Anti-Ramsey Multiplicities

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January 3, 2018

Abstract

The Ramsey multiplicity constant of a graph H is the minimum proportion of copies of H in the complete graph which are monochromatic under an edge-coloring of K_n as n goes to infinity. Graphs for which this minimum is asymptotically achieved by taking a random coloring are called *common*, and common graphs have been studied extensively, leading to the Burr-Rosta conjecture and Sidorenko's conjecture. Erdős and Sós asked what the maximum number of rainbow triangles is in a 3-coloring of the edge set of K_n, a rainbow version of the Ramsey multiplicity question. A graph H is called r-anti-common if the maximum proportion of rainbow copies of H in any r-coloring of $E(K_n)$ is asymptotically achieved by taking a random coloring. In this paper, we investigate anti-Ramsey multiplicity for several families of graphs. We determine classes of graphs which are either anti-common or not. Some of these classes follow the same behavior as the monochromatic case, but some of them do not. In particular the rainbow equivalent of Sidorenko's conjecture, that all bipartite graphs are anti-common, is false.

1 Introduction

A *graph* $G = (V, E)$ consists of a set $V = V(G) = \{v_1, \dots, v_n\}$ of *vertices* and a set $E = E(G) = \{vv' : v, v' \in V\}$ of *edges*, where vv' is an unordered pair of vertices. All graphs considered here are finite and simple so that $vv' \in E$ implies $v \neq v'$ and each pair of vertices has at most one edge between them. Let $e(G) = |E(G)|$ denote the *size* of the graph G. Given a vertex $v \in V(G)$, the *degree* of v is the number of edges of G which contain v, i.e. $d(v) := |\{e \in E : v \in e\}|$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, we write $H \subseteq G$ and we say G contains a *copy* of H.

We will be considering many classes of graphs, including the complete graph, the cycle, and subclasses of bipartite graphs. The *complete graph* K_n is a graph on n vertices such that every pair of vertices has an edge between them. A *cycle* on n vertices, denoted C_n, has a vertex set which can be ordered $v_1, \ldots, v_n$ such that $E(C_n) = \{v_i v_{i+1} : i \in \{1, \ldots, n-1\}\} \cup \{v_1 v_n\}$. A graph $G = (V, E)$ is *bipartite* if its vertex set can be partitioned into two disjoint sets $V = X \sqcup Y$ such that every edge connects a vertex in X to one in Y. Note that the cycle C_n is bipartite if and only if n is even. The *complete bipartite graph* $K_{n,m}$ has parts X, Y such that $|X| = n$ and $|Y| = m$ and edge set $E(K_{n,m}) = \{xy : x \in X, y \in Y\}$. *Stars* are complete bipartite graphs of the form $K_{1,m}$; we call the vertex with degree m the *center* and a vertex of degree 1 a *leaf.* A *matching* of size m, is a disjoint set of m edges and is denoted mK_2. Removing an edge from the cycle C_n gives a *path* on n vertices which we write as P_n. Matchings and paths are both subclasses of the set of bipartite graphs. Figure 1 contains examples of each of these graphs.

An *r-edge-coloring* of a graph G is a function with domain $E(G)$ and codomain a set of r colors, $\{1, \ldots, r\}$. Given an edge coloring c of G, a subgraph H of G is said to be *monochromatic* if for every $e, f \in E(H)$ $c(e) = c(f)$. A subgraph is monochromatic if all its edges are the same color (e.g., Figure 2).

Given a complete graph K_n and a subgraph H of K_n, it is an interesting question to determine how many monochromatic copies of H are we guaranteed to find in any r-edge-coloring of K_n. The maximum number we can guarantee is known as the Ramsey multiplicity. In particular, the *Ramsey multiplicity* $M_r(H; n)$ is the minimum over all r-edge-colorings of K_n of the number of monochromatic copies of H. We consider the Ramsey multiplicity of a graph H with m vertices relative to the number of copies of H in K_n via the ratio

$$C_r(H; n) = \frac{M_r(H; n)}{\binom{n}{m} \frac{m!}{|\mathrm{Aut}(H)|}}.$$

The denominator is the number of copies of H in K_n where $\mathrm{Aut}(H)$ is the set of automorphisms of H. Intuitively, this ratio can be thought of as the probability a randomly chosen copy of H in K_n is monochromatic. We can obtain an immediate bound on $C_r(H; n)$ by coloring each edge of K_n color i independently with probability $\frac{1}{r}$. Under this random coloring, any copy of H in K_n is monochromatic with probability $r^{1-e(H)}$. This gives an upper bound on $C_r(H; n)$ of $r^{1-e(H)}$. In [15], Jagger, Šťovíček, and Thomason show that $C_r(H; n)$ is nondecreasing in n and so since it is also bounded the limit

$$C_r(H) = \lim_{n \to \infty} C_r(H; n),$$

exists and is known as the *Ramsey multiplicity constant* of H [10].

