Independent sets in polarity graphs

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Abstract

Given a projective plane Σ and a polarity θ of Σ, the corresponding polarity graph is the graph whose vertices are the points of Σ, and two distinct points p_1 and p_2 are adjacent if p_1 is incident to p_2^θ in Σ. A well-known example of a polarity graph is the Erdős-Rényi orthogonal polarity graph ER_q, which appears frequently in a variety of extremal problems. Eigenvalue methods provide an upper bound on the independence number of any polarity graph. Mubayi and Williford showed that in the case of ER_q, the eigenvalue method gives the correct upper bound in order of magnitude. We prove that this is also true for other families of polarity graphs. This includes a family of polarity graphs for which the polarity is neither orthogonal nor unitary. We conjecture that any polarity graph of a projective plane of order q has an independent set of size $\Omega(q^{3/2})$. Some related results are also obtained.

1 Introduction

The use of finite geometry to construct graphs with interesting properties has a rich history in graph theory. One of the most well-known constructions is due to Brown [6], and Erdős, Rényi, and Sós [11] who used an orthogonal polarity of a Desarguesian projective plane to give examples of graphs that give an asymptotically tight lower bound on the Turán number of the 4-cycle. Later these same graphs were used to solve other extremal problems in a variety of areas such as Ramsey theory [3], [15], hypergraph Turán theory [16], and even the Cops and Robbers game on graphs [5]. While our focus is not on the graphs of [6] and [11], we take a moment to define them. Let q be a power of a prime and $PG(2,q)$ be the projective geometry over the 3-dimensional vector space $\mathbb{F}_q^3$. We represent the points of $PG(2,q)$ as non-zero vectors (x_0,x_1,x_2) where $x_i \in \mathbb{F}_q$. Two vectors (x_0,x_1,x_2) and (y_0,y_1,y_2) are equivalent if $\lambda(x_0,x_1,x_2) = (y_0,y_1,y_2)$ for some $\lambda \in \mathbb{F}_q \backslash \{0\}$. The *Erdős-Rényi orthogonal polarity graph*, denoted ER_q, is the graph whose vertices are the points of $PG(2,q)$. Two distinct vertices (x_0,x_1,x_2) and (y_0,y_1,y_2) are adjacent if and only if $x_0y_0 + x_1y_1 + x_2y_2 = 0$.

The graph ER_q has been studied as an interesting graph in its own right. Parsons [19] determined the automorphism group of ER_q and obtained several other results. In particular, Parsons showed that for $q \equiv 1 (\mathrm{mod}\ 4)$, ER_q contains a $\frac{1}{2}(q+1)$-regular graph on $\binom{q}{2}$ vertices with girth 5. This construction gives one of the best known lower bounds

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on the maximum number of edges in an n-vertex graph with girth 5. It is still an open problem to determine this maximum, and for more on this problem, see [1]; especially their Conjecture 1.7 and the discussion preceding it. Bachratý and Širáň [4] reproved several of the results of [19] and we recommend [4] for a good introduction to the graph ER_q. They also used ER_q to construct vertex-transitive graphs with diameter two.

Along with the automorphism group of a graph, two other important graph parameters are the independence number and the chromatic number. At this time we transition to a more general setting as many of the upper bounds on the independence number of ER_q are true for a larger family of graphs. Let $\Sigma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane of order q. A bijection $\theta : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ is a *polarity* if $\theta(\mathcal{P}) = \mathcal{L}$, $\theta(\mathcal{L}) = \mathcal{P}$, θ^2 is the identity map, and $p\mathcal{I}l$ if and only if $l^\theta \mathcal{I} p^\theta$. A point $p \in \mathcal{P}$ is called an *absolute point* if $p\mathcal{I}p^\theta$. A classical result of Baer is that any polarity of a projective plane of order q has at least $q+1$ absolute points. A polarity with $q+1$ points is called an *orthogonal polarity*, and such polarities exist in the Desarguesian projective plane as well as in other non-Desarguesian planes. For more on polarities see [14], Chapter 12. Given a projective plane $\Sigma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ of order q and an orthogonal polarity θ, the corresponding *orthogonal polarity graph* $G(\Sigma, \theta)$ is the graph with vertex set $\mathcal{P}$ where two distinct vertices p_1 and p_2 are adjacent if and only if $p_1 \mathcal{I} p_2^\theta$. Let $G^\circ(\Sigma, \theta)$ be the graph obtained from $G(\Sigma, \theta)$ by adding loops to the absolute points of θ. The integer $q+1$ is an eigenvalue of $G^\circ(\Sigma, \theta)$ with multiplicity 1, and all other eigenvalues are $\sqrt{q}$ or $-\sqrt{q}$. A result of Hoffman [13] implies

$$\alpha(G^\circ(\Sigma, \theta)) \leq \frac{-(q^2+q+1)(-q^{1/2})}{q+1-\sqrt{q}} \tag{1}$$

which gives $\alpha(G(\Sigma, \theta)) \leq q^{3/2} + q^{1/2} + 1$. An improved estimate in the case that q is even was obtained by Hobart and Williford [12] using association schemes. They conjectured that the upper bound (1) can be improved to

$$\alpha(G(\Sigma, \theta)) \leq q(q^{1/2}+1) - 2(q^{1/2}-1)(q+q^{1/2})^{1/2}$$

but this is still open. Mubayi and Williford [18] showed that the upper bound (1) gives the correct order of magnitude for $\alpha(ER_q)$. One of the results of [18] is that

$$\alpha(ER_{q^2}) \geq \frac{1}{2}q^3 + \frac{1}{2}q^2 + 1 \tag{2}$$

whenever q is a power of an odd prime. Their construction can be adapted in a straightforward manner to obtain the following lower bound on the independence number of wider class of orthogonal polarity graphs which we introduce now and will be the focus of much of our investigations. We remark that the study of polarity graphs coming from non-Desarguesian planes was suggested in [4].

