Remarks on classification theory for abstract elementary classes with applications to abelian group theory and ring theory

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Abstract

This thesis has two parts. The first part deals with the classification theory of abstract elementary classes and the second part deals with links and applications of this theory to algebra.

Part I: Remarks on classification theory for abstract elementary classes

This part of the thesis is made up of three chapters based on the corresponding papers: [Ch. 2], [Ch. 3] (a joint work with S. Vasey), and [Ch. 4] (a joint work with R. Grossberg).

Chapter 2, *Non-forking w-good frames.* We introduce and study the notion of a *w-good λ-frame* which is a weakening of Shelah's notion of a good λ-frame. *w-good λ-frames* are useful as they imply the existence of larger models. We show that if $\mathbf{K}$ has a w-good λ-frame, then $\mathbf{K}$ has a model of size λ^{++}. This result extends [Sh:h, §II.4.13.3], [JaSh13, 3.1.9], and [Vas16a, 8.9].

Chapter 3, *Universal classes near* $\aleph_1$ (a joint work with S. Vasey). Shelah has provided sufficient conditions for an $L_{\omega_1,\omega}$-sentence ψ to have arbitrarily large models and for a Morley-like theorem to hold of ψ. These conditions involve structural and set-theoretic assumptions on all the $\aleph_n$'s. Using tools of Boney, Shelah, and Vasey, we give assumptions on $\aleph_0$ and $\aleph_1$ which suffice when ψ is restricted to be universal.

Chapter 4, *Simple-like independence relations in abstract elementary classes* (a joint work with R. Grossberg). We introduce and study simple and supersimple independence relations in the context of AECs with a monster model. We show that if $\mathbf{K}$ has a simple independence relation with the $(< \aleph_0)$-witness property for singletons, then $\mathbf{K}$ does not have the tree property. We characterize supersimple independence relations by finiteness of the Lascar rank under locality assumptions on the independence relation.

Part II: Applications to abelian group theory and ring theory

This part of the thesis is made up of seven chapters based on the corresponding papers: [Ch. 5], [Ch. 6], [Ch. 7] (a joint work with T.G. Kucera), [Ch. 8], [Ch. 9], [Ch. 10], and [Ch. 11]. Chapters 5 and 6 deal with abelian groups and Chapters 7 - 11 with modules over associative rings with unity.

Chapter 5, *Algebraic description of limit models in classes of abelian groups.* We study limit models in the class of abelian groups with the subgroup relation and in

the class of torsion-free abelian groups with the pure subgroup relation. We show that the former are divisible groups. As for the latter, we show that long limit models are pure-injective while short ones are not pure-injective. This is the first place where explicit examples of limit models are studied.

Chapter 6, *A model theoretic solution to a problem of László Fuchs.* Problem 5.1 in page 181 of [Fuc15] asks to find the cardinals λ such that there is a universal abelian p-group for purity of cardinality λ, i.e., an abelian p-group U_λ of cardinality λ such that every abelian p-group of cardinality $\leq \lambda$ purely embeds in U_λ. In this chapter we use ideas from the theory of abstract elementary classes to show:

Theorem. Let p be a prime number. If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then there is a universal abelian p-group for purity of cardinality λ. Moreover for $n \geq 2$, there is a universal abelian p-group for purity of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.

Chapter 7, *On universal modules with pure embeddings* (a joint work with T.G. Kucera). We show that if T is a first-order theory (not necessarily complete) with an infinite model extending the theory of R-modules and $\mathbf{K}^T = (Mod(T), \leq_p)$ has joint embedding and amalgamation, then $\mathbf{K}^T$ has a universal model of cardinality λ if $\lambda^{|T|} = \lambda$ or $\forall \mu < \lambda(\mu^{|T|} < \lambda)$. A corollary of this result is [Sh820, 1.2] which asserts the existence of universal models in the class of reduced torsion-free groups with pure embeddings.

Chapter 8, *Superstability, noetherian rings and pure-semisimple rings.* We uncover a connection between the model-theoretic notion of superstability and that of noetherian rings and pure-semisimple rings. We show:

Theorem. Let R be an associative ring with unity.

1. R is left noetherian if and only if the class of left R-modules with embeddings is superstable.

2. R is left pure-semisimple if and only if the class of left R-modules with pure embeddings is superstable.

Chapter 9, *On superstability in the class of flat modules and perfect rings.* We study the notion of superstability in the class of flat modules with pure embeddings. We show:

Theorem. Let R be an associative ring with unity. R is left perfect if and only if the class of flat left R-modules with pure embeddings is superstable.

It is worth mentioning that the class of flat left R-modules is not first-order axiomatizable for most rings.

Chapter 10, *A note on torsion modules with pure embeddings.* We study $\mathfrak{s}$-tosion modules with pure embeddings as an abstract elementary class. We analyse its limit models and determine when the class is superstable under the assumption that the ring is right semihereditary. In order to fulfill this goal, we develop relative notions of pure-injectivity and Σ-pure-injectivity. As a corollary, we show that the class of torsion abelian groups with pure embeddings is strictly stable, i.e., stable not superstable.

Chapter 11, *Some stable non-elementary classes of modules*. We address the question of whether every AEC of modules with pure embeddings is stable. We show that many non-elementary classes are stable, among them: absolutely pure modules, locally pure-injective modules, flat modules, and $\mathfrak{s}$-torsion modules. As an application of these results we give new characterizations of noetherian rings, pure-semisimple rings, dedekind domains, and fields via superstability. We also show that these results can be used to obtain universal models.

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Contents

Ch. 1. Introduction 1

1.1 General introduction to AECs . . . 1

1.2 Stability . . . 5

1.3 Superstability . . . 7

1.4 Organization . . . 11

I Remarks on classification theory for abstract elementary classes **12**

Ch. 2. Non-forking w-good frames 13

2.1 Introduction . . . 13

2.2 Preliminaries . . . 16

2.2.1 Basic notions . . . 16

2.2.2 Galois-types . . . 17

2.3 w-good frames . . . 18

2.3.1 Frames . . . 18

2.3.2 Inside a w-good $[\lambda, \mu)$-frame . . . 24

2.3.3 Extending w-good λ-frames . . . 26

2.4 Applications . . . 28

2.4.1 Definition and basic properties . . . 28

2.4.2 Weak density . . . 32

2.4.3 Existence of non-forking extension . . . 34

2.4.4 Conclusion . . . 41

Ch. 3. Universal classes near $\aleph_1$ 44

3.1 Introduction . . . 45

3.2 Preliminaries . . . 48
3.3 Main results . . . 53
3.4 Open questions and generalizations . . . 55

Ch. 4. Simple-like independence relations in abstract elementary classes 57
4.1 Introduction . . . 58
4.2 Preliminaries . . . 61
4.2.1 Basic concepts . . . 62
4.2.2 Independence relations and the witness property . . . 64
4.3 The basic notions . . . 65
4.4 Stable independence relations . . . 68
4.4.1 Almost-stable independence relations . . . 71
4.5 Simple independence relations . . . 75
4.6 Simple independent relations with the witness property . . . 81
4.6.1 Failure of the tree property . . . 81
4.6.2 Simple independence relations with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property . . . 83
4.7 Supersimple independence relations and the U-rank . . . 85
4.7.1 Lascar rank . . . 86
4.7.2 A family of classes with the $(< \aleph_0)$-witness property . . . 89
4.8 Future work . . . 91

