. Finally, the main result of [7] shows that $\mathbb{F}_3^4$ can be partitioned into a single point and four disjoint maximal 1-caps of size 20.

Proof of Theorem 3.6. Since translates of Sidon sets are also Sidon sets, for each $a \in \mathbb{F}_3^n$ the set $S_a := \{(x, x^2 + a) : x \in \mathbb{F}_{3^{n/2}}\}$ is a maximal 2-cap. Since $(x, x^2 + a) = (y, y^2 + b)$ implies $x = y$ and hence $a = b$, we have that S_a and S_b are disjoint for $a \neq b$. Therefore, as a ranges over $\mathbb{F}_{3^{n/2}}$ the sets S_a cover 3^n points and thus there is the claimed partition. □

Question 3.7. By Corollary 2.3, all maximal 2-caps in $\mathbb{F}_3^2$ are affinely equivalent. Is this true in $\mathbb{F}_3^n$ when n is even?

We remark that when $n = 4$, a computer program verified that all maximal 2-caps sum to $\mathbf{0}$. If a set of nine points sums to $\mathbf{0}$ in $\mathbb{F}_3^n$, then its image under any affine transformation will likewise sum to $\mathbf{0}$, so this is a necessary condition for all maximal 2-caps in $\mathbb{F}_3^4$ to be affinely equivalent.

3.2. **Odd dimension.**

Lemma 3.8. *If $C = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is a 2-cap of size four in $\mathbb{F}_3^n$ then $D = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}\}$ is a 2-cap of size five.*

Proof. First we note that the points of D are distinct since if, without loss of generality, $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{a}$, this implies that $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are collinear, which is impossible since C is a 2-cap.

Now, suppose for contradiction that D is not a 2-cap, so there exist some $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in D$ with $\mathbf{x} + \mathbf{y} = \mathbf{z} + \mathbf{w}$. Since C is a 2-cap, we may assume that $\mathbf{x} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$. Without loss of generality, we then have that $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) + \mathbf{a} = \mathbf{b} + \mathbf{c}$ or $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) + \mathbf{a} = \mathbf{b} + \mathbf{b}$ or $2(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \mathbf{b} + \mathbf{c}$.

If $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) + \mathbf{a} = \mathbf{b} + \mathbf{c}$, then $\mathbf{a} = \mathbf{d}$ which is impossible since C has size four. If $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) + \mathbf{a} = \mathbf{b} + \mathbf{b}$, then $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$ contradicting the fact that C is a 2-cap. Finally, if $2(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \mathbf{b} + \mathbf{c}$, then $\mathbf{a} + \mathbf{d} = \mathbf{b} + \mathbf{c}$, again contradicting the fact that C is a 2-cap.

□

Theorem 3.9. *In $\mathbb{F}_3^3$, a maximal 2-cap has size 5, that is, $r(2, \mathbb{F}_3^3) = 5$. Further, all complete 2-caps are maximal and all maximal 2-caps are affinely equivalent.*

Proof. Since $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an affinely independent set in $\mathbb{F}_3^3$, by Lemma 3.8, $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ is a 2-cap in $\mathbb{F}_3^3$. Hence, $r(2, \mathbb{F}_3^3) \geq 5$. But by Proposition 3.3, $r(2, \mathbb{F}_3^3) < 6$ and hence $r(2, \mathbb{F}_3^3) = 5$.

Let C be any complete 2-cap in $\mathbb{F}_3^3$. Since $\mathbb{F}_3^3$ is a three-dimensional affine space, if $|C| \leq 3$, then $\mathbb{F}_3^3$ contains a point which is affinely independent from the points of C, so C cannot be complete. Hence, $|C| \geq 4$. But if $|C| = 4$ then by Lemma 3.8, C is not complete. Hence, $|C| = 5$, and any complete 2-cap in $\mathbb{F}_3^3$ is already maximal.

For the final claim, suppose C is a maximal 2-cap in $\mathbb{F}_3^3$. Pick any four points in C. Since these points are affinely independent, there exists an invertible affine transformation mapping these points to the set $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, we need only show that all maximal 2-caps containing $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are affinely equivalent.

It is easy to verify that there are exactly five such maximal 2-caps, namely:

(1) $C_1 = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 1, 1)\}$,
(2) $C_2 = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1, 2, 2)\}$,
(3) $C_3 = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (2, 1, 2)\}$,
(4) $C_4 = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (2, 2, 1)\}$, and
(5) $C_5 = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (2, 2, 2)\}$.

It suffices to exhibit an invertible affine transformation T_i mapping C_1 to C_i for $i = 2, 3, 4, 5$. We provide these T_i explicitly, writing $T_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{b}_i$ for an invertible matrix A_i and $\mathbf{b}_i \in \mathbb{F}_3^3$.

(1) $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

(2) $A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

(3) $A_4 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and

(4) $A_5 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $\mathbf{b}_5 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. □

Theorem 3.10. *A maximal* 2*-cap in* $\mathbb{F}_3^5$ *has size* 13*, that is,* $r(2, \mathbb{F}_3^5) = 13$.