Let q be a power of an odd prime and $f(X) \in \mathbb{F}_q[X]$. The polynomial $f(X)$ is a *planar polynomial* if for each $a \in \mathbb{F}_q^*$, the map

$$x \mapsto f(x+a) - f(x)$$

is a bijection on $\mathbb{F}_q$. Planar polynomials were introduced by Dembowski and Ostrom [10] in their study of projective planes of order q that admit a collineation group of order

q^2. Given a planar polynomial $f(X) \in \mathbb{F}_q[X]$, one can construct a projective plane as follows. Let $\mathcal{P} = \{(x,y) : x,y \in \mathbb{F}_q\} \cup \{(x) : x \in \mathbb{F}_q\} \cup \{(\infty)\}$. For $a,b,c \in \mathbb{F}_q$, let

$$\begin{aligned}
[a,b] &= \{(x, f(x-a)+b) : x \in \mathbb{F}_q\} \cup \{(a)\}, \\
[c] &= \{(c,y) : y \in \mathbb{F}_q\} \cup \{(\infty)\}, \\
[\infty] &= \{(c) : c \in \mathbb{F}_q\} \cup \{(\infty)\}.
\end{aligned}$$

Let $\mathcal{L} = \{[a,b] : a,b \in \mathbb{F}_q\} \cup \{[c] : c \in \mathbb{F}_q\} \cup \{[\infty]\}$. Define Π_f to be the incidence structure whose points are $\mathcal{P}$, whose lines are $\mathcal{L}$, and incidence $\mathcal{I}$ is given by containment. When f is a planar polynomial, Π_f is a projective plane. For instance if $f(X) = X^2$ and q is any power of an odd prime, Π_f is isomorphic to the Desarguesian plane $PG(2,q)$. For other examples, see [7].

Assume that $f(X) \in \mathbb{F}_q[X]$ is a planar polynomial. The plane Π_f possesses an orthogonal polarity ω given by

$$\begin{gathered}
(\infty)^\omega = [\infty], \quad [\infty]^\omega = (\infty), \quad (c)^\omega = [-c], \quad [c]^\omega = (-c) \\
(x,y)^\omega = [-x,-y], \quad \text{and} \quad [a,b]^\omega = (-a,-b)
\end{gathered}$$

where $a,b,c \in \mathbb{F}_q$. We write G_f for the corresponding orthogonal polarity graph. This is the graph whose vertices are the points of Π_f and two distinct vertices p_1 and p_2 are adjacent in G_f if and only if p_1 is incident to p_2^ω in Π_f. For vertices of the form (x,y) the adjacency relation is easily described in terms of f. The distinct vertices (x_1,y_1) and (x_2,y_2) are adjacent if and only if

$$f(x_1+x_2) = y_1 + y_2.$$

Our first result is a generalization of (2) to orthogonal polarity graphs which need not come from a Desarguesian plane.

Theorem 1.1 *If q is a power of an odd prime and $f(X) \in \mathbb{F}_{q^2}[X]$ is a planar polynomial all of whose coefficients belong to the subfield $\mathbb{F}_q$, then*

$$\alpha(G_f) \geq \frac{1}{2} q^2 (q-1)$$

Even though we have the restriction that the coefficients of f belong to $\mathbb{F}_q$, many of the known examples of planar functions have this property. Most of the planar functions discussed in [7], including those that give rise to the famous Coulter-Matthews plane, satisfy our requirement.

It is still an open problem to determine an asymptotic formula for the independence number of ER_p for odd prime p. However, given the results of [18] and Theorem 1.1, it would be quite surprising to find an orthogonal polarity graph of a projective plane of order q whose independence number is $o(q^{3/2})$. We believe that the lower bound $\Omega(q^{3/2})$ is a property shared by all polarity graphs, including polarity graphs that come from polarities which are not orthogonal.

Conjecture 1.2 *If $G(\Sigma, \theta)$ is a polarity graph of a projective plane of order q, then*

$$\alpha(G(\Sigma, \theta)) = \Omega(q^{3/2}).$$

There are polarity graphs which are not orthogonal polarity graphs for which Conjecture 1.2 holds. If $G(\Sigma, \theta)$ is a polarity graph where θ is unitary and Σ has order q, then $\alpha(G(\Sigma, \theta)) \geq q^{3/2} + 1$. Indeed, the absolute points of any polarity graph form an independent set and a unitary polarity has $q^{3/2} + 1$ absolute points. In Section 3 we show that there is a polarity graph $G(\Sigma, \theta)$ where θ is neither orthogonal or unitary and Conjecture 1.2 holds.

Theorem 1.3 *Let p be an odd prime, $n \geq 1$ be an integer, and $q = p^{2n}$. There is a polarity graph $G(\Sigma, \theta)$ such that Σ has order q, θ is neither orthogonal nor unitary, and*

$$\alpha(G(\Sigma, \theta)) \geq \frac{1}{2} q(\sqrt{q} - 1).$$

In connection with Theorem 1.1 and Conjecture 1.2, we would like to mention the work of De Winter, Schillewaert, and Verstraëte [8] and Stinson [21]. In these papers the problem of finding large sets of points and lines such that there is no incidence between these sets is investigated. Finding an independent set in a polarity graph is related to this problem as an edge in a polarity graph corresponds to an incidence in the geometry. The difference is that when one finds an independent set in a polarity graph, choosing the points determines the lines. In [8] and [21], one can choose the points and lines independently.

As mentioned above, Conjecture 1.2 holds for unitary polarity graphs as the absolute points form an independent set. Mubayi and Williford [18] asked whether or not there is an independent set in the graph U_q of size $\Omega(q^{3/2})$ that contains no absolute points. For q a square of a prime power, the graph U_q has the same vertex set as ER_q and two distinct vertices (x_0, x_1, x_2) and (y_0, y_1, y_2) are adjacent if and only if $x_0 y_0^{\sqrt{q}} + x_1 y_1^{\sqrt{q}} + x_2 y_2^{\sqrt{q}} = 0$. We could not answer their question, but we were able to produce an independent set of size $\Omega(q^{5/4})$ that contains no absolute points. We remark that a lower bound of $\Omega(q)$ is trivial.

Theorem 1.4 *Let q be an even power of an odd prime. The graph U_q has an independent set I that contains no absolute points and*

$$|I| \geq 0.19239 q^{5/4} - O(q).$$

Related to the independence number is the chromatic number. In [20], it is shown that $\chi(ER_{q^2}) \leq 2q + O(\frac{q}{\log q})$ whenever q is a power of an odd prime. Here we prove that this upper bound holds for another family of orthogonal polarity graphs.

Definition 1.5 *Let p be an odd prime. Let n and s be positive integers such that $s < 2n$ and $\frac{2n}{s}$ is an odd integer. Let $d = p^s$ and $q = p^n$. We call the pair $\{q, d\}$ an* admissible pair*.*

If $\{q,d\}$ is an admissible pair, then the polynomial $f(X)=X^{d+1}\in\mathbb{F}_{q^2}[X]$ is a planar polynomial. For a nice proof, see Theorem 3.3 of [7].