II Applications to abelian group theory and ring theory 93

Ch. 5. Algebraic description of limit models in classes of abelian groups 94
5.1 Introduction . . . 95
5.2 Preliminaries . . . 98
5.2.1 Basic notions . . . 98
5.2.2 Limit models . . . 100
5.3 Abelian groups . . . 101
5.4 Torsion-free abelian groups . . . 103
5.4.1 Basic properties . . . 103
5.4.2 Limit models . . . 105
5.5 Finitely Butler Groups . . . 111

Ch. 6. A model theoretic solution to a problem of László Fuchs 115
6.1 Introduction . . . 116

6.2 An introduction to AECs from an algebraic point of view 117
6.3 Main result 122
6.4 Some generalizations 126

Ch. 7. On universal modules with pure embeddings 128
7.1 Introduction 129
7.2 Preliminaries 131
7.2.1 Abstract elementary classes 131
7.2.2 Model theory of modules 134
7.2.3 Torsion-free groups 136
7.3 Universal models in classes of R-modules 136
7.3.1 Stability 141
7.3.2 Universal models 142
7.3.3 Reduced torsion-free abelian groups 143
7.4 Limit models in classes of R-modules 145
7.4.1 Torsion-free abelian groups 149

Ch. 8. Superstability, noetherian rings and pure-semisimple rings 151
8.1 Introduction 152
8.2 Preliminaries 156
8.2.1 Basic concepts 156
8.2.2 Limit models, saturated models and superlimits 158
8.2.3 Module theory 160
8.2.4 Notation 162
8.3 A new characterization of noetherian rings 163
8.4 A new characterization of pure-semisimple rings 167
8.4.1 The theory $\tilde{T}$ 168
8.4.2 Superstability in classes closed under direct sums 171
8.4.3 Algebraic characterizations of superstability and pure-semisimple rings 175
8.4.4 Superstable classes 178

Ch. 9. On superstability in the class of flat modules and perfect rings 182
9.1 Introduction 183
9.2 Preliminaries 185
9.2.1 Abstract elementary classes 185
9.2.2 Module Theory 188
9.3 The main case 190
9.4 A special case 195

Ch. 10. A note on torsion modules with pure embeddings 200
10.1 Introduction 201
10.2 Preliminaries 203
10.2.1 Module theory 203
10.2.2 Abstract elementary classes 204
10.3 Relative pure-injective and Σ-pure-injective modules 206
10.4 $\mathfrak{s}$-torsion modules as an AEC 211
10.4.1 Basic properties 211
10.4.2 Limit models and superstability 212
10.5 Torsion abelian groups 216

Ch. 11. Some stable non-elementary classes of modules 220
11.1 Introduction 221
11.2 Preliminaries 223
11.2.1 Abstract elementary classes 223
11.2.2 Independence relations 225
11.2.3 Module Theory 226
11.3 Classes closed under pure-injective envelopes 226
11.3.1 Stability 228
11.3.2 Limit models and superstability 229
11.3.3 Characterizing several classes of rings 231
11.4 Classes closed under pure epimorphic images 236
11.4.1 Stability 237
11.4.2 Classes satisfying Hypothesis 11.3.1 243
11.5 Classes that admit intersections 244
11.5.1 Stability 245

Bibliography 247

Chapter 1

Introduction

This thesis is in model theory, an area of mathematical logic. We focus on abstract elementary classes. Abstract elementary classes (AECs for short) were introduced by Shelah in [Sh88] to study those classes of structures that can not be axiomatized in first-order logic. The setting is general enough to encompass many interesting examples (see Example 1.1.2), but it still allows the development of a rich theory as witnessed by Shelah's two volume book [Sh:h], Baldwin's book [Bal09], Grossberg's book [Gro2X], and more than a hundred publications.

1.1 General introduction to AECs

Let us begin by recalling the definition of an abstract elementary class. Given a model M, we write $|M|$ for its underlying set and $\|M\|$ for its cardinality.

Definition 1.1.1 (Definition 1.2 in [Sh88])**.** An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

1. K is a class of τ-structures, for some fixed language $\tau = \tau(\mathbf{K})$.
2. $\leq_{\mathbf{K}}$ is a partial ordering on K.
3. $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.

4. If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.

5. Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$.

6. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:

   (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.

   (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.

7. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

Below we introduce many examples of AECs in the context of algebra, some of these examples were first identified as AECs in this thesis. Hopefully the naturalness of such classes together with the fact that none of them are axiomatizable by a complete first-order theory and that most of them are not even first-order axiomatizable helps convince the reader of the importance of developing non-elementary model theory.

Recall that ϕ is a positive primitive formula, *pp*-formula for short, if ϕ is an existentially quantified finite system of linear equations. For R-modules M and N, we say that M is a *pure submodule* of N if *pp*-formulas are preserved between M and N and we denote it by $M \leq_p N$. Equivalently, M is a pure submodule of N if for every L right R-module $L \otimes M \to L \otimes N$ is a monomorphism. We write $M \leq N$ to express that M is a submodule of N. In all the examples below the language will be the language of modules, i.e., for a ring R we take $L_R = \{0, +, -\} \cup \{r\cdot : r \in R\}$.

Example 1.1.2. We begin by giving some examples of abstract elementary classes contained in the class of abelian groups:

- $(Ab, \leq)$ where Ab is the class of abelian groups.

- $(Ab, \leq_p)$ where Ab is the class of abelian groups.

- $(K^{\mathrm{Div}}, \leq)$ where K^{Div} is the class of divisible abelian groups. A group G is divisible if for every $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ such that $nh = g$.

- $(K^{tf}, \leq_p)$ where K^{tf} is the class of torsion-free abelian groups. A group G is a torsion-free group if every element has infinite order.

- $(K^{tf}, \leq)$ where K^{tf} is the class of torsion-free abelian groups.
- $(K^{rtf}, \leq_p)$ where K^{rtf} is the class of reduced torsion-free abelian groups. A group G is reduced if it does not have non-trivial divisible subgroups.
- $(K^{\text{tf-Div}}, \leq)$ where $K^{\text{tf-Div}}$ is the class of torsion-free divisible abelian groups.
- $(K^{p\text{-grp}}, \leq_p)$ where $K^{p\text{-grp}}$ is the class of abelian p-groups for p a prime number. A group G is a p-group if every element $g \neq 0$ has order p^n for some $n \in \mathbb{N}$.
- $(\mathrm{Tor}, \leq_p)$ where Tor is the class of abelian torsion groups. A group G is a torsion group if every element $g \neq 0$ has finite order.
- $(\mathrm{Tor}, \leq)$ where Tor is the class of abelian torsion groups.
- $(K^{\text{p-Div}}, \leq_p)$ where $K^{\text{p-Div}}$ is the class of divisible abelian p-groups.
- $(K^{B_0}, \leq_p)$ where K^{B_0} is the class of finitely Butler groups. A group G is a finitely Butler group if G is torsion-free and every pure subgroup of finite rank is a pure subgroup of a finite rank completely decomposable group.
- $(\aleph_1\text{-free}, \leq_p)$ where $\aleph_1$-free is the class of $\aleph_1$-free groups. A group G is $\aleph_1$-free if it is torsion-free and every countable subgroup of G is free.