The earliest result in this area was by Goodman in 1959 who proved $C_2(K_3) = \frac{1}{4}$ [11]. In 1962, Erdős conjectured that $C_2(K_n) = 2^{1-\binom{n}{2}}$ for all cliques [8]. Burr and Rosta later conjectured that for all graphs H, $C_2(H) = 2^{1-e(H)}$ [4]. We call a graph *common* if it satisfies the Burr-Rosta conjecture. Sidorenko disproved the Burr-Rosta conjecture by showing that a triangle with a pedant edge is not common [18]. Thomason disproved the initial conjecture of Erdős by showing

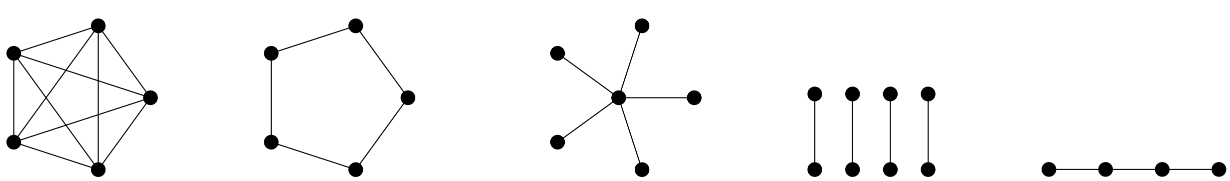

Figure 1: K_5, C_5, $K_{1,5}$, $4K_2$, and P_4 respectively.

that for $p \geq 4$, K_p is not common [20]. Sidorenko conjectured instead that all bipartite graphs are common [17], this conjecture is well-known and is referred to as Sidorenko's conjecture. Much work has been done on the both the Burr-Rosta conjecture (see, e.g., [15, 11, 4, 18, 19, 14]) and on Sidorenko's conjecture (c.f. [2, 6, 13, 16]). If we instead consider $r > 2$, we call H is called *r-common* if $C_r(H) = r^{1-e(H)}$. Jagger et. al. showed that if a graph G is not r-common, then it is not $(r+1)$-common [15]. In 2011, Cummings and Young proved that no graph containing K_3 is 3-common [1]. There are many open questions which remain for $r > 2$.

We will consider a similar parameter to the Ramsey multiplicity constant by searching for rainbow subgraphs as opposed to monochromatic subgraphs. Given an edge coloring c of G, a subgraph H of G is said to be *rainbow* if for every pair of distinct edges $e, f \in E(H)$, $c(e) \neq c(f)$. In Figure 2, the edges 13 and 34 form a rainbow copy of P_2. Under this umbrella, a minimization problem is uninteresting since it is possible to color all edges the same color and hence contain no rainbow copy of H (assuming $e(H) > 1$). Instead, we ask what is the maximum number of rainbow copies of H we can find amongst all edge colorings of K_n. Let $rb_r(H; n)$ be the maximum over all r-edge-colorings of K_n of the number of rainbow copies of H and call this the *anti-Ramsey multiplicity* of H. In this paper, we will build the theory of the anti-Ramsey multiplicity constant and prove/disprove r-anti-commonality of various classes of graphs.

2 The anti-Ramsey multiplicity constant

Before we define the anti-Ramsey multiplicity constant, we will first prove that given a graph H, the maximum probability a copy of H is rainbow under a coloring of K_n is bounded and monotone as a function of n. As in the Ramsey case, we will consider the anti-Ramsey multiplicity of a graph H with m vertices relative to the number of copies of H in K_n via the ratio

$$rbC_r(H; n) = \frac{rb_r(H; n)}{\binom{n}{m}\frac{m!}{|\mathrm{Aut}(H)|}}.$$

For the remainder of this section, fix a graph $H = (V, E)$ with $|V| = m$ and $e(H) = e$.