Theorem 1.6 *Let q be a power of an odd prime and $\{q,d\}$ be an admissible pair. If $f(X)=X^{d+1}$, then*

$$\chi(G_f)\leq 2q+O\left(\frac{q}{\log q}\right).$$

The eigenvalue bound (1) gives a lower bound of $\chi(G_f)\geq\frac{q^4+q^2+1}{q^3+q+1}$ so that the leading term in the upper bound of Theorem 1.6 is best possible up to a constant factor. Not only does this bound imply that $\alpha(G_f)\geq\frac{1}{2}q^3-o(q^3)$, but shows that most of the vertices of G_f can be partitioned into large independent sets.

The technique that is used to prove Theorem 1.6 is the same as the one used in [20] and can be applied to other orthogonal polarity graphs. In Section 6, we sketch an argument that the bound of Theorem 1.6 also holds for a plane coming from a Dickson commutative division ring (see [14]). It is quite possible that the technique applies to more polarity graphs, but in order to obtain a general result, some new ideas will be needed. Furthermore, showing that every polarity graph of a projective plane of order q has chromatic number at most $O(\sqrt{q})$ is a significant strengthening of Conjecture 1.2. When p is prime, it is still unknown whether or not $\chi(ER_p)=O(\sqrt{p})$.

2 Proof of Theorem 1.1

Let q be a power of an odd prime and $f(X)\in\mathbb{F}_{q^2}[X]$ be a planar polynomial, all of whose coefficients are in the subfield $\mathbb{F}_q$. Let G_f be the orthogonal polarity graph whose construction is given before the statement of Theorem 1.1. Partition $\mathbb{F}_q^*$ into two sets $\mathbb{F}_q^+$ and $\mathbb{F}_q^-$ where $a\in\mathbb{F}_q^+$ if and only if $-a\in\mathbb{F}_q^-$. Let μ be a root of an irreducible quadratic over $\mathbb{F}_q$ and so $\mathbb{F}_{q^2}=\{a+\mu b: a,b\in\mathbb{F}_q\}$. Let

$$I=\{(x,y+z\mu):x,y\in\mathbb{F}_q,z\in\mathbb{F}_q^+\}.$$

Note that $|I|=\frac{1}{2}q^2(q-1)$ and we claim that I is an independent set. Suppose $(x_1,y_1+z_1\mu)$ and $(x_2,y_2+z_2\mu)$ are distinct vertices in I and that they are adjacent. Then

$$f(x_1+x_2)=y_1+y_2+(z_1+z_2)\mu. \tag{3}$$

The left-hand side of (3) belongs to $\mathbb{F}_q$ since the coefficients of f are in $\mathbb{F}_q$ and $x_1+x_2\in\mathbb{F}_q$. The right-hand side of (3) is not in $\mathbb{F}_q$ since $z_1+z_2\neq 0$. We have a contradiction so no two vertices in I are adjacent.

3 Proof of Theorem 1.3

Let p be an odd prime, $n\in\mathbb{N}$, and $q=p^{2n}$. Let $\{1,\lambda\}$ be a basis for a 2-dimensional vector space over $\mathbb{F}_q$. Let $\sigma:\mathbb{F}_q\to\mathbb{F}_q$ be the map $x^\sigma=x^{p^n}$. Observe that σ is a field automorphism of order 2, and the fixed elements of σ are precisely the elements of the subfield $\mathbb{F}_{p^n}$ in $\mathbb{F}_q$. Let θ be a generator of $\mathbb{F}_q^*$ which is the group of non-zero elements of $\mathbb{F}_q$ under multiplication. Let D be the division ring whose elements are $\{x + \lambda y : x, y \in \mathbb{F}_q\}$ where addition is done componentwise, and multiplication is given by the rule

$$(x + \lambda y) \cdot (z + \lambda t) = xz + \theta t y^\sigma + \lambda(yz + x^\sigma t).$$

Here we are following the presentation of [14]. Define the map $\alpha : D \to D$ by

$$(x + \lambda y)^\alpha = x + \lambda y^\sigma.$$

Let $\Pi_D = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be the plane coordinatized by D. That is,

$$\mathcal{P} = \{(x, y) : x, y \in D\} \cup \{(x) : x \in D\} \cup \{(\infty)\}$$

and

$$\mathcal{L} = \{[m, k] : m, k \in D\} \cup \{[m] : m \in D\} \cup \{[\infty]\}$$

where

$$\begin{aligned} [m, k] &= \{(x, y) : m \cdot x + y = k\} \cup \{(m)\}, \\ [k] &= \{(k, y) : y \in D\} \cup \{(\infty)\}, \\ [\infty] &= \{(m) : m \in D\} \cup \{(\infty)\}. \end{aligned}$$

The incidence relation $\mathcal{I}$ is containment. A polarity of Π_D is given by the map ω where

$$(\infty)^\omega = [\infty], \quad [\infty]^\omega = (\infty), \quad (m)^\omega = [m^\alpha], \quad [k]^\omega = (k^\alpha)$$

and

$$(x, y)^\omega = [x^\alpha, -y^\alpha], \qquad [m, k] = (m^\alpha, -k^\alpha).$$

The polarity ω has $|D|^{5/4}+1$ absolute points. Let $G(\Pi_D, \omega)$ be the corresponding polarity graph.

We now derive an algebraic condition for when the distinct vertices

$$u = (x_1 + \lambda x_2, y_1 + \lambda y_2) \text{ and } v = (z_1 + \lambda z_2, t_1 + \lambda t_2)$$

are adjacent. The vertex u is adjacent to v if and only if $u \mathcal{I} v^\omega$. This is equivalent to

$$(x_1 + \lambda x_2, y_1 + \lambda y_2) \mathcal{I} [z_1 + \lambda(z_2)^\sigma, -t_1 + \lambda(-(t_2)^\sigma)]$$

which in turn, is equivalent to

$$(z_1 + \lambda(z_2)^\sigma) \cdot (x_1 + \lambda x_2) = -y_1 - t_1 + \lambda(-y_2 - (t_2)^\sigma). \tag{4}$$

Using the definition of multiplication in D, (4) can be rewritten as

$$z_1 x_1 + \theta x_2 z_2 + \lambda((z_2)^\sigma x_1 + (z_1)^\sigma x_2) = -y_1 - t_1 + \lambda(-y_2 - (t_2)^\sigma).$$