Below are some examples of AECs in classes of modules:

- $(R\text{-Mod}, \leq)$ where R-Mod is the class of R-modules.
- $(R\text{-Mod}, \leq_p)$ where R-Mod is the class of R-modules.
- $(R\text{-Flat}, \leq_p)$ where R-Flat is the class of flat R-modules. A module F is flat if $(-) \otimes F$ is an exact functor.
- $(R\text{-Absp}, \leq)$ where R-Absp is the class of absolutely pure R-modules. A module M is absolutely pure, if for every N, if $M \leq N$, then $M \leq_p N$.
- $(R\text{-l-inj}, \leq)$ where R-l-inj is the class of locally injective R-modules (also called finitely injective modules). A module M is locally injective if given $\bar{a} \in M^{<\omega}$ there is an injective submodule of M containing $\bar{a}$.
- $(R\text{-l-pi}, \leq_p)$ where R-l-pi is the class of locally pure-injective R-modules. A module M is locally pure-injective if given $\bar{a} \in M^{<\omega}$ there is a pure-injective pure submodule of M containing $\bar{a}$.
- $(\mathfrak{s}\text{-Tor}, \leq_p)$ where $\mathfrak{s}$-Tor is the class of $\mathfrak{s}$-torsion R-modules. A module M is $\mathfrak{s}$-torsion, if for every $m \in M$ there is a low formula $\phi(x)$ such that $M \vDash \phi[m]$. $\phi(x)$ is low if for every $r \in R$, if $R \vDash \phi[r]$ then $r = 0$.

Remark 1.1.3. We will say more about these examples in this introduction and later in this thesis. In particular, the reader can refer to Figure 1.3.1 at the end of this introduction.

The main objective in the study of AECs is to develop a classification theory analogous to the one for first-order theories. The main test question is *Shelah's categoricity conjecture* which is a substantial generalization of *Morley's categoricity theorem*. A class is categorical in λ if there is a unique model of cardinality λ in the class up to isomorphisms.

Shelah's categoricity conjecture. If $\mathbf{K}$ is categorical in *some* cardinal greater than or equal to $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$, then it is categorical in *all* cardinals greater than or equal to $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$.

Many partial results have been obtained in this direction as witnessed by for example [Sh87a], [Sh87b], [Sh394], [Sh:h], [GrVan06b], [GrVan06c], [Bon14b], [Vas17b], [Vas17d], [Vas17c], [Vas19], and [ShVas]. In this thesis we obtain an instance of Shelah's categoricity conjecture for universal $L_{\omega_1,\omega}$-sentences in Chapter 3, which is a joint work with Sebastien Vasey.

Another important test question is the following question posed by Grossberg in the eighties.

Grossberg's question. Assume $\mathbf{K}$ is an AEC with $\lambda \geq \mathrm{LS}(\mathbf{K})$, if $\mathbf{K}$ is categorical in λ and λ^+, must $\mathbf{K}$ have a model of size λ^{++}?

The above question is interesting when $\lambda < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ as it follows from Shelah's presentation theorem that if an AEC has a model of size greater than or equal to $\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ then it has arbitrarily large models. The best approximation is [Sh:h, §VI.0.2] which is a revision of the main theorem of [Sh576]. Shelah claims, under certain cardinal arithmetic hypothesis, that if $\mathbf{K}$ is categorical in λ, λ^+ and has a model of cardinality λ^{++} but essentially less than $2^{\lambda^{++}}$ models in λ^{++}, then there is a model of size λ^{+++}. We revisit Shelah result in Chapter 2. Assuming that the class is (λ, λ^+)-tame[1] we are able to avoid using the set-theoretic machinery developed first in [Sh576, §3] and then revised in [Sh:h, §VII] and used in Shelah's original proof.

A fundamental notion in trying to settle Shelah's categoricity conjecture and Grossberg's question is that of a *good λ-frame*. These were introduced by Shelah in [Sh:h, §II.2] and are the central notion of Shelah's two volume book on AECs [Sh:h]. The intuitive idea is that if $\mathbf{K}$ has a good λ-frame, then the models of size λ are well-behaved and there is a non-forking like relation on types over model of size λ akin to superstability. The reader can consult the definition in [Ch. 2, 3.2].

[1]Intuitively $\mathbf{K}$ is (λ, λ^+)-tame if Galois-types over models of size λ^+ are determined by their restrictions to models of size λ.

Since good λ-frames are hard to build, some weaker notions which only satisfy a subset of the axioms of good λ-frames were consider in [JaSh13], [JaSh940], [Vas16a], and [Vas16c]. In Chapter 2, we introduce a weaker notion than all the ones previously studied called *w-good λ-frames* (Definition [Ch. 2, 3.7]). These frames are useful as they imply the existence of larger models.

Good frames correspond to *local* independence relations. Recently, *global* independence relations have been introduced for *nice* AECs and even for *nice* categories. See for example [BoGr17], [BGKV16], [Vas16a], and [LRV19]. By an *independence relation*, it is understood a *forking-like relation* in the sense of Shelah which in turn generalizes linear independence in vector spaces.

In Chapter 4, which is a joint work with Rami Grossberg, we introduce *simple* and *supersimple* independence relations. The main difference between the independence notions previously studied in AECs and the ones that we introduce is that we do not assume uniqueness of non-forking extensions and instead assume the type-amalgamation property. Although this may seem like a minor change, based on our knowledge of forking in first-order theories this is a significant one.

Another important component in the classification theory of AECs corresponds to finding dividing lines. A dividing line is a property such that the classes satisfying such a property have some nice behaviour while those not satisfying it have a wild one. These were introduced by Shelah in [Sh1] and are central in both elementary and non-elementary model theory. An introduction to dividing lines for mathematicians not working in mathematical logic can be found in [Sh1151, Part I] and [Bal20].

In this thesis, we study four dividing lines. The aforementioned simplicity and supersimplicity in addition to stability and superstability. Our treatment of stability and superstability differs significantly with our treatment of simplicity and supersimplicity. Stability and superstability were first considered for AECs by Shelah in [Sh394] and are well-understood notions in this context. So what we do in this thesis is to find links and applications to abelian group theory and ring theory. In the rest of this introduction we will explain what is known and the results we obtain for both of these dividing lines.

1.2 Stability

Stability was introduced by Shelah for first-order theories in [Sh1] and for AECs in [Sh394]. Under additional hypothesis that are satisfied by most of the classes studied in this thesis[2] stability is a well-understood concept as witnessed by [GrVan06] and [Vas18].

[2]The hypothesis are amalgamation, joint embedding, no maximal models, and $\mathrm{LS}(\mathbf{K})$-tameness.

Since an AEC is a semantic object, the notion of a type as a set of formulas is not the correct generalization of a first-order type. Instead, Shelah introduced a semantic notion of type in [Sh300], we call them Galois-types following [Gro02]. Intuitively, a *Galois-type* over a model M can be identified with an orbit of the group of automorphisms of the monster model which fix M point-wise. The reader can consult the full definition in [Ch. 6, 2.6]. If $M \in K$ we denote by $\mathbf{gS}(M)$ the set of all Galois-type over M and by $\mathbf{K}_\lambda$ the models of $\mathbf{K}$ of cardinality λ.

Definition 1.2.1. An AEC $\mathbf{K}$ is λ-*stable* if for any $M \in \mathbf{K}_\lambda$, $|\mathbf{gS}(M)| \leq \lambda$. We say that $\mathbf{K}$ is *stable* if $\mathbf{K}$ is λ-stable for some $\lambda \geq \mathrm{LS}(\mathbf{K})$.