Proposition 2.1.

$$rbC_r(H; n) \geq \frac{\binom{r}{e}e!}{r^e}$$

Proof. We will color the edges of K_n uniformly and independently at random from the set $\{1, \ldots, r\}$. In particular, each edge is colored color i with probability $\frac{1}{r}$ for $i = 1, \ldots, r$. The number of possible rainbow edge assignments of a graph with e edges is $\binom{r}{e}e!$ and a given edge

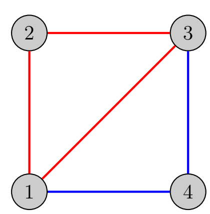

Figure 2: The vertices $\{1, 2, 3\}$ form a monochromatic K_3.

assignment occurs with probability $\left(\frac{1}{r}\right)^e$. Thus the expected probability that a randomly selected copy of H in K_n is rainbow is given by $\frac{\binom{r}{e}e!}{r^e}$. Therefore there exists a coloring such that this probability is at least $\frac{\binom{r}{e}e!}{r^e}$ and since $rbC_r(C;n)$ is the maximum over all such probabilities, the inequality follows. □

Proposition 2.2.

$$rbC_r(H;n) \leq rbC_r(H;n-1)$$

Proof. The inequality is clear if $rbC_r(H;n) = 0$ and so we suppose otherwise. Equivalently, we must show

$$\frac{rb_r(H;n)}{\binom{n}{m}} \leq \frac{rb_r(H;n-1)}{\binom{n-1}{m}} \iff$$

$$\frac{rb_r(H;n)}{\frac{n!}{m!(n-m)!}} \leq \frac{rb_r(H;n-1)}{\frac{(n-1)!}{m!(n-m-1)!}} \iff$$

$$\frac{rb_r(H;n)}{\frac{n}{n-m}} \leq rb_r(H;n-1) \iff$$

$$(n-m)rb_r(H;n) \leq rb_r(H;n-1)n$$

Let c_r be an r-edge-coloring of K_n such that the number of rainbow copies of H in K_n under coloring c_r is exactly $rb_r(H;n)$. We will count the order of the set

$$H_n := \{(G,H) : G \text{ is a } K_{n-1} \subseteq K_n \text{ and } H \subseteq G \text{ is rainbow}\}$$

in two ways. First, note that each rainbow copy of H is contained in $n-m$ different K_{n-1} by removing any vertex in K_n that is not a vertex of H. Since there are exactly $rb_r(H;n)$ copies of H in K_n, $|H_n| = (n-m)rb_r(H;n)$. Now each K_{n-1} in K_n contains at most $rb_r(H;n-1)$ rainbow copies of H and so $|G_n| \leq rb_r(G;n-1)n$. Therefore

$$(n-m)rb_r(H;n) = |H_n| \leq rb_r(H;n-1)n,$$

which implies the result. □

We are now ready to define the anti-Ramsey multiplicity constant.

Corollary 2.3. *The anti-Ramsey multiplicity constant, given by*

$$rbC_r(H) = \lim_{n\to\infty} rbC_r(H;n),$$

exists and is finite.

Proof. By Propositions 2.1 and 2.2, the sequence $\{rbC_r(H;n)\}_{n=m}^{\infty}$ is bounded and monotone. Hence by the Monotone Convergence Theorem, the limit exists and is finite. □

Note that the anti-Ramsey multiplicity constant has the same lower bound as that of Proposition 2.1, motivating the following definition.

Definition 2.4. For $r \geq m$, we say that H is r-anti-common if

$$rbC_r(H) = \frac{\binom{r}{e}e!}{r^e}.$$

If H is r-anti-common for all $r \geq m$, H is called anti-common.

3 Anti-common graphs

In this section we will prove anti-commonality for matchings and disjoint unions of stars. We will state but not prove the number of automorphisms for each graph in question and for more details regarding automorphisms of graphs see [3]. Since the anti-Ramsey multiplicity constant is a limit as $n \to \infty$, the order of term relative to n is especially important. Because of this, we will introduce *big O notation.* Suppose $f(n)$ and $g(n)$ are two real-valued functions. We say

$$f(n) = O(g(n))$$

if and only if there exist positive constants C, N such that $|f(n)| \leq C|g(n)|$ for all $n > N$. In the following lemma, will use the fact that if $f(n) = O(n^{m-1})$, then

$$\lim_{n\to\infty} \frac{f(n)}{\binom{n}{m}} = 0.$$

Lemma 3.1. *If $H = (V, E)$ has order m and size e such that for sufficiently large n*

$$rb_r(H; n) \leq \frac{n^m \binom{r}{e} e!}{|Aut(H)| r^e} + O(n^{m-1}),$$

then H is r-anti-common.

