Motivation

Recall

Thm: (Bondy, Simonovits) $ex(n, C_{2k}) \leq c_k n^{1+\frac{1}{k}}$

this gives an upper bound.

Our goal: Construct a graph, $H_k(p)$, which has $2p^k$ vertices, p^{k+1} edges, and no C_{2k}. for $2k = 4, 6, 10$ and p a prime.

note:
$$(\#\text{ vtxs})^{1+\frac{1}{k}} = 2^{1+\frac{1}{k}} \cdot (p^k)^{\frac{k+1}{k}} = c_k p^{k+1}$$

Construction: (BPT)

vtxs: Let the vertex set be $A \cup B$

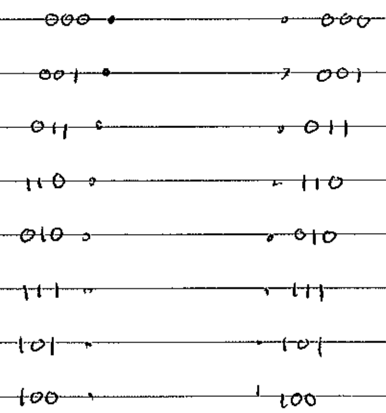
each vertex $a \in A$ gets a unique label $(a_0, a_1, \ldots, a_{k-1})$

each vertex $b \in B$ gets a unique label $(b_0, b_1, \ldots, b_{k-1})$

where $a_j, b_j \in \{0, \ldots, p-1\}$

so each side has p^k vertices. ✓

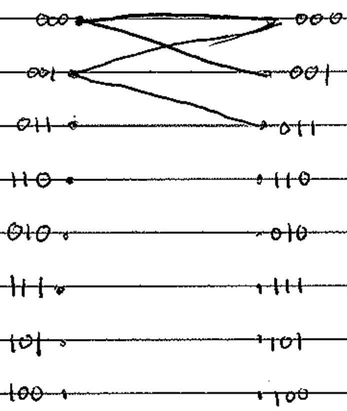
eg: $p = 2$, $k = 3$



(Of course the ordering of the vertices in this picture is arbitrary, but this order makes the picture pretty later) edges: draw an edge from $(a_0, a_1, \dots, a_{k-1})$ to $(b_0, \dots, b_{k-1})$ iff

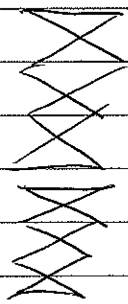
$$b_j \equiv a_j + a_{j+1} b_{k-1} \pmod p \qquad \text{for } j = 0, \dots, k-2$$

eg:



$b_0 \equiv a_0 + a_1 b_2$
$b_1 \equiv a_1 + a_2 b_2$

etc.



note: if we fix a and b_{k-1}, we can read off the other digits of b. Since there are p choices for b_{k-1}, a has [redacted] degree p, so $|E| = p \cdot |A| = p^{k+1}$ ✓

edges ...

Now we must show that $H_k(p)$ avoids C_{2k}.

lemma: If $H_k(p)$ contains a cycle of length $2k$, $\Theta = (a^{(0)}, b^{(0)}, a^{(1)}, b^{(1)}, \dots, a^{(k-1)}, b^{(k-1)})$ then for each $b \in \Theta$ there is some $b' \in \Theta$ such that $b_{k-1} = b'_{k-1}$. $(b, b' \in B,\ b \neq b')$

(ie, the final digits of the bs are not unique.
ie, if you divided the bs into equivalence classes by their last digit, none would be alone)

Lemma 2: if $b, b' \in B$ are both adjacent to $a \in A$, then $b_{k-1} \neq b_k$

So lemma 1 tells us that each b has a b' with the same final digit, but lemma 2 tells us that that b' is not right next to b.

Thm: $H_k(p)$ does not contain C_{2k} for $k = 2, 3, 5$.

Pf: $2k = 4$: a 4cycle $(a^{(0)}, b^{(0)}, a^{(1)}, b^{(1)})$

~~$b^{(0)}_{k-1}$ would have to match $b^{(1)}_{k-1}$ but they can't be equal by lemma 2.~~

lemma1 $\Rightarrow$ $b^{(0)}_{k-1} = b^{(1)}_{k-1}$

lemma2 $\Rightarrow$ $b^{(0)}_{k-1} \neq b^{(1)}_{k-1}$

$2k = 6$: a 6-cycle: $(a^{(0)}, b^{(0)}, a^{(1)}, b^{(1)}, a^{(2)}, b^{(2)})$

lemma1 $\Rightarrow$ $b^{(0)}_{k-1} = b^{(1)}_{k-1} = b^{(2)}_{k-1}$

lemma2 $\Rightarrow$ $b^{(0)}_{k-1} \neq b^{(1)}_{k-1} \neq b^{(2)}_{k-1}$

$2k = 10$: $(a^{(0)}, b^{(0)}, a^{(1)}, b^{(1)}, a^{(2)}, b^{(2)}, a^{(3)}, b^{(3)}, a^{(4)}, b^{(4)})$

lemma1 $\Rightarrow$ ~~these~~ b's break up into at most 2 equivalence classes.