A classical result from first-order model theory is that if T is a complete first-order theory extending the theory of modules, then the class of models of T with pure embeddings is stable. This was shown independently by Fisher [Fis75] and Baur [Bau75, Theo 1] in the seventies. A natural question to ask is if the same is true for any AEC of modules with pure embeddings.

Question 1.2.2 ([Ch. 6, 2.12])**.** Let R be an associative ring with unity. If $(K, \leq_p)$ is an abstract elementary class such that $K \subseteq R$-Mod, is $(K, \leq_p)$ stable? Is this true if $R = \mathbb{Z}$? Under what conditions on R is this true?

In this thesis, we give many instances where the solution to the above question is affirmative, but we are unable to give a complete answer to the question. In particular, all the classes introduced in Example 1.1.2 are stable with respect to pure embeddings. The way we approach the problem is by showing that if a class has some *nice* algebraic properties then it has to be stable. Our best result in this direction is obtained in Chapter 11 and generalizes some of the results obtained in the previous chapters.

Theorem 1.2.3 ([Ch. 11, 3.7, 4.16])**.** *Assume $(K, \leq_p)$ is an AEC with $K \subseteq R$-Mod such that K is closed under direct sums. If K is closed under pure-injective envelopes and direct summands or under pure epimorphic images and pure submodules, then $(K, \leq_p)$ is stable.*

An interesting consequence of stability is that it implies the existence of many universal models. A model $M \in K$ is a *universal model of cardinality* λ, if M is of size λ and if for every $N \in K$ of size λ there is a $\mathbf{K}$-embedding[3] $f : N \to M$. The search for universal models in classes of structures dates back to the early twentieth century when Hausdorff showed that there is a universal linear order of cardinality $\aleph_{n+1}$ if $2^{\aleph_n} = \aleph_{n+1}$. A detailed history outlining the study of universal models in various classes of structures is presented in [Bal20, §2.1].

[3] $f : N \to M$ is a $\mathbf{K}$-*embedding* if $f : N \cong f[N] \leq_{\mathbf{K}} M$.

In this thesis we focus on the existence of universal models in classes of modules with respect to pure embeddings. The main problem we adress is Problem 5.1 in page 181 of [Fuc15]. The problem stated by Fuchs is the following:

Problem. For which cardinals λ is there a universal abelian p-group for purity? We mean an abelian p-group U_λ of cardinality λ such that every abelian p-group of cardinality $\leq \lambda$ embeds in U_λ as a pure subgroup. The same question for torsion-free abelian groups.

By using that the class of p-groups with pure embeddings is a stable AEC with amalgamation, we give sufficient conditions for the existence of universal models. Moreover, we obtain a complete solution to Fuchs' problem below $\aleph_\omega$ with the exception of $\aleph_0$ and $\aleph_1$.

Theorem 1.2.4 ([Ch. 6, 3.6, 3.8])**.** *Let p be a prime number. If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then there is a universal abelian p-group for purity of cardinality λ. Moreover for $n \geq 2$, there is a universal abelian p-group for purity of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.*

The question for torsion-free abelian groups has been thoroughly studied by Kojman and Shelah [KojSh95], [Sh3], [Sh820], and [Sh1151]. Using the methods described in this section, we first extend the positive results to classes of modules that are first-order axiomatizable and have amalgamation and joint embedding in [Ch. 7] and later we extend the positive results to classes of flat modules for arbitrary rings in [Ch. 11].

1.3 Superstability

Superstability was introduced by Shelah for first-order theories in [Sh1]. Extensions of superstability for non-first-order theories were first studied in Grossberg's PhD thesis and published in [GrSh86]. In the context of AECs, superstability was first considered in [Sh394], but it was not until the work of Grossberg and Vasey ([GrVas17], [Vas18]) that it was fully grasped. In [GrVas17, 1.3] and [Vas18], it was shown (under additional hypothesis that are satisfied by most of the classes studied in this thesis[4]) that superstability is a well-behaved concept and many conditions that were believed to characterize superstability were found to be equivalent. Grosberg's and Vasey's work builds on significant results of Boney, Shelah, Villaveces, and VanDieren.[5]

[4]The hypothesis are amalgamation, joint embedding, no maximal models and LS($\mathbf{K}$)-tameness.

[5]A more detailed account of the development of the notion of superstability in AECs can be consulted in the introduction of [GrVas17].

Before introducing superstability, we recall the concepts of universal extensions and limit models. These were originally introduced in [KolSh96, 3.6, 4.1]. A model M is *universal over* N if and only if $\|N\| = \|M\| = \lambda$ and for every $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \xrightarrow[N]{} M$.

Definition 1.3.1. Let λ be an infinite cardinal and $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that:

1. $M_0 = N$.
2. $M = \bigcup_{i<\alpha} M_i$.
3. M_{i+1} is universal over M_i for each $i < \alpha$.

M is a (λ, α)-limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. M is a λ-limit model if there is a limit ordinal $\alpha < \lambda^+$ such that M is a (λ, α)-limit model.

The next result characterizes the existence of limit models.

Fact 1.3.2 ([Sh:h, §II.1.16], [GrVan06, 2.9])**.** Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation, and no maximal models. $\mathbf{K}$ is λ-stable if an only if $\mathbf{K}$ has a λ-limit model.

We say that $\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has λ-limit models and if any two λ-limit models are isomorphic. Determining whether an AEC has uniqueness of limit models for a fixed cardinal is an interesting question that has been studied thoroughly [ShVi99], [Van06], [GVV16], [Bon14], [Van16], [BoVan], and [Vas19].

Definition 1.3.3. $\mathbf{K}$ is a *superstable* AEC if and only if $\mathbf{K}$ has uniqueness of limit models on a tail of cardinals.

It is worth mentioning that the above definition is equivalent to classical first-order superstability. For a complete first-order theory T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory[6]. The definition of superstability we just introduced appears for the first time in [GrVas17].

Superstability may seem like a purely model-theoretic property. However in this thesis we show that it is equivalent to classical ring theoretic notions. More specifically, we will characterize noetherian rings, pure-semisimple rings, and perfect rings via superstability.

[6]A first-order theory T is superstable if and only if T is λ-stable for every $\lambda \geq 2^{|T|}$.

Noetherian rings are rings such that every increasing chain of (left) ideals is stationary. These were introduced by Noether in [Noe21] and play a prominent role in commutative algebra.

Theorem 1.3.4 ([Ch. 8, 3.12], [Ch. 11, 3.18]). *Let R be an associative ring with unity. The following are equivalent.*

1. *R is left noetherian.*
2. *The class of left R-modules with embeddings is superstable.*
3. *The class of absolutely pure left R-modules with embeddings is superstable*
4. *The class of locally injective left R-modules with embeddings is superstable.*

Pure-semisimple rings are rings such that every (left) module is pure-injective. These were given a name by Simson [Sim77], but studied since the sixties [Cha60].

Theorem 1.3.5 ([Ch. 8, 4.28], [Ch. 11, 3.21]). *Let R be an associative ring with unity. The following are equivalent.*

1. *R is left pure-semisimple*
2. *The class of left R-modules with pure embeddings is superstable*
3. *The class of locally pure-injective left R-modules with pure embeddings is superstable.*

Perfect rings are rings such that every flat (left) module is a projective module. These were introduced by Bass in [Bas60] and play an important role in homological algebra.