Proof. Assume that for n large enough we have $rb_r(H; n) \leq \frac{n^m \binom{r}{e} e!}{|\text{Aut}(H)| r^e} + O(n^{m-1})$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{rb_r(H; n)}{\binom{n}{m} \frac{m!}{|\text{Aut}(H)|}} &\leq \lim_{n\to\infty} \frac{\frac{n^m \binom{r}{e} e!}{|\text{Aut}(H)| r^e} + O(n^{m-1})}{\binom{n}{m} \frac{m!}{|\text{Aut}(H)|}} \\
&= \frac{\binom{r}{e} e!}{r^e} \lim_{n\to\infty} \frac{n^m + O(n^{m-1})}{\binom{n}{m} m!} \\
&= \frac{\binom{r}{e} e!}{r^e} \lim_{n\to\infty} \frac{n^m + O(n^{m-1})}{n(n-1)\cdots(n-m+1)} \\
&= \frac{\binom{r}{e} e!}{r^e}.
\end{aligned}$$

□

We will also use the following inequality, often referred to as Maclaurin's inequality.

Fact 3.2. *Given positive integers $r \leq n$ and positive real numbers $x_1, \ldots, x_n$,*

$$\sum_{\{i_1, i_2, \ldots, i_r\} \subseteq [n]} x_{i_1} x_{i_2} \cdots x_{i_r} \leq \binom{n}{r} \left(\frac{\sum_{i=1}^n x_i}{n} \right)^r$$

Proposition 3.3. *Matchings are anti-common.*

Proof. Let k be the size of the matching and note $|\text{Aut}(kK_2)| = 2^k k!$. Using Lemma 3.1, it suffices to show for n sufficiently large,

$$rb_r(kK_2; n) = \frac{n^{2k} \binom{r}{k}}{2^k r^k} + O(n^{2k-1})$$

Let q_i be the number of edges of color i, then there are

$$\sum_{\{i_1,i_2,\ldots,i_k\}\subseteq[r]} q_{i_1}q_{i_2}\cdots q_{i_k}$$

rainbow subgraphs of size k. These rainbow subgraphs form a matching of size k if and only if they have order $2k$. The complete graph K_n contains $O(n^j)$ subgraphs of size k on j vertices. Hence the number of sets of k edges which form a rainbow subgraph on fewer than $2k$ vertices is

$$\sum_{j=0}^{2k-1} O(n^j) = O(n^{2k-1}).$$

Therefore there are

$$\sum_{\{i_1,i_2,\ldots,i_k\}\subseteq[r]} q_1q_2\cdots q_k + O(n^{2k-1})$$

rainbow matchings of size k. Note that $\sum_{i=1}^r q_i = \binom{n}{2}$ and so by Fact 3.2 we have

$$\begin{aligned} rb_r(kK_2;n) &= \sum_{\{i_1,i_2,\ldots,i_k\}\subseteq[r]} q_{i_1}q_{i_2}\cdots q_{i_k} + O(n^{2k-1}) \\ &\leq \binom{r}{k}\left(\frac{\binom{n}{2}}{r}\right)^k + O(n^{2k-1}) \\ &\leq \frac{n^{2k}\binom{r}{k}}{2^k r^k} + O(n^{2k-1}). \end{aligned}$$

□

We will use a similar proof to show that stars $K_{1,m-1}$ are anti-common. This lemma will be used in the proof of Theorem 3.5 which generalizes the result to disjoint unions of stars.

Lemma 3.4. *Stars are anti-common.*

Proof. Consider $S = K_{1,m-1}$ and note that

$$|\mathrm{Aut}(S)| = (m-1)!.$$

By Lemma 3.1, It suffices to prove that for sufficiently large n,

$$rb_r(K_{1,m-1};n) = \frac{\binom{r}{m-1}n^m}{r^{m-1}} + O(n^{m-1})$$

Given a vertex v of K_n, let q_i be the number of edges of color i incident with v. Then the number of rainbow copies of S with center v is

$$\sum_{\{i_1,i_2,\cdots,i_{m-1}\}\subseteq[r]} q_{i_1}q_{i_2}\cdots q_{i_{m-1}}.$$

Vertices of K_n have degree $n-1$, so by Fact 3.2 we have

$$\sum_{\{i_1,i_2,\cdots,i_{m-1}\}\subseteq[r]} q_{i_1}q_{i_2}\cdots q_{i_{m-1}} \leq \left(\frac{n-1}{r}\right)^{m-1}\binom{r}{m-1}.$$

Stars with centers v and v' are distinct if $v \neq v'$, therefore the total number of rainbow copies of S in K_n is at most

$$n\left(\frac{n-1}{r}\right)^{m-1}\binom{r}{m-1} = \frac{\binom{r}{m-1}n^m}{r^{m-1}} + O(n^{m-1}).$$