This gives the pair of equations

$$x_1 z_1 + \theta x_2 z_2 = -y_1 - t_1 \tag{5}$$

and

$$x_1(z_2)^\sigma + x_2(z_1)^\sigma = -y_2 - (t_2)^\sigma.$$

Let $\Box_q$ be the set of nonzero squares in $\mathbb{F}_q$. Note that any element of $\mathbb{F}_{p^n}$ is a square in $\mathbb{F}_q$. Define

$$I = \{(x_1 + \lambda x_2, y_1 + \lambda y_2) : x_1, y_1 \in \mathbb{F}_{p^n}, x_2 \in \Box_q, y_2 \in \mathbb{F}_q\}.$$

Then $|I| = \frac{1}{2}(q-1)q(p^n)^2 = \frac{1}{2}q^2(q-1)$. We now show that I is an independent set. Suppose that $(x_1 + x_2\lambda, y_1 + y_2\lambda)$ and $(z_1 + z_2\lambda, t_1 + t_2\lambda)$ are distinct vertices in I that are adjacent. Then (5) holds so

$$\theta = (x_2 z_2)^{-1}(-y_1 - t_1 - x_1 z_1). \tag{6}$$

The left hand side of (6) is not a square in $\mathbb{F}_q$. Since x_2 and z_2 belong to $\Box_q$, we have that $(x_2 z_2)^{-1}$ is a square in $\mathbb{F}_q$. Since $y_1, t_1, x_1, z_1 \in \mathbb{F}_{p^n}$, we have that $-y_1 - t_1 - x_1 z_1$ is in $\mathbb{F}_{p^n}$ and thus is a square in $\mathbb{F}_q$. We conclude that the right hand side of (6) is a square. This is a contradiction and so I must be an independent set. This shows that

$$\alpha(G(\Pi_D, \omega)) \geq \frac{1}{2}q^2(q-1).$$

4 Proof of Theorem 1.4

Let p be an odd prime, $n \in \mathbb{N}$, and $q = p^{2n}$. Let θ be a generator of $\mathbb{F}_q^*$. The field $\mathbb{F}_q$ contains a subfield with $\sqrt{q}$ elements and we write $\mathbb{F}_{\sqrt{q}}$ for this subfield. We will use the fact that $x^{\sqrt{q}} = x$ if and only if $x \in \mathbb{F}_{\sqrt{q}}$ and that the characteristic of $\mathbb{F}_q$ is a divisor of $\sqrt{q}$ without explicitly saying so.

Let U_q be the graph whose vertex set is $V(ER_q)$ and two vertices (x_0, x_1, x_2) and (y_0, y_1, y_2) are adjacent if and only if

$$x_0 y_0^{\sqrt{q}} + x_1 y_1^{\sqrt{q}} + x_2 y_2^{\sqrt{q}} = 0.$$

In [18], it is shown that U_q has an independent set J of size $q^3 + 1$. This independent set consists of the absolute points in U_q; namely

$$J = \{(x_0, x_1, x_2) : x_0^{\sqrt{q}+1} + x_1^{\sqrt{q}+1} + x_2^{\sqrt{q}+1} = 0\}.$$

To find an independent set in U_q with no absolute points and size $\Omega(q^{5/4})$, we will work with a graph that is isomorphic to U_q. Let U_q^* be the graph whose vertex set is $V(ER_q)$ where (x_0, x_1, x_2) and (y_0, y_1, y_2) are adjacent if and only if

$$x_0 y_2^{\sqrt{q}} + x_2 y_0^{\sqrt{q}} = x_1 y_1^{\sqrt{q}}.$$

The proof of Proposition 3 of [18] is easily adapted to prove the following.

Lemma 4.1 *The graph U_q is isomorphic to the graph U_q^*.*

For any $\mu \in \mathbb{F}_q \backslash \mathbb{F}_{\sqrt{q}}$, we have $\mathbb{F}_q = \{a + b\mu : a, b \in \mathbb{F}_{\sqrt{q}}\}$. The next lemma shows that we can find a μ that makes many of our calculations significantly easier.

Lemma 4.2 *There is a $\mu \in \mathbb{F}_q \backslash \mathbb{F}_{\sqrt{q}}$ such that $\mu^{\sqrt{q}} + \mu = 0$.*

Proof. Let $\mu = \theta^{\frac{1}{2}(\sqrt{q}+1)}$. Since $\mathbb{F}^*_{\sqrt{q}} = \langle \theta^{\sqrt{q}+1} \rangle$, we have that $\mu \notin \mathbb{F}_{\sqrt{q}}$. Using the fact that $-1 = \theta^{\frac{1}{2}(q-1)}$, we find that

$$\begin{aligned} \mu^{\sqrt{q}} + \mu &= \theta^{\frac{1}{2}\sqrt{q}(\sqrt{q}+1)} + \theta^{\frac{1}{2}(\sqrt{q}+1)} = \theta^{\frac{1}{2}\sqrt{q}(\sqrt{q}+1)} - \theta^{\frac{1}{2}(q-1)+\frac{1}{2}(\sqrt{q}+1)} \\ &= \theta^{\frac{1}{2}(q+\sqrt{q})} - \theta^{\frac{1}{2}(q+\sqrt{q})} = 0. \end{aligned}$$

■

For the rest of this section we fix a $\mu \in \mathbb{F}_q \backslash \mathbb{F}_{\sqrt{q}}$ that satisfies the statement of Lemma 4.2. Given $c \in \mathbb{F}_{\sqrt{q}}$, define

$$X_c = \{(1, a, b + c\mu) : a, b \in \mathbb{F}_{\sqrt{q}}\}.$$

Lemma 4.3 *If c_1 and c_2 are elements of $\mathbb{F}_{\sqrt{q}}$ with $c_1 \neq c_2$, then the graph U^*_q has no edge with one endpoint in X_{c_1} and the other in X_{c_2}.*

Proof. Suppose that $(1, a_1, b_1 + c_1\mu)$ is adjacent to $(1, a_2, b_2 + c_2\mu)$ where $a_i, b_i, c_i \in \mathbb{F}_{\sqrt{q}}$. By definition of adjacency in U^*_q,

$$b_2 + c_2\mu^{\sqrt{q}} + b_1 + c_1\mu = a_1a_2.$$

By Lemma 4.2, this can be rewritten as

$$(c_1 - c_2)\mu = a_1a_2 - b_1 - b_2. \tag{7}$$

The right hand side of (7) belongs to the subfield $\mathbb{F}_{\sqrt{q}}$. Therefore, $c_1 - c_2 = 0$ since $\mu \notin \mathbb{F}_{\sqrt{q}}$. ■