Theorem 1.3.6 ([Ch. 9, 3.15]). *Let R be an associative ring with unity. R is left perfect if and only if the class of flat left R-modules with pure embeddings is superstable.*

We study the question of when the class of $\mathfrak{s}$-torsion modules with pure embeddings is superstable in [Ch. 10]. In that case superstability does not seem to correspond to a well-studied class of rings.

The proof for each class of rings is different, but a key step in all of them is to characterize the limit models in algebraic terms. In the first case they are injective modules, in the second case they are pure-injective modules, in the third case they are cotorsion modules, and in the last case they are relative pure-injective modules.

To finish this introduction we present the next diagram. In it, we display where all the classes introduced in Example 1.1.2 are in the stability hierarchy. The classes that are on the red cloud are stable, but it is not known if they are strictly stable or superstable. Many of the results needed to classify the classes of the diagram are first obtained in this thesis.

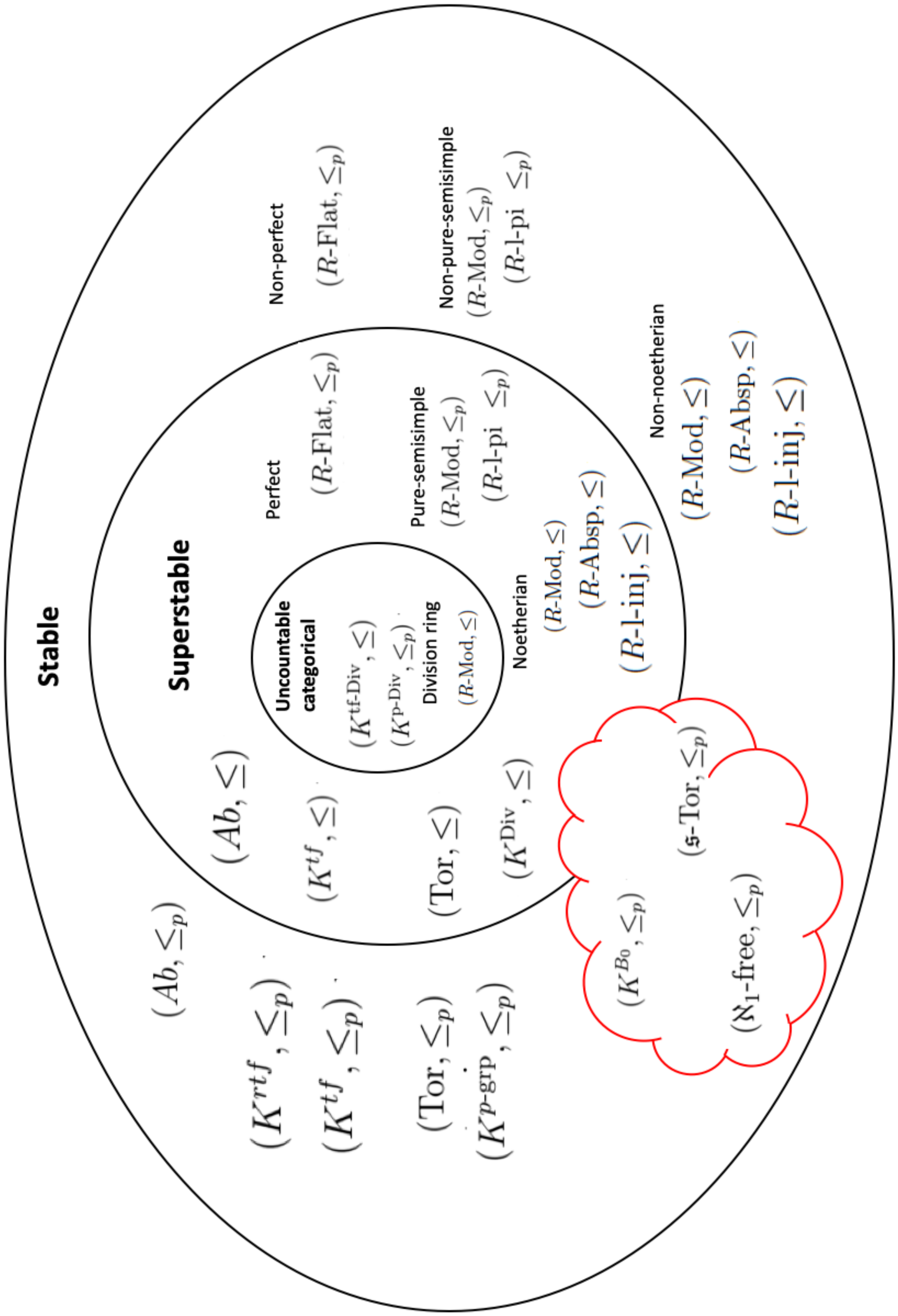

FIGURE 1.3.1: Known universe of AECs of modules

1.4 Organization

Besides this introduction, this thesis has two parts. Each part is divided into chapters and each chapter corresponds to a paper written by the author either with collaborators or by himself. Each chapter has its own introduction and preliminaries (which sometimes overlap). For a reader familiar with algebra but unfamiliar with model theory or AECs we would recommend the preliminaries of Chapter 6. If the reader has the possibility, we would advice him to look at the published papers.

The thesis is organized thematically rather than on the order the papers were written. For this reason some of the chapters refer to results from later chapters.

The first part of the thesis deals with the classification theory of abstract elementary classes. It is made up of three chapters based on the corresponding papers: [Ch. 2], [Ch. 3] (a joint work with Sebastien Vasey), and [Ch. 4] (a joint work with Rami Grossberg). The three papers that make up this part have been accepted for publication.

The second part of the thesis is made up of seven chapters based on the corresponding papers: [Ch. 5], [Ch. 6], [Ch. 7] (a joint work with Thomas G. Kucera), [Ch. 8], [Ch. 9], [Ch. 10], and [Ch. 11]. Chapters 5 and 6 deal with abelian groups and Chapters 7 - 11 with modules over associative rings with unity. Chapters 5 - 9 have been accepted for publication while Chapters 10 and 11 have been submitted for publication.

An overview of what is done in each chapter can be consulted in the abstract of this thesis.

Part I

Remarks on classification theory for abstract elementary classes

Chapter 2

Non-forking w-good frames

This chapter is based on [Ch. 2].

Abstract

We introduce the notion of a w-good λ-frame which is a weakening of Shelah's notion of a good λ-frame. Existence of a w-good λ-frame implies existence of a model of size λ^{++}. Tameness and amalgamation imply extension of a w-good λ-frame to larger models. As an application we show:

Theorem 2.0.1. *Suppose* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and* $2^{\lambda^+} > \lambda^{++}$. *If* $\mathbb{I}(\mathbf{K}, \lambda) = \mathbb{I}(\mathbf{K}, \lambda^+) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$ *and* $\mathbf{K}$ *is* (λ, λ^+)*-tame, then* $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$.

The proof presented clarifies some of the details of the main theorem of [Sh576] and avoids using the heavy set-theoretic machinery of [Sh:h, §VII] by replacing it with tameness.

2.1 Introduction

The central notion of Shelah's two volume book [Sh:h] is that of a *good* λ*-frame*, which is a forking-like notion for types of singletons in abstract elementary classes. It is crucial in transferring existence of models and categoricity to other cardinalities.