□

Theorem 3.5. *Disjoint unions of stars are anti-common.*

Proof. Fix positive integers $k \leq m$ and let $\mathcal{P}_k^{\geq 2}(m)$ denote the set of integer partitions of m into k parts with each part having size at least 2. For $P = \{\{m_1, \ldots, m_k\}\} \in \mathcal{P}_k^{\geq 2}(m)$, let S_P be a disjoint union of k stars with components $S_{P,i} = K_{1,m_i-1}$ for $i = 1, \ldots, k$. Let $m_{i_1} \leq \cdots \leq m_{i_{j(P)}}$ be the $j(P)$ distinct sizes of the stars in S_P and let M_s be the number of stars in S_P of size m_{i_s}. Then defining $\gamma(P) = \prod_{i=1}^{j(P)} M_i!$, we have the number of automorphisms of S_P is given by

$$|\mathrm{Aut}(S_P)| = \gamma(P)\prod_{i=1}^{k}(m_i - 1)!.$$

Given $P \in \mathcal{P}_k^{\geq 2}(m)$, let

$$\binom{m-k}{P-1} = \binom{m-k}{m_1 - 1, \ldots, m_k - 1}$$

then we want to show for sufficiently large n

$$rb_r(S_P; n) = \binom{m-k}{P-1}\frac{\binom{r}{m-k}\binom{n}{m}m!}{\gamma(P)r^{m-k}} + O(n^{m-1}).$$

Claim 3.6.

$$\sum_{P \in \mathcal{P}_k(m)} \gamma(P) rb_r(S_P; n) \leq \sum_{P \in \mathcal{P}_k(m)} \binom{m-k}{P-1}\frac{\binom{n}{m}m!\binom{r}{m-k}}{r^{m-k}}$$

Proof. Let $\mathcal{C}_k(n)$ denote the collection of sets of k distinguishable vertices in K_n. Given $C \in \mathcal{C}_k(n)$, we will count all the number of rainbow disjoint unions of k stars with exactly m vertices and with C the set of centers. Let $q_i(C)$ denote the number of edges of color i incident to any vertex in C, except those edges between two vertices in C. Then the number of rainbow disjoint unions of k stars with m vertices and distinguishable centers C is exactly

$$\sum_{\{i_1,\ldots,i_{m-k}\} \subseteq [r]} q_{i_1}(C) \cdots q_{i_{m-k}}(C). \tag{1}$$

Note that $\sum_{i=1}^{r} q_i(C) = k(n-1) - \binom{k}{2}$ and so by Fact 3.2 the sum in (1) is at most

$$\binom{r}{m-k}\left(\frac{k(n-1) - \binom{k}{2}}{r}\right)^{m-k}.$$

The lefthand size of the inequality of this claim counts rainbow subgraphs such that given P. if $S_{P,i}$ and $S_{P,j}$ have the same order they will be distinguishable in the count above. Therefore since

$|\mathcal{C}_k(n)| = \binom{n}{k}k!$, we have

$$\sum_{P\in\mathcal{P}_k(m)} \gamma(P) rb_r(S_P;n) \le \binom{n}{k}k!\binom{r}{m-k}\left(\frac{k(n-1)-\binom{k}{2}}{r}\right)^{m-k}$$
$$\le \frac{\binom{r}{m-k}n^m}{r^{m-k}}k^{m-k} + O(n^{m-1})$$

Notice that

$$\{\{\{m_1-1,\dots,m_k-1\}\} : \{\{m_1,\dots,m_k\}\} \in \mathcal{P}_k^{\ge 2}(m)\}$$

is the set of integer partitions of $m-k$ into k parts. Therefore, by the Multinomial Theorem, we can rewrite

$$\frac{\binom{r}{m-k}n^m}{r^{m-k}}k^{m-k} + O(n^{m-1}) = \frac{\binom{r}{m-k}n^m}{r^{m-k}} \sum_{\{\{m_1,\dots,m_k\}\}\in\mathcal{P}_k(m)} \binom{m-k}{m_1-1,\dots,m_k-1} + O(n^{m-1})$$
$$= \sum_{P\in\mathcal{P}_k^{\ge 2}(m)} \binom{m-k}{P-1}\frac{\binom{r}{m-k}\binom{n}{m}m!}{r^{m-k}} + O(n^{m-1})$$

which proves the claim. □

By Proposition 2.1, we have for each $P = \{\{m_1,\dots,m_k\}\} \in \mathcal{P}_k^{\ge 2}(m)$,

$$\gamma(P) rb_r(S_P;n) \ge \frac{(m-k)!\binom{r}{m-k}\binom{n}{m}m!}{\prod_{i=1}^k (m_i-1)!r^{m-k}} + O(n^{m-1}) \tag{2}$$
$$= \binom{m-k}{P-1}\frac{\binom{r}{m-k}\binom{n}{m}m!}{r^{m-k}} + O(n^{m-1}). \tag{3}$$