Pigeonhole $\Rightarrow$ one class has three elements $\Rightarrow$ 2 ~~are~~ are adjacent to the same a.

lemma 2 $\Rightarrow$ $\times$

~~Pf. (Lemma 1)~~

Lemma 1: if $H_k(p)$ contains a cycle of length $2k$ $\Theta = (a^{(0)}, b^{(0)}, \ldots, a^{(k-1)}, b^{(k-1)})$ then
For each $b \in \Theta$, there is some $b' \in \Theta$ such that $b_{k-1} = b'_{k-1}$
$(b, b' \in B, \ b \neq b')$

Pf: assume a, b, a' are adjacent in Θ $(a, a' \in A, b \in B)$
then

$$b_j \equiv a_j + a_{j+1} b_{k-1} \equiv a'_j + a'_{j+1} b_{k-1} \qquad (\forall_j)$$

$$a_j - a'_j \equiv (a'_{j+1} - a_{j+1})\, b_{k-1} \qquad (\forall_j)$$

using this recursively, we get

$$a_j - a'_j \equiv (a'_{k-1} - a_{k-1}) \cdot (b_{k-1})^{k-1-j} \qquad (*)$$

if $a'_{k-1} = a_{k-1}$ then $a_j - a'_j \equiv 0$ $\forall_j$, so $a = a'$.
thus $a'_{k-1} \neq a_{k-1}$ (this is similar to lemma 2, but with a's and b's swapped)

let $S_i = b^{(i)}_{k-1}$ (the last component of each b in the cycle)
$X_i = a^{(i+1 \bmod p)}_{k-1} - a^{i}_{k-1}$ (note that $X_i \neq 0$)

so $(*)$ becomes $a^{(i)}_j - a^{(i+1)}_j \equiv X_i S_i^{k-1-j}$
this is true for all $a \in \Theta \cap A$ so

$$a^{(0)}_j - a^{(1)}_j \equiv X_0 S_0^{k-1-j}$$
$$a^{(1)}_j - a^{(2)}_j \equiv X_1 S_1^{k-1-j}$$
$$\vdots$$
$$+\ a^{(k-1)}_j - a^{(0)}_j \equiv X_{k-1} S_{k-1}^{k-1-j}$$

$$0 \equiv X_0 S_0^{k-1-j} + X_1 S_1^{k-1-j} + \ldots + X_{k-1} S_{k-1}^{k-1-j} \qquad \forall_j$$

or ~~$0 \equiv$~~

$$0 \equiv x_0 + x_1 + x_2 + \dots + x_{k-1}$$
$$0 \equiv s_0 x_0 + s_1 x_1 + \dots + s_{k-1} x_{k-1}$$
$$0 \equiv s_0^2 x_0 + s_1^2 x_1 + \dots + s_{k-1}^2 x_{k-1}$$
$$\vdots$$
$$0 \equiv s_0^{k-1} x_0 + \dots + s_{k-1}^{k-1} x_{k-1}$$

Now consider the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{k-1} \\ s_0^2 & & \cdots & s_{k-1}^2 \\ & & \vdots & \\ s_0^{k-1} & & \cdots & s_{k-1}^{k-1} \end{bmatrix}$$

is a vandermonde matrix. A Vandermonde matrix over a field that if the i^{th} column can be written as a linear combination of the others, then $s_i = s_{i'}$ for some $i' \neq i$.

since $x_i \neq 0$, every column can be written as a linear combination of the others, so each i has an $i' \neq i$ such that $s_i = s_{i'}$, ie $b_{k-1}^{(i)} = b_{k-1}^{(i')}$.

$\blacksquare$

lemma 2: if $b, b' \in B$ $(b \neq b')$ are both adjacent to $a \in A$ then $b_{k-1} \neq b'_{k-1}$

Pf: ~~assume not.~~

$$b_j \equiv a_j + a_{j+1} b_{k-1}$$
$$b'_j \equiv a_j + a_{j+1} b'_{k-1}$$

if $b_{k-1} = b'_{k-1}$, then $b_j = b'_j \; \forall j$

so $b = b'$.

$\blacksquare$