Since it is hard to build good λ-frames, several weaker notions have been studied. Jarden and Shelah introduced *semi-good λ-frames* in [JaSh13] and *almost-good λ-frames* in [JaSh940]. Vasey worked with *good$^{-(S)}\lambda$-frames* in [Vas16a] and with *good$^{-}\lambda$-frames* in [Vas16c].[1] These notions have been particularly useful in deriving existence of models in larger cardinalities.

In this paper we introduce the notion of a *w-good λ-frame* (see Definition 2.3.6), which is a weaker notion than all the ones mentioned above. A w-good λ-frame satisfies all the properties of a good λ-frame except that the density requirement is weakened and stability, symmetry and local character are not assumed.

In [Sh:h, §III.0] Shelah introduced pre-λ-frames which are a weaker notion than that of a w-good λ-frame, but they are so weak that no interesting statement follows from their existence. The next diagram exhibits the relationship in strength between all the frames presented above. In the diagram, the source of an arrow is stronger than its target.[2]

$$\begin{tikzcd}
 & \boxed{\text{semi-good } \lambda\text{-frame}} \arrow[r] & \boxed{\textit{good}^{-}\lambda\text{-frame}} \arrow[d] & \\
\boxed{\text{good } \lambda\text{-frame}} \arrow[ur] \arrow[r] \arrow[dr] & \boxed{\textit{good}^{-(S)}\lambda\text{-frame}} \arrow[ur] & \boxed{\text{w-good } \lambda\text{-frame}} \arrow[r] & \boxed{\text{pre-}\lambda\text{-frame}} \\
 & \boxed{\text{almost-good } \lambda\text{-frame}} \arrow[ur] & &
\end{tikzcd} \tag{2.1.1}$$

A w-good λ-frame is useful as it allows us to construct larger models. More precisely we show the following theorem which generalizes [Sh:h, §II.4.13.3], [JaSh13, 3.1.9] and [Vas16a, 8.9]:

Theorem 2.1.1. *If $\mathfrak{s}$ is a w-good $[\lambda, \mu)$-frame, then $\mathbf{K}_\kappa \neq \emptyset$ for all $\kappa \in [\lambda, \mu^+]$.*

Under the hypothesis of tameness and the amalgamation property w-good λ-frames can be extended to larger models. The technique used to show this is similar

[1] See Definition 2.3.5 for the definitions of all these notions and Diagram 2.1.1 for their comparison in strength.

[2] In Section 3.1 we present a more detailed discussion regarding the implications in the other direction.

to that of [Bon14, 1.1], we only need to show that weak density and no maximal models transfer up.

Theorem 2.3.24. *Assume* $\mathbf{K}$ *is an AEC with the* $[\lambda, \mu^+)$*-amalgamation property. If* $\mathfrak{s}$ *is a w-good* λ*-frame and* $\mathbf{K}$ *is* $(\lambda, \leq \mu)$*-tame, then* $\mathfrak{s}$ *can be extended to a w-good* $[\lambda, \mu^+)$*-frame.*

After presenting the above theorem, we apply the results obtained for w-good λ-frames to prove the following:

Theorem 2.4.2. *Suppose* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and* $2^{\lambda^+} > \lambda^{++}$*. If* $\mathbb{I}(\mathbf{K}, \lambda) = \mathbb{I}(\mathbf{K}, \lambda^+) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$ *and* $\mathbf{K}$ *is* (λ, λ^+)*-tame, then* $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$*.*

The proof presented in the paper is an exposition of the ideas displayed in [Sh576] with the following key feature. Using the assumption that $2^{\lambda^+} > \lambda^{++}$, that $\mathbf{K}$ is (λ, λ^+)-tame and the results obtained for w-good frames, we are able to avoid using the set-theoretic machinery developed in [Sh576, §3] and in [Sh:h, §VII] and used in Shelah's original proof. The set-theoretic machinery was initially developed by Shelah in a 20 pages section of [Sh576, §3], ten years later Shelah redid this section in a 200 pages chapter of his book [Sh:h, §VII]. In Shelah words "Compared to [Sh576, §3], the present version [Chapter VII] is hopefully more transparent". This newer version was not refereed and we were still unable to verify Shelah's assertions.

Another interesting consequence of Theorem 2.4.2 is that it gives a 200 pages shorter proof for the main theorem of [Sh576] (see Fact 2.1.2(1) below), with the extra hypothesis that $2^{\lambda^+} > \lambda^{++}$, in the case $\mathbf{K}$ is a universal class (see Definition 2.4.30).

Lastly, we would like to point out that Theorem 2.4.2 is not the best possible result in this direction, since the main theorem of [Sh:h, §VI.0.(2)] (which is a revised version of [Sh576]) is the following.

Fact 2.1.2. Suppose $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. If $\mathbb{I}(\mathbf{K}, \lambda) = \mathbb{I}(\mathbf{K}, \lambda^+) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < \mu_{unif}(\lambda^{++}, 2^{\lambda^+})$[3], then

1. $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$.
2. There is an almost-good λ-frame on $\mathbf{K}$.[4]

The paper is organized as follows. Section 2 presents necessary background. Section 3 introduces the notion of a w-good frame, shows that w-good frames imply the existence of larger models and shows how to extend w-good frames under tameness and amalgamation. Section 4 presents an exposition of the proof of the main theorem of [Sh576], with the additional hypothesis that $2^{\lambda^+} > \lambda^{++}$ and (λ, λ^+)-tameness.

[3]See [Sh:h, §VII.0.4] for a definition of μ_{unif} and some of its properties.

[4]Combining further results of Shelah, [Vas20, 7.1] actually gets a good λ-frame and a good λ^+-frame.

The proof presented avoids using the set-theoretic machinery of [Sh:h, §VII] by using (λ, λ^+)-tameness and the results of Section 3.

This paper was written while the author was working on a Ph.D. under the direction of Rami Grossberg at Carnegie Mellon University and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work in particular. I thank Hanif Cheung for helpful conversations. I thank Sebastien Vasey for very useful comments on an early version. I thank the referee for valuable comments that helped improve the paper.

2.2 Preliminaries

We present the basic concepts of abstract elementary classes that we will need for the development of this paper. These are further studied in [Bal09, §4 - 8] and [Gro2X, §2, §4.4].

2.2.1 Basic notions

First we will fix some notation.

Notation 2.2.1.

- Given $M \in \mathbf{K}$ we denote the universe of M by $|M|$ and its cardinality by $\|M\|$.
- Let $LS(\mathbf{K}) \leq \lambda < \mu$ such that λ is an infinite cardinals and μ is an infinite cardinal or infinity. Let $[\lambda, \mu) = \{\kappa \in card : \lambda \leq \kappa < \mu\}$. Given an abstract elementary class $\mathbf{K}$ and $[\lambda, \mu)$ an interval of cardinals, $\mathbf{K}_{[\lambda,\mu)} = \{M \in \mathbf{K} : \|M\| \in [\lambda, \mu)\}$. In particular we let $\mathbf{K}_{\{\lambda\}} = \mathbf{K}_{[\lambda,\lambda^+)} = \mathbf{K}_\lambda$.

Let us recall the following three properties. They play an important role in this paper, although not every AEC satisfies them.

Definition 2.2.2. Let $LS(\mathbf{K}) \leq \lambda < \mu$ such that λ is an infinite cardinals and μ is an infinite cardinal or infinity.