Therefore, Claim 3.6 and the inequality (3) above implies for each $P \in \mathcal{P}_k^{\ge 2}(m)$,

$$rb_r(S_P;n) = \binom{m-k}{P-1}\frac{\binom{r}{m-k}\binom{n}{m}m!}{\gamma(P)r^{m-k}}.$$

□

4 Graphs which are not anti-common

Not all graphs are r-anti-common for all r, and here we will prove in particular that complete graphs and K_4 without an edge are not anti-common. We will also give sufficient conditions, based on the number of edges, for a graph to not be anti-common.

4.1 Specific graphs which are not anti-common

In order to show that a graph is not anti-common for some r, we will construct a coloring with more rainbow subgraphs than that guaranteed in Proposition 2.1. Our arguments will start with a fixed coloring of some K_m for m small and we will use a blow-up argument to construct a coloring of a larger K_n.

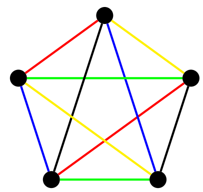

Figure 3: A 5-edge-coloring of K_5 with 10 rainbow copies of $K_4\backslash e$.

Definition 4.1. A *blow-up* is an inductive coloring of K_n, where the edges are colored as follows. Pick $m \leq n$ and fix a coloring of K_m with labeled vertices $v_1, \dots, v_m$. Divide the vertices of K_n into m disjoint sets of size $\lfloor \frac{n}{m} \rfloor$ and $\lceil \frac{n}{m} \rceil$, namely $V_1, \dots, V_m$. For $u_i \in V_i$ and $u_j \in V_j$, color the edge $u_i u_j$ the same color as the edge $v_i v_j$ in the coloring of K_m. Repeat this process with each V_i until there are no vertices left to be split into m disjoint sets. We call this a blow-up of the initial coloring of K_m with n vertices.

Proposition 4.2. *The graph $K_4\backslash e$ is not 5-anti-common.*

Proof. Note that the 5-edge-coloring of K_5 in Figure 4.1 contains 10 rainbow copies of $K_4\backslash e$. Given $n = 5^k$ for k a positive integer, let $F(n)$ be the number of rainbow copies of $K_4\backslash e$ contained in a blow-up of the coloring in Figure 4.1 on n vertices. Within each of the 5 parts, there are $5F\left(\frac{n}{5}\right)$ rainbow copies of $K_4\backslash e$ and there are $10\left(\frac{n}{5}\right)^4$ with one vertex in each part. Therefore

$$F(n) \geq 5F\left(\frac{n}{5}\right) + 10\left(\frac{n}{5}\right)^4$$

and solving this recurrence gives

$$F(n) \geq \frac{n^4}{62} + O(n^3).$$

There are 4 automorphisms of $K_4\backslash e$, hence

$$\begin{aligned} rb_r(K_4\backslash e; n) &\geq \frac{n^4}{62} + O(n^3) \\ &> \frac{6n^4}{625} + O(n^3) \\ &= \frac{\binom{n}{4} 4! \binom{5}{5} 5!}{4 \cdot 5^5} + O(n^3). \end{aligned}$$

□

In [9], it was shown that K_3 is not 3-anti-common. We will now prove for $a \geq 4$, K_a is not $\binom{a}{2}$-anti-common.

Theorem 4.3. *The complete graph K_a is not $\binom{a}{2}$-anti-common.*

Proof. Consider a rainbow K_a, i.e. let c be an $\binom{a}{2}$-edge-coloring of K_a such that each edge is a different color. Given $n = a^k$ for k a positive integer, let $F(n)$ denote the number of rainbow copies of K_a contained in a blow-up of the coloring c on n vertices. There are $aF\left(\frac{n}{a}\right)$ rainbow copies of K_a within each of the a parts, and there are $\left(\frac{n}{a}\right)^a$ rainbow copies of K_a with exactly one vertex from each part. Therefore

$$F(n) \geq aF\left(\frac{n}{a}\right) + \left(\frac{n}{a}\right)^a$$

and solving this recurrence gives

$$F(n) \geq \frac{n^a}{a^a - a} + O(n^{a-1}).$$

Therefore, since the number of automorphisms of K_a is $a!$, in order to show

$$\frac{n^a}{a^a - a} + O(n^{a-1}) > \frac{\binom{n}{a}\binom{a}{2}!}{\binom{a}{2}^{\binom{a}{2}}}$$

we will prove

$$\frac{a!}{a^a - a} > \frac{\binom{a}{2}!}{\binom{a}{2}^{\binom{a}{2}}}. \tag{4}$$