Now we consider the subgraph $U^*_q[X_c]$ where $c \in \mathbb{F}_{\sqrt{q}}$. The vertex set of $U^*_q[X_c]$ is

$$\{(1, a, b + c\mu) : a, b \in \mathbb{F}_{\sqrt{q}}\}$$

and two vertices $(1, a_1, b_1 + c\mu)$ and $(1, a_2, b_2 + c\mu)$ are adjacent if and only if

$$b_2 + c(\mu^{\sqrt{q}} + \mu) + b_1 = a_1a_2.$$

By Lemma 4.2, this is equivalent to

$$b_2 - a_1a_2 + b_1 = 0. \tag{8}$$

Let $ER^*_{\sqrt{q}}$ be the graph whose vertex set is $V(ER_{\sqrt{q}})$ and (x_0, x_1, x_2) is adjacent to (y_0, y_1, y_2) if and only if

$$x_0y_2 - x_1y_1 + x_2y_0 = 0.$$

Proposition 3 of [18] shows that $ER^*_{\sqrt{q}}$ is isomorphic to $ER_{\sqrt{q}}$. It follows from (8) that the graph $U_q^*[X_c]$ is isomorphic to the subgraph of $ER^*_{\sqrt{q}}$ induced by $\{(1,x_1,x_2) : x_1, x_2 \in \mathbb{F}_{\sqrt{q}}\}$. Note that $ER^*_{\sqrt{q}}$ has exactly $\sqrt{q}+1$ vertices more than $U_q^*[X_c]$. By Theorem 5 of [18], we can find an independent set in $U_q^*[X_c]$ with at least $.19239q^{3/4} - q^{1/2} - 1$ vertices. Call this independent set I_c.

We want to throw away the absolute points in U_q^* that are in I_c. In $U_q^*[X_c]$, the vertex $(1, a, b + c\mu)$ is an absolute point if and only if

$$b + c\mu + b + c\mu^{\sqrt{q}} = a^2$$

which, again by Lemma 4.2, is equivalent to

$$2b = a^2.$$

There are $\sqrt{q}$ choices for a and a given a uniquely determines b. Thus I_c contains at most $q^{1/2}$ absolute points in U_q. Let J_c be the set I_c with the absolute points removed so that $|J_c| \geq .19239q^{3/4} - 2q^{1/2} - 1$.

Define

$$I = \bigcup_{c \in \mathbb{F}_{\sqrt{q}}} J_c.$$

By Lemma 4.3, I is an independent set in U_q^*. Observe that

$$|I| \geq \sqrt{q}(0.19239q^{3/4} - 2q^{1/2} - 1) = 0.19239q^{5/4} - O(q)$$

and I contains no absolute points.

We note that when q is a fourth power, the coefficient 0.19239 may be raised to $\frac{1}{2}$, as Theorem 5 in [18] is stronger in this case.

5 Proof of Theorem 1.6

Let s and n be positive integers with $\frac{2n}{s} = r \geq 3$ an odd integer. Let $q = p^n$, $d = p^s$, and note that $\{q, d\}$ is an admissible pair. Let $\mathbb{F}_{q^2}^*$ be the non-zero elements of $\mathbb{F}_{q^2}$ and θ be a generator of the cyclic group $\mathbb{F}_{q^2}^*$. Write $\mathbb{F}_q$ and $\mathbb{F}_d$ for the unique subfields of $\mathbb{F}_{q^2}$ of order q and d, respectively. An identity that will be used is

$$\frac{p^{2n}-1}{p^s-1} = (p^s)^{r-1} + (p^s)^{r-2} + \cdots + p^s + 1.$$

It will be convenient to let

$$t = \frac{p^{2n}-1}{p^s-1} \tag{9}$$

and observe that t is odd since $r = \frac{2n}{s}$ is odd.

Lemma 5.1 *There is a $\mu \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ such that when μ^d is written in the form $\mu^d = u_1 + u_2\mu$ with $u_1, u_2 \in \mathbb{F}_q$, the element u_2 is a $(d-1)$-th power.*

Proof. Let $h(X) = X^d + (\theta^{q+1})^{d-1}X$. We claim that the roots of h are the elements in the set $Z = \{0\} \cup \{\theta^{q+1+it} : 0 \le i \le d-2\}$. Clearly 0 is a root. Let $0 \le i \le d-2$. Note that since $2n = sr$,

$$\begin{aligned} dt &\equiv p^s\left((p^s)^{r-1} + (p^s)^{r-2} + \cdots + p^s + 1\right) \\ &\equiv p^{2n} + (p^s)^{r-1} + \cdots + p^{2s} + p^s \\ &\equiv 1 + (p^s)^{r-1} + \cdots + p^{2s} + p^s \equiv t (\text{mod } p^{2n} - 1). \end{aligned}$$

This implies $d(q+1+it) \equiv (q+1)(d-1) + (q+1) + it (\text{mod } q^2 - 1)$ so that

$$(\theta^{q+1+it})^d - (\theta^{q+1})^{d-1}\theta^{q+1+it} = 0.$$

We conclude that the roots of h are the elements in Z.

Let $\mu = \theta^{q+1+t}$. The non-zero elements of the subfield $\mathbb{F}_q$ are the elements of the subgroup $\langle \theta^{q+1} \rangle$ in $\mathbb{F}_{q^2}^*$. Since t is odd and $q+1$ is even, $q+1+t$ is not divisible by $q+1$ thus $\mu \notin \mathbb{F}_q$. Let $u_2 = (\theta^{q+1})^{d-1}$ and $u_1 = 0$. We have

$$0 = h(\mu) = \mu^d - (\theta^{q+1})^{d-1}\mu = \mu^d - u_1 - u_2\mu$$

so $\mu^d = u_1 + u_2\mu$. By construction, $\mu \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$, $\mu^d = u_1 + u_2\mu$ with $u_1, u_2 \in \mathbb{F}_q$, and u_2 is a $(d-1)$-th power. ■

The next lemma is known (see Exercise 7.4 in [17]). A proof is included for completeness.

Lemma 5.2 *If $u_2, \delta \in \mathbb{F}_{q^2}^*$ and u_2 is a $(d-1)$-th power, then for any $\xi \in \mathbb{F}_{q^2}$, the equation*

$$X^d + u_2\delta^{d-1}X = \xi$$

has a unique solution in $\mathbb{F}_{q^2}$.