1. $\mathbf{K}_{[\lambda,\mu)}$ has the *amalgamation property* (or $\mathbf{K}$ has the $[\lambda, \mu)$-amalgamation property): if for every $M, N, R \in \mathbf{K}_{[\lambda,\mu)}$ such that $M \leq_{\mathbf{K}} N, R$, there are f $\mathbf{K}$-embedding and $R^* \in \mathbf{K}$ such that $f : N \underset{M}{\longrightarrow} R^*$ and $R \leq_{\mathbf{K}} R^*$.
2. $\mathbf{K}_{[\lambda,\mu)}$ has the *joint embedding property* (or $\mathbf{K}$ has the $[\lambda, \mu)$-joint embedding property): if for every $M, N \in \mathbf{K}_{[\lambda,\mu)}$, there are f $\mathbf{K}$-embedding and $R^* \in \mathbf{K}$ such that $f : M \to R^*$ and $N \leq_{\mathbf{K}} R^*$.

3. $\mathbf{K}_{[\lambda,\mu)}$ has *no maximal models*: if for every $M \in \mathbf{K}_{[\lambda,\mu)}$, there is $M^* \in \mathbf{K}$ such that $M <_{\mathbf{K}} M^*$.

The following fact was first proven in [Sh88], but a more straightforward proof appears in [Gro02, 4.3].

Fact 2.2.3. Assume $2^\lambda < 2^{\lambda^+}$. Let $\mathbf{K}$ be an AEC. If $\mathbb{I}(\mathbf{K}, \lambda) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}_\lambda$ has the amalgamation property.

2.2.2 Galois-types

Let us begin by reviewing the concept of pre-type and some of its basic properties, pre-types will play a very important role in Section 4.

Definition 2.2.4.

1. The class of *pre-types* is:

$$K^3_\lambda = \{(a, M, N) : M \leq_{\mathbf{K}} N, a \in |N|\backslash|M| \text{ and } M, N \in \mathbf{K}_\lambda\}.$$

2. Given $(a_0, M_0, N_0), (a_1, M_1, N_1) \in K^3_\lambda$ we define:
   (a) $(a_0, M_0, N_0) \leq (a_1, M_1, N_1)$ if and only if $M_0 \leq_{\mathbf{K}} M_1$, $N_0 \leq_{\mathbf{K}} N_1$ and $a_0 = a_1$.
   (b) $(a_0, M_0, N_0) < (a_1, M_1, N_1)$ if and only if $(a_0, M_0, N_0) \leq (a_1, M_1, N_1)$ and $M_0 \neq M_1$.

3. Given $(a_0, M_0, N_0), (a_1, M_1, N_1) \in K^3_\lambda$ and $h : N_0 \to N_1$, we define $(a_0, M_0, N_0) \leq_h (a_1, M_1, N_1)$ if and only if $h \restriction_{M_0}: M_0 \to M_1$ is a $\mathbf{K}$-embedding, $h : N_0 \to N_1$ is a $\mathbf{K}$-embedding and $h(a_0) = a_1$.

We will also use the following property of pre-types, which is introduced in [Sh576, 2.5]. This will only be used in Section 4.

Definition 2.2.5. $(a_0, M_0, N_0) \in K^3_\lambda$ is *reduced* if for any $(a_1, M_1, N_1) \in K^3_\lambda$ such that $(a_0, M_0, N_0) \leq (a_1, M_1, N_1)$ we have that $M_1 \cap N_0 = M_0$.

The following appears as [Sh576, 2.6(1)] without a proof and it is proven in [JaSh13, 3.3.4].

Fact 2.2.6. For every $(a_0, M_0, N_0) \in K^3_\lambda$ there is $(a_1, M_1, N_1) \in K^3_\lambda$ such that $(a_0, M_0, N_0) \leq (a_1, M_1, N_1)$ and (a_1, M_1, N_1) is reduced. In that case, we say that reduced pre-types are dense in K^3_λ.

Let us recall the concept of Galois-type, this was introduced by Shelah in [Sh300].

Definition 2.2.7.

1. Given $(a_0, M_0, N_0), (a_1, M_1, N_1) \in K^3_\lambda$ we say $(a_0, M_0, N_0)E_{at}(a_1, M_1, N_1)$ if $M := M_0 = M_1$ and there are f_0, f_1 and $N \in \mathbf{K}$ such that $f_l : N_l \underset{M}{\longrightarrow} N$ for each $l \in \{0, 1\}$ and $f_0(a_0) = f_1(a_1)$.

2. Let E be the transitive closure of E_{at}.

3. Given $(a, M, N) \in K^3_\lambda$, we define the *Galois-type* (also referred to as orbital type in the literature) as $\mathbf{tp}(a/M, N) = [(a, M, N)]_E$.

4. Given $M \in \mathbf{K}_\lambda$, let $\mathbf{gS}(M) = \{\mathbf{tp}(a/M, N) : M \leq_{\mathbf{K}} N \in \mathbf{K}_\lambda \text{ and } a \in |N|\}$ and $\mathbf{gS}^{na}(M) = \{\mathbf{tp}(a/M, N) : (a, M, N) \in K^3_\lambda\}$. $\mathbf{gS}^{na}(M)$ is the set of nonalgebraic types.

The following is straightforward.

Fact 2.2.8. If $\mathbf{K}$ is an AEC and $\mathbf{K}_\lambda$ has the amalgamation property, then E_{at} is transitive. Hence $E_{at} = E$.

The concept of tameness was introduced by Grossberg and VanDieren in [GrVan06]. We use this property to avoid using the set-theoretic machinery of [Sh:h, §VII] mentioned in the introduction. The idea of using tameness instead of set-theoretic ideas traces back to [GrVan06] and [GrVan06b].

Definition 2.2.9. We say $\mathbf{K}$ is $(< \kappa)$-*tame* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}(M)$, there is $N \leq_{\mathbf{K}} M$ such that $\|N\| < \kappa$ and $p \restriction_N \neq q \restriction_N$. $\mathbf{K}$ is κ-tame, if $\mathbf{K}$ is $(< \kappa^+)$-tame. If we write $(\kappa, \leq \lambda)$-tame we restrict to $M \in \mathbf{K}_{[\kappa, \lambda^+)}$ and if we write (κ, λ)-tame we restrict to $M \in \mathbf{K}_\lambda$.

2.3 w-good frames

2.3.1 Frames

The concept of a good λ-frame is introduced in [Sh:h, §II.2, p. 259-263]. We will follow the simplification and generalization given in [Vas16c] and [BoVas17a].

First let us recall the notion of a pre-frame.