We will use the following bounds on the factorial function

$$e\left(\frac{\binom{a}{2}}{e}\right)^{\binom{a}{2}} \leq \binom{a}{2}! \leq e\binom{a}{2}\left(\frac{\binom{a}{2}}{e}\right)^{\binom{a}{2}}$$

where e is the base of the natural logarithm. From this we have

$$\frac{\binom{a}{2}!}{\binom{a}{2}^{\binom{a}{2}}} \leq \frac{\binom{a}{2}}{e^{\binom{a}{2}-1}}$$

and also using the inequality from (4), $\frac{a!}{a^a-a} \geq \frac{1}{e^{a-1}}$ and therefore it's enough to show

$$\frac{\binom{a}{2}}{e^{\binom{a}{2}-1}} < \frac{1}{e^{a-1}}.$$

One can check that this inequality holds for $a \geq 4$ which concludes the proof. $\square$

4.2 Sufficient conditions for not anti-commonality

In what follows log represents the natural logarithm. We will also be using both sides of the Stirling's approximation given below.

Theorem 4.4 (Stirling's Approximation)**.**

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

Theorem 4.5. *Suppose H is a graph with m vertices and e edges and let c be a constant such that $2\pi m(1-c) > 1$ and*

$$c + (1-c)\log(1-c) \geq \frac{2}{m-1} + \frac{1}{\binom{m}{2}^2 12}.$$

If $e \geq c\binom{m}{2}$, then H is not $\binom{m}{2}$-anti-common.

Proof. Let H be a graph which satisfies the hypothesis above and consider a rainbow coloring of H. Blow-up this coloring to n vertices and similar work as that in the proof of Theorem 4.3 gives that the number of rainbow copies of H in K_n is at least

$$\frac{n^m m!}{m^m} + O(n^{m-1}).$$

From the relationships between c and m we have

$$c\binom{m}{2} - \frac{1}{\binom{m}{2}12} + (1-c)\binom{m}{2}\log(1-c) - m \geq 0$$

and so raising both sides by the base of the logarithm e gives

$$e^{c\binom{m}{2} - \frac{1}{\binom{m}{2}12} - m}(1-c)^{\binom{m}{2}(1-c)} \geq 1.$$

Then since $2\pi m(1-c) > 1$ we have

$$\begin{aligned}
\sqrt{2\pi m(1-c)}e^{c\binom{m}{2} - \frac{1}{\binom{m}{2}12} - m}(1-c)^{\binom{m}{2}(1-c)} &> 1 \\
\frac{\sqrt{2\pi m}}{e^m} &> \frac{e^{\frac{1}{\binom{m}{2}12}}}{\sqrt{1-c}e^{c\binom{m}{2}}(1-c)^{\binom{m}{2}(1-c)}} \\
&= \frac{e^{\frac{1}{\binom{m}{2}12}}\left(\frac{\binom{m}{2}}{e}\right)^{\binom{m}{2}}}{\binom{m}{2}^{c\binom{m}{2}}\left(\frac{\binom{m}{2}(1-c)}{e}\right)^{\binom{m}{2}(1-c)}\sqrt{1-c}} \\
&\geq \frac{\binom{m}{2}!}{\binom{m}{2}^{c\binom{m}{2}}\left(\binom{m}{2} - c\binom{m}{2}\right)!\sqrt{1-c}} \\
&= \frac{\binom{\binom{m}{2}}{c\binom{m}{2}}\left(c\binom{m}{2}\right)!}{\binom{m}{2}^{c\binom{m}{2}}} \\
&\geq \frac{\binom{\binom{m}{2}}{e}e!}{\binom{m}{2}^e}.
\end{aligned}$$

Using Stirling's approximation, we have

$$\frac{\sqrt{2\pi m}}{e^m} \leq \frac{m!}{m^m}.$$

and therefore

$$\frac{n^m m!}{m^m} + O(n^{m-1}) > \frac{n^m\binom{\binom{m}{2}}{e}e!}{\binom{m}{2}^e} + O(n^{m-1})$$

∎

Corollary 4.6. *Let H be a graph on m vertices and e edges such that*

$$e > m\sqrt{m-1}.$$

Then for $m \geq 6$, H is not $\binom{m}{2}$-anti-common.