Proof. Let $u_2, \delta \in \mathbb{F}_{q^2}^*$ and $g(X) = X^d + u_2\delta^{d-1}X$. The polynomial g is a permutation polynomial if and only if the only root of g is 0 (see Theorem 7.9 of [17]). If $g(X) = 0$, then $X(X^{d-1} + u_2\delta^{d-1}) = 0$. It suffices to show that $-u_2\delta^{d-1}$ is not a $(d-1)$-th power of any element of $\mathbb{F}_{q^2}$ as this would imply that the equation $X^{d-1} + u_2\delta^{d-1} = 0$ has no solutions. By hypothesis, $u_2 = w^{d-1}$ for some $w \in \mathbb{F}_{q^2}^*$. Since -1 is not a $(d-1)$-th power, the product $-u_2\delta^{d-1} = -(w\delta)^{d-1}$ is not a $(d-1)$-th power. We conclude that g is a permutation polynomial on $\mathbb{F}_{q^2}$. In particular, given any $\xi \in \mathbb{F}_{q^2}$, there is a unique solution to the equation $X^d + u_2\delta^{d-1}X = \xi$. ■

For the rest of this section, we fix a $\mu \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ that satisfies the conclusion of Lemma 5.1; that is,

$$\mu^d = u_1 + u_2\mu$$

where $u_1, u_2 \in \mathbb{F}_q$ and u_2 is a $(d-1)$-th power in $\mathbb{F}_{q^2}$. Let

$$\mu^{d+1} = w_1 + w_2\mu$$

where $w_1, w_2 \in \mathbb{F}_q$. We fix a partition of $\mathbb{F}_q^*$ into two sets

$$\mathbb{F}_q^* = \mathbb{F}_q^+ \cup \mathbb{F}_q^- \tag{10}$$

where $a \in \mathbb{F}_q^+$ if and only if $-a \in \mathbb{F}_q^-$.

It will be convenient to work with a graph that is isomorphic to a large induced subgraph of G_f. By Lemma 5.2, the map $x \mapsto x^d + x$ is a permutation on $\mathbb{F}_{q^2}$. Therefore, every element of $\mathbb{F}_{q^2}$ can be written in the form $a^d + a$ for some $a \in \mathbb{F}_{q^2}$ and this representation is unique. Let $\mathcal{A}_{q^2,d}$ be the graph with vertex set $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ where distinct vertices $(a^d + a, x)$ and $(b^d + b, y)$ are adjacent if and only if

$$a^d b + ab^d = x + y.$$

Working with this equation defining our adjacencies will be particularly helpful for the rather technical Lemma 5.8 below.

Lemma 5.3 *The graph $\mathcal{A}_{q^2,d}$ is isomorphic to the subgraph of G_f induced by $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$.*

Proof. One easily verifies that the map $\tau : V(\mathcal{A}_{q^2,d}) \to \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ defined by

$$\tau((a^d + a, x)) = (a, x + a^{d+1})$$

is a graph isomorphism from $\mathcal{A}_{q^2,d}$ to the subgraph of G_f induced by $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$. ∎

Lemma 5.4 *If*

$$I^+ = \{(a^d + a, x_1 + x_2\mu) : a, x_1 \in \mathbb{F}_q, x_2 \in \mathbb{F}_q^+\},$$

then I^+ is an independent set in the graph $\mathcal{A}_{q^2,d}$. The same statement holds with I^- and $\mathbb{F}_q^-$ in place of I^+ and $\mathbb{F}_q^+$, respectively.

Proof. Suppose that $(a^d + a, x_1 + x_2\mu)$ is adjacent to $(b^d + b, y_1 + y_2\mu)$ where $a, b \in \mathbb{F}_q$. The left hand side of

$$a^d b + ab^d = (x_1 + y_1) + (x_2 + y_2)\mu$$

is in $\mathbb{F}_q$ so $x_2 + y_2 = 0$. If $x_2, y_2 \in \mathbb{F}_q^+$, then $x_2 + y_2 \neq 0$ and so no two vertices in I^+ can be adjacent. Similarly, no two vertices in I^- can be adjacent. ∎

Lemma 5.5 *For any $k = \alpha^d + \alpha \in \mathbb{F}_{q^2}$, the map*

$$\phi_k((a^d + a, x)) = (a^d + a + k, x + a^d\alpha + a\alpha^d + \alpha^{d+1})$$

is an automorphism of the graph $\mathcal{A}_{q^2,d}$.

Proof. The vertex $\phi_k((a^d + a, x))$ is adjacent to $\phi_k((b^d + b, y))$ if and only if

$$v = (a^d + a + \alpha^d + \alpha, x + a^d\alpha + a\alpha^d + \alpha^{d+1})$$

is adjacent to

$$w = (b^d + b + \alpha^d + \alpha, y + b^d\alpha + b\alpha^d + \alpha^{d+1}).$$

Since d is a power of p, $v = ((a+\alpha)^d + (a+\alpha), x + a^d\alpha + a\alpha^d + \alpha^{d+1})$. Similarly,

$$w = ((b+\alpha)^d + (b+\alpha), y + b^d\alpha + b\alpha^d + \alpha^{d+1}).$$

From this we see that v is adjacent to w if and only if

$$\begin{array}{c} (a+\alpha)^d(b+\alpha) + (a+\alpha)(b+\alpha)^d = \\ x + a^d\alpha + a\alpha^d + \alpha^{d+1} + y + b^d\alpha + b\alpha^d + \alpha^{d+1}. \end{array} \tag{11}$$

A routine calculation shows that (11) is equivalent to the equation $a^d b + ab^d = x + y$ which holds if and only if $(a^d + a, x)$ is adjacent to $(b^d + b, y)$ in $\mathcal{A}_{q^2,d}$. ■

Let $J = I^+ \cup I^-$ and observe that $J = \{(a^d + a, x_1 + x_2\mu) : a, x_1 \in \mathbb{F}_q, x_2 \in \mathbb{F}_q^*\}$. Let

$$K = \bigcup_{\beta \in \mathbb{F}_q} \phi_{(\beta\mu)^d + (\beta\mu)}(J).$$

Lemma 5.6 *If $\mathcal{A}_{q^2,d}[K]$ is the subgraph of $\mathcal{A}_{q^2,d}$ induced by K, then*

$$\chi(\mathcal{A}_{q^2,d}[K]) \leq 2q.$$

Proof. By Lemma 5.4, the vertices in J may be colored using at most 2 colors. By Lemma 5.5, the vertices in $\phi_k(J)$ can also be colored using at most 2 colors. Since K is the union of q sets of the form $\phi_k(J)$ where $k \in \mathbb{F}_{q^2}$, we may color K using at most $2q$ colors. ■

Lemma 5.6 shows that we can color all but at most $O(q^3)$ vertices of $\mathcal{A}_{q^2,d}$ with at most $2q$ colors. We now show that the remaining vertices can be colored with $o(q)$ colors. Before stating the next lemma we recall that $\mu^d = u_1 + u_2\mu$ and we let $\mu^{d+1} = w_1 + w_2\mu$ where $u_1, u_2, w_1, w_2 \in \mathbb{F}_q$.