Proof. Let C be a maximal 2-cap in $\mathbb{F}_3^5$. By Theorem 3.4 $r(2,\mathbb{F}_3^4) = 9$ so by Lemma 2.4 we may assume that $|C| \geq 9$. We will apply a sequence of affine transformations to C to conclude that lexicographically, the first points in C are $\{\mathbf{0}, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_2\}$ or $\{\mathbf{0}, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2\}$.

Given any four affinely independent points, there exists an invertible affine transformation mapping them to $\mathbf{0}$, $\mathbf{e}_5$, $\mathbf{e}_4$, and $\mathbf{e}_3$, so without loss of generality we may assume that C contains the subset $\{\mathbf{0}, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3\}$. These points all lie in the $(0,0)$-affine subspace of $\mathbb{F}_3^n$. Since $r(2,\mathbb{F}_3^3) = 5$, the $(0,0)$-affine subspace contains four points or five points of C. If it contains five points, then by Theorem 3.9, we may apply an affine transformation (using a block matrix) and assume that the fifth point is $\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5$.

Consider any other point $\mathbf{a} \in C$. Since $\mathbf{a}$ is not in the $(0,0)$-affine subspace, therefore $\{\mathbf{0}, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{a}\}$ is an affinely independent set so there exists an affine transformation T fixing $\mathbf{0}$, $\mathbf{e}_5$, $\mathbf{e}_4$, and $\mathbf{e}_3$ and mapping $\mathbf{a}$ to $\mathbf{e}_2$. Notice that if T is given by multiplication by the invertible matrix A followed by addition by $\mathbf{b} \in \mathbb{F}_3^5$, we have that

$$T(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) = A(\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) + \mathbf{b} = T(\mathbf{0}) + T(\mathbf{e}_3) + T(\mathbf{e}_4) + T(\mathbf{e}_5) = \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5$$

so T fixes $\mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5$.

Hence, up to affine equivalence, we may assume that the lexicographically earliest points in C are $\{\mathbf{0}, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5, \mathbf{e}_2\}$ or $\{\mathbf{0}, \mathbf{e}_5, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_2\}$. A computer program was used to enumerate all possible complete 2-caps beginning with these sets of points. This verified that $r(2,\mathbb{F}_3^5) = 13$. The C++ code for the program is available on the third author's professional website. □

Remark 3.11. The maximal 2-cap in $\mathbb{F}_3^5$ that is lexicographically earliest is explicitly given by the points: $(0,0,0,0,0)$, $(0,0,0,0,1)$, $(0,0,0,1,0)$, $(0,0,1,0,0)$, $(0,0,1,1,1)$, $(0,1,0,0,0)$, $(0,1,1,1,2)$, $(0,2,1,2,0)$, $(0,2,2,1,2)$, $(1,0,0,0,0)$, $(1,0,1,2,1)$, $(2,0,1,0,2)$, $(2,2,0,2,2)$.

We conclude by giving bounds on $r(2,\mathbb{F}_3^7)$.

Proposition 3.12. *One has that* $33 \leq r(2,\mathbb{F}_3^7) \leq 47$.

Proof. The upper bound on $r(2,\mathbb{F}_3^7)$ is a consequence of Proposition 3.3. For the lower bound, we constructed a 2-cap of size 33 by first embedding a maximal 2-cap in $\mathbb{F}_3^6$ as a 2-cap C of size 27 in $\mathbb{F}_3^7$. We then used a computer program to enumerate all complete 2-caps containing C as a subset. The largest of these complete 2-caps has size 33. The lexicographically earliest one is given by the points: $(0,0,0,0,0,0,0)$, $(1,0,0,1,0,0,1)$, $(0,0,0,2,0,0,1)$, $(0,0,1,0,1,0,0)$, $(0,0,1,1,1,2,1)$, $(0,0,1,2,1,1,1)$, $(0,0,2,0,1,0,0)$, $(0,0,2,1,1,1,1)$, $(0,0,2,2,1,2,1)$, $(0,1,0,0,1,2,0)$, $(0,1,0,1,0,2,1)$,

$(0,1,0,2,2,2,1)$, $(0,1,1,0,2,1,1)$, $(0,1,1,1,1,0,2)$, $(0,1,1,2,0,2,2)$, $(0,1,2,0,2,0,2)$, $(0,1,2,1,1,1,0)$, $(0,1,2,2,0,2,0)$, $(0,2,0,0,1,2,0)$, $(0,2,0,1,2,2,1)$, $(0,2,0,2,0,2,1)$, $(0,2,1,0,2,0,2)$, $(0,2,1,1,0,2,0)$, $(0,2,1,2,1,1,0)$, $(0,2,2,0,2,1,1)$, $(0,2,2,1,0,2,2)$, $(0,2,2,2,1,0,2)$, $(1,0,0,0,0,0,0)$, $(1,0,0,0,0,0,1)$, $(2,0,0,1,0,2,0)$, $(2,0,0,1,1,0,1)$, $(2,0,0,1,1,1,2)$, and $(2,0,0,1,1,2,2)$. □

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