Proof. Let H be a graph that satisfies the hypothesis and set $c = \frac{2}{\sqrt{m-1}}$. Since $2\pi m(1-c) > 1$ for $m \geq 6$, we can apply Proposition 4.5 and thus it suffices to show

$$c + (1-c)\log(1-c) \geq \frac{2}{m-1} + \frac{1}{\binom{m}{2}^2 12}.$$

For $m \geq 6$ we also have $|c| < 1$, so we can expand the log function as follows

$$\begin{aligned} c + (1-c)\log(1-c) &= c + (1-c)\left(-c - \frac{c^2}{2} - \frac{c^3}{3} - \cdots\right) \\ &= \sum_{i=2}^{\infty} \frac{1}{i(i-1)} c^i \\ &= \frac{2}{m-1} + \frac{4}{3(m-1)^{3/2}} + \sum_{i=4}^{\infty} \frac{1}{i(i-1)} \left(\frac{2}{\sqrt{m-1}}\right)^i \\ &> \frac{2}{m-1} + \frac{1}{\binom{m}{2}^2 12}. \end{aligned}$$

□

Using Corollary 4.6, we are able to show that a similar conjecture to that of Sidorenko's in the Ramsey setting does not hold in the anti-Ramsey setting.

Corollary 4.7. *Not all bipartite graphs are anti-common.*

Proof. Consider the complete bipartite graph $K_{8,8}$. This graph has 16 vertices and 64 edges, and in particular

$$e(K_{8,8}) = 64 > 62 > 16\sqrt{15}.$$

By Proposition 4.6, $K_{8,8}$ is not $\binom{16}{2}$-anti-common and is therefore not anti-common. □

Indeed, Corollary 4.6 shows that almost all bipartite graphs are not anti-common. If Sidorenko's conjecture is true, this is very different behavior from the monochromatic situation.

5 Future directions

As in the Ramsey case, we wish to establish an implication between a graph being r-anti-common and $(r+1)$-anti-common. Through our investigation of this problem, we have shown the following inequality.

Proposition 5.1. *Let H be a graph with e edges, then*

$$rb_{r+1}(H;n) \geq rb_r(H;n) \geq \left(\frac{(r+e)(r+1-e)}{r(r+1)}\right) rb_{r+1}(H;n).$$

Proof. Since the set of $(r+1)$-edge-colorings contains the set of r-edge-colorings, the left inequality follows immediately. Now consider an $(r+1)-$edge-coloring of K_n such that the number of rainbow copies of H is exactly $rb_{r+1}(H;n)$. Randomly choose a color from $[r+1]$ and call it r'. For all edges colored r', recolor them randomly from the set of colors $[r+1]\setminus\{r'\}$. In the initial coloring, the expected number of rainbow copies of H with one edge colored r' is

$$\frac{rb(G,n,r+1)e}{r+1}.$$

With probability $\frac{r-e+1}{r}$, each of these rainbow subgraphs will remain rainbow in the new coloring. Therefore the expected number of rainbow copies of H in the new coloring is

$$\left(rb_{r+1}(H;n) - \frac{rb_{r+1}(H;n)e}{r+1}\right) + \frac{rb_{r+1}(H;n)e(r-e+1)}{r(r+1)} = \left(\frac{(r+e)(r+1-e)}{r(r+1)}\right) rb_{r+1}(H;n).$$

This implies that there exists such a coloring of K_n with r colors and hence

$$\left(\frac{(r+e)(r+1-e)}{r(r+1)}\right) rb_{r+1}(H;n) \leq rb_r(H;n).$$

□

This inequality leads us to believe that the implication below is in fact true.

Conjecture 5.2. *If H is not r-anti-common, then H is not $(r+1)$-anti-common.*

There are also many other classes of graphs whose anti-commonality have yet to be studied. Preliminary results on cycles lead us to believe that for $k \geq 3$, cycles of length k are not k-anti-common. One can show using the blow-up method in Section 4 that C_4 is not 4-anti-common and that C_5 is not 5-anti-common.

6 Acknowledgments

We would like to thank Carnegie Mellon University for supporting the Summer Undergraduate Applied Mathematics Institute. Additionally, we gratefully acknowledge financial support for this research from the following grants: NSF DGE-1041000 (Jessica De Silva), NSF DMS-1606351 (Michael Tait), and NSF DMS-1719841 (Michael Young).

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