Lemma 5.7 *If $X = (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}) \backslash K$, then*

$$X = \{((a+\beta\mu)^d + (a+\beta\mu), t_1 + (a^d\beta + a\beta^d u_2 + \beta^{d+1}w_2)\mu) : a, \beta, t_1 \in \mathbb{F}_q\}.$$

Proof. For any $\beta \in \mathbb{F}_q$, the set $\phi_{(\beta\mu)^d + (\beta\mu)}(J)$ can be written as

$$\{(a^d + a + (\beta\mu)^d + (\beta\mu), x_1 + x_2\mu + a^d(\beta\mu) + a(\beta\mu)^d + (\beta\mu)^{d+1}) : a, x_1 \in \mathbb{F}_q, x_2 \in \mathbb{F}_q^*\}.$$

Let $(s_1 + s_2\mu, t_1 + t_2\mu) \in \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ where $s_1, s_2, t_1, t_2 \in \mathbb{F}_q$. The vertex $(s_1 + s_2\mu, t_1 + t_2\mu)$ is in K if we can find $a, x_1, \beta \in \mathbb{F}_q$ and $x_2 \in \mathbb{F}_q^*$ such that

$$s_1 + s_2\mu = (a+\beta\mu)^d + a + \beta\mu, \tag{12}$$
$$t_1 + t_2\mu = x_1 + x_2\mu + a^d(\beta\mu) + a(\beta\mu)^d + (\beta\mu)^{d+1}. \tag{13}$$

Since every element of $\mathbb{F}_{q^2}$ can be written as $z^d + z$ for some $z \in \mathbb{F}_{q^2}$, we can write $s_1 + s_2\mu = z^d + z$ and then choose a and β in $\mathbb{F}_q$ so that $z = a + \beta\mu$. With this choice of a and β, equation (12) holds.

Since $\mu^{d+1} = w_1 + w_2\mu$, equation (13) can be rewritten as

$$t_1 + t_2\mu = (x_1 + a\beta^d u_1 + \beta^{d+1} w_1) + (x_2 + a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2)\mu. \tag{14}$$

Let $x_1 = t_1 - a\beta^d u_1 - \beta^{d+1} w_1$. If $t_2 \neq a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2$, then we can take $x_2 = t_2 - a^d\beta - a\beta^d u_2 - \beta^{d+1} w_2$ and (13) holds. Therefore, the vertices in $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ not in K are those vertices in the set

$$\{((a + \beta\mu)^d + (a + \beta\mu), t_1 + (a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2)\mu) : a, \beta, t_1 \in \mathbb{F}_q\}.$$

∎

Lemma 5.8 *If $\mathcal{A}_{q^2,d}[X]$ is the subgraph of $\mathcal{A}_{q^2,d}$ induced by $X = (\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}) \backslash K$, then*

$$\chi(\mathcal{A}_{q^2,d}[X]) = O\left(\frac{q}{\log q}\right).$$

Proof. For $\beta \in \mathbb{F}_q$, partition X into the sets X_β where

$$X_\beta = \{((a + \beta\mu)^d + (a + \beta\mu), t_1 + (a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2)\mu) : a, t_1 \in \mathbb{F}_q\}.$$

Fix a $\beta \in \mathbb{F}_q$ and a vertex

$$v = ((a + \beta\mu)^d + (a + \beta\mu), t_1 + (a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2)\mu)$$

in X_β. Let $\gamma \in \mathbb{F}_q$. We want to count the number of vertices

$$w = ((x + \gamma\mu)^d + (x + \gamma\mu), y_1 + (x^d\gamma + x\gamma^d u_2 + \gamma^{d+1} w_2)\mu)$$

in X_γ that are adjacent to v. The vertices v and w are adjacent if and only if

$$\begin{aligned} &(a + \beta\mu)^d(x + \gamma\mu) + (a + \beta\mu)(x + \gamma\mu)^d \\ = t_1 + y_1 + (a^d\beta + a\beta^d u_2 &+ \beta^{d+1} w_2 + x^d\gamma + x\gamma^d u_2 + \gamma^{d+1} w_2)\mu. \end{aligned} \tag{15}$$

If $\gamma = \beta$, then we can choose $x \in \mathbb{F}_q$ in q different ways and the above equation uniquely determines y_1. We conclude that the vertex $v \in X_\beta$ has at most q other neighbors in X_β.

Assume now that $\gamma \neq \beta$. We need to count how many $x, y_1 \in \mathbb{F}_q$ satisfy (15). A computation using the relations $\mu^d = u_1 + u_2\mu$ and $\mu^{d+1} = w_1 + w_2\mu$ shows that (15) is equivalent to

$$\begin{aligned} a^d x + a^d\gamma\mu + \beta^d x(u_1 + u_2\mu) + \beta^d\gamma(w_1 + w_2\mu) \\ + ax^d + a\gamma^d(u_1 + u_2\mu) + \beta x^d\mu + \beta\gamma^d(w_1 + w_2\mu) \\ = t_1 + y_1 + (a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2 + x^d\gamma + x\gamma^d u_2 + \gamma^{d+1} w_2)\mu. \end{aligned}$$

Equating the coefficients of μ gives

$$a^d\gamma + \beta^d x u_2 + \beta^d\gamma w_2 + a\gamma^d u_2 + \beta x^d + \beta\gamma^d w_2 = a^d\beta + a\beta^d u_2 + \beta^{d+1} w_2 + x^d\gamma + x\gamma^d u_2 + \gamma^{d+1} w_2.$$

This equation can be rewritten as

$$x^d(\gamma-\beta)+x(\gamma^d-\beta^d)u_2=\xi \tag{16}$$

for some $\xi \in \mathbb{F}_q$ that depends only on a, γ, β, and μ. Since $\gamma-\beta \neq 0$, equation (16) is equivalent to

$$x^d+u_2(\gamma-\beta)^{d-1}x=\xi(\gamma-\beta)^{-1}. \tag{17}$$

By Lemma 5.2, (17) has a unique solution for x since u_2 is a $(d-1)$-power and $\gamma-\beta \in \mathbb{F}_q^*$. Once x is determined, (15) gives a unique solution for y_1. Therefore, v has at most one neighbor in X_β. We conclude that the degree of v in X is at most $q+(q-1)<2q$.

The graph $\mathcal{A}_{q^2,d}[X]$ does not contain a 4-cycle and has maximum degree at most $2q$. This implies that the neighborhood of any vertex contains at most q edges. By a result of Alon, Krivelevich, and Sudakov [2], the graph $\mathcal{A}_{q^2,d}[X]$ can be colored using $O\left(\frac{q}{\log q}\right)$ colors. ∎

Proof of Theorem 1.6. Partition the vertex set of $\mathcal{A}_{q^2,d}$ as

$$V(\mathcal{A}_{q^2,d})=K\cup X.$$

By Lemmas 5.6 and 5.8, we can color the vertices in $K\cup X$ using $2q+O\left(\frac{q}{\log q}\right)$ colors. This gives a coloring of the vertices in $\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$ in G_f and it only remains to color the vertices in the set $\{(m): m\in\mathbb{F}_{q^2}\}\cup\{(\infty)\}$.

The vertex (∞) is adjacent to (m) for every $m\in\mathbb{F}_{q^2}$. Since G_f is C_4-free, the subgraph of G_f induced by the neighborhood of (∞) induces a a graph with maximum degree at most 1. We may color the vertices in $\{(m): m\in\mathbb{F}_{q^2}\}\cup\{(\infty)\}$ using three new colors not used to color $\mathbb{F}_{q^2}\times\mathbb{F}_{q^2}$ to obtain a $2q+O(\frac{q}{\log q})$ coloring of G_f. ∎

6 Dickson Commutative Division Rings

Let p be an odd prime, $n>1$ be an integer, $q=p^n$, and a be any element of $\mathbb{F}_q$ that is not a square. Let $1\le r<n$ be an integer. Let D be a 2-dimensional vector space over $\mathbb{F}_q$ with basis $\{1,\lambda\}$. Define a product $\cdot$ on D by the rule

$$(x+\lambda y)\cdot(z+\lambda t)=xz+ay^{p^r}t^{p^r}+\lambda(yz+st).$$

With this product and the usual addition, D is a commutative division ring (see [14], Theorem 9.12 and note that it is common to call such a structure a semifield). We can use D to define a projective plane Π (see [14], Theorem 5.2). This plane also has an orthogonal polarity (see [14], page 248). Let $\mathcal{GD}_{q^2}$ be the corresponding orthogonal polarity graph. Using the argument of Section 5, one can prove that

$$\chi(\mathcal{GD}_{q^2})\le 2q+\left(\frac{q}{\log q}\right).$$

A rough outline is as follows. Let $\mathcal{AD}_{q^2}$ be the subgraph of $\mathcal{GD}_{q^2}$ induced by the vertices

$$\{((x_1+\lambda x_2,y_1+\lambda y_2): x_1,x_2,y_1,y_2\in\mathbb{F}_q\}.$$

Partition $\mathbb{F}_q^*$ into the sets $\mathbb{F}_q^+$ and $\mathbb{F}_q^-$ where $a \in \mathbb{F}_q^+$ if and only if $-a \in \mathbb{F}_q^-$. The sets

$$I^+ = \{(x_2\lambda, y_1 + y_2\lambda) : x_2, y_1 \in \mathbb{F}_q, y_2 \in \mathbb{F}_q^+\}$$

and

$$I^- = \{(x_2\lambda, y_1 + y_2\lambda) : x_2, y_1 \in \mathbb{F}_q, y_2 \in \mathbb{F}_q^-\}$$

are independent sets in $\mathcal{GD}_{q^2}$. For any $k \in \mathbb{F}_q$, the map

$$\phi_k(x_1 + \lambda x_2, y_1 + \lambda y_2) = (x_1 + \lambda x_2 + k, y_1 + \lambda y_2 + kx_1 + 2^{-1}k^2 + \lambda x_2 k)$$

is an automorphism of $\mathcal{AG}_{q^2}$.

Let $J = I^+ \cup I^-$ and $K = \bigcup_{k \in \mathbb{F}_q} \phi_k(J)$ and observe that

$$K = \{(k + x_2\lambda, y_1 + y_2\lambda + 2^{-1}k^2 + \lambda x_2 k) : x_1, y_1, k \in \mathbb{F}_q, y_2 \in \mathbb{F}_q^*\}.$$

If $X = (D \times D)\backslash K$, then

$$X = \{(s_1 + s_2\lambda, t_1 + (s_2 s_1)\lambda) : s_1, s_2, t_1 \in \mathbb{F}_q\}.$$

It can then be shown that the subgraph of $\mathcal{GD}_{q^2}$ induced by X has maximum degree at most $2q$. The remaining details are left to the reader.

7 Concluding Remark

The argument used to prove Theorem 1.4 can be extended to other unitary polarity graphs. We illustrate with an example. Let a and e be integers with $a \not\equiv \pm(\text{mod } 2e)$, $e \equiv 0(\text{mod } 4)$, and $\gcd(a, e) = 1$. Let $f : \mathbb{F}_q \to \mathbb{F}_q$ be the polynomial $f(X) = X^n$ where $n = \frac{1}{2}(3^a + 1)$ and $q = 3^e$. The map f is a planar polynomial and the corresponding plane is the Coulter-Matthews plane [7]. This plane has a unitary polarity whose action on the affine points and lines is given by

$$(x, y)^\theta = [-x^{\sqrt{q}}, -y^{\sqrt{q}}] \quad \text{and} \quad (a, b)^\theta = (-a^{\sqrt{q}}, -b^{\sqrt{q}}).$$

The proof of Theorem 1.4 can be modified to show that the corresponding unitary polarity graph has an independent set of size $\frac{1}{2}q^{5/4} - o(q^{5/4})$ that contains no absolute points. The reason for the condition $e \equiv 0(\text{mod } 4)$ instead of $e \equiv 0(\text{mod } 2)$, which is the condition given in [7] for f to be planar, is that we need $\sqrt{q}$ to be a square in order to apply Theorem 1.1 to the subgraphs that correspond to the $U_q^*[X_c]$ in the proof of Theorem 1.4.

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