Definition 2.3.1. Let $\lambda < \mu$ where λ is an infinite cardinal and μ is an infinite cardinal or infinity. A *pre-*$[\lambda, \mu)$*-frame* is a triple $(\mathbf{K}, \mathop{\smile}\limits^{\textstyle|}, \mathbf{gS}^{bs})$ where the following properties hold:

1. $\mathbf{K}$ is an abstract elementary class with $\lambda \geq \mathrm{LS}(\mathbf{K})$ and $\mathbf{K}_\lambda \neq \emptyset$.
2. $\mathbf{gS}^{bs} \subseteq \bigcup_{M \in \mathbf{K}_{[\lambda,\mu)}} \mathbf{gS}^{na}(M)$. Let $\mathbf{gS}^{bs}(M) = \mathbf{gS}(M) \cap \mathbf{gS}^{bs}$.
3. $\mathop{\smile}\limits^{\textstyle|}$ is a relation on quadruples (M_0, M_1, a, N), where $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$, $a \in N$ and $M_0, M_1, N \in \mathbf{K}_{[\lambda,\mu)}$. We write $a \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\textstyle|}}} M_1$ or $\mathbf{tp}(a/M_1, N)$ does not fork over M_0 (which is well-defined by the next three properties).
4. <u>Invariance:</u> If $f : N \cong N'$ and $a \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\textstyle|}}} M_1$, then $f(a) \underset{f[M_0]}{\overset{N'}{\mathop{\smile}\limits^{\textstyle|}}} f[M_1]$. If $\mathbf{tp}(a/M_1, N) \in \mathbf{gS}^{bs}(M_1)$, then $\mathbf{tp}(f(a)/f[M_1], N') \in \mathbf{gS}^{bs}(f[M_1])$.
5. <u>Monotonicity:</u> If $a \underset{M_0}{\overset{N}{\mathop{\smile}\limits^{\textstyle|}}} M_1$ and $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M_1' \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N \leq_{\mathbf{K}} N''$ with $N'' \in \mathbf{K}_{[\lambda,\mu)}$ and $a \in N'$, then $a \underset{M_0'}{\overset{N'}{\mathop{\smile}\limits^{\textstyle|}}} M_1'$ and $a \underset{M_0'}{\overset{N''}{\mathop{\smile}\limits^{\textstyle|}}} M_1'$.
6. <u>Non-forking types are basic:</u> If $a \underset{M}{\overset{N}{\mathop{\smile}\limits^{\textstyle|}}} M$, then $\mathbf{tp}(a/M, N) \in \mathbf{gS}^{bs}(M)$.

To simplify the comparison between the different kinds of frames we will introduce below, we recall the notion of good frame.

Definition 2.3.2. Let $\lambda < \mu$ where λ is an infinite cardinal and μ is an infinite cardinal or infinity. A *good* $[\lambda, \mu)$*-frame* is a triple $(\mathbf{K}, \mathop{\smile}\limits^{\textstyle|}, \mathbf{gS}^{bs})$ where the following properties hold:

1. $(\mathbf{K}, \mathop{\smile}\limits^{\textstyle|}, \mathbf{gS}^{bs})$ is a pre-$[\lambda, \mu)$-frame.
2. $\mathbf{K}_{[\lambda,\mu)}$ has amalgamation, joint embedding and no maximal models.
3. <u>bs-Stability:</u> $|\mathbf{gS}^{bs}(M)| \leq ||M||$ for all $M \in \mathbf{K}_{[\lambda,\mu)}$.
4. <u>Density of basic types:</u> If $M <_{\mathbf{K}} N$ are both in $\mathbf{K}_{[\lambda,\mu)}$, then there is an $a \in |N|$ such that $\mathbf{tp}(a/M, N) \in \mathbf{gS}^{bs}(M)$.
5. <u>Existence of non-forking extension:</u> If $p \in \mathbf{gS}^{bs}(M)$ and $M \leq_{\mathbf{K}} N$ with $N \in \mathbf{K}_{[\lambda,\mu)}$, then there is $q \in \mathbf{gS}^{bs}(N)$ that does not fork over M and extends p.

6. Uniqueness: If $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_{[\lambda,\mu)}$, $p, q \in \mathbf{gS}(N)$ do not fork over M and $p \restriction_M = q \restriction_M$, then $p = q$.

7. Symmetry: If $a_1 \underset{M_0}{\overset{N}{\downarrow}} M_2$, $a_2 \in M_2$ and $\mathbf{tp}(a_2/M_0, N) \in \mathbf{gS}^{bs}(M_0)$, then there are M_1 and $N' \geq_{\mathbf{K}} N$ with $a_1 \in M_1$ and $M_1, N' \in \mathbf{K}_{[\lambda,\mu)}$ such that $a_2 \underset{M_0}{\overset{N'}{\downarrow}} M_1$.

8. Local character: If $\delta < \mu$ is a limit ordinal, $\{M_i : i < \delta\} \subseteq \mathbf{K}_{[\lambda,\mu)}$ is an increasing continuous chain and $p \in \mathbf{gS}^{bs}(M_\delta)$, then there is an $i < \delta$ such that p does not fork over M_i.

9. Continuity: If $\delta < \mu$ is a limit ordinal, $\{M_i : i < \delta\} \subseteq \mathbf{K}_{[\lambda,\mu)}$ is an increasing continuous chain, $\{p_i : i < \delta\}$ with $p_i \in \mathbf{gS}^{bs}(M_i)$ and for $i < j < \delta$ implies that $p_i = p_j \restriction_{M_i}$ and $p \in \mathbf{gS}^{na}(M_\delta)$ is an upper bound for $\{p_i : i < \delta\}$, then $p \in \mathbf{gS}^{bs}(M_\delta)$. Moreover, if each p_i does not fork over M_0, then neither does p.

10. Transitivity: If $M_0 \leq M_1 \leq M_2$ with $M_0, M_1, M_2 \in \mathbf{K}_{[\lambda,\mu)}$, $p \in \mathbf{gS}(M_2)$ does not fork over M_1 and $p \restriction_{M_1}$ does not fork over M_0, then p does not fork over M_0.

Recall the following notation which was introduced in [Vas16c].

Notation 2.3.3. Given $\mathbb{L}$ a list of properties a *good*$^{-\mathbb{L}}$$\lambda$-frame is a pre-$\lambda$-frame that satisfies all the properties of a good λ-frame except possibly the properties listed in $\mathbb{L}$. We abbreviate stability by St, density by D, symmetry by S and local character by Lc.

In [JaSh940] Jarden and Shelah introduced the following weakening of local character.

Definition 2.3.4. A $(\mathbf{K}, \downarrow, \mathbf{gS}^{bs})$ pre-λ-frame satisfies *weak local character* if there is a 2-ary relation $\leq^*$ in $\mathbf{K}_\lambda$ such that:

- If $M \leq^* N$ both in $\mathbf{K}_\lambda$, then $M \leq_{\mathbf{K}} N$.
- For every $M \in \mathbf{K}_\lambda$, there is $N \in \mathbf{K}_\lambda$ such that $M <^* N$.
- If $M \leq^* N \leq_{\mathbf{K}} R$ all in $\mathbf{K}_\lambda$, then $M \leq^* R$.
- If $\delta < \lambda^+$ is a limit ordinal and $\{M_i : i \leq \delta+1\} \subseteq \mathbf{K}_\lambda$ is an $\leq^*$-increasing continuous chain, then there are $a \in |M_{\delta+1}| \backslash |M_\delta|$ and $\alpha < \delta$ such that $\mathbf{tp}(a/M_\delta, M_{\delta+1}) \in \mathbf{gS}^{bs}(M_\delta)$ and does not fork over M_α.

Since good λ-frames were introduced several weaker notions have been studied. In the definition below we recall all of them and write in parenthesis the paper in which they were introduced.

Definition 2.3.5.

1. ([JaSh13]) A *semi-good λ-frame* is a $good^{-(St)}\lambda$-frame with the additional property that for any $M \in \mathbf{K}_\lambda(|\mathbf{gS}^{bs}(M)| \leq \|M\|^+)$.
2. ([JaSh940]) An *almost-good λ-frame* is a $good^{-(Lc)}\lambda$-frame with the additional property that it satisfies weak local character.
3. ([Vas16c]) A *$good^{-(S)}\lambda$-frame* is a good λ-frame without symmetry.
4. ([Vas16a]) A *$good^{-}\lambda$-frame* is a $good^{-(St,S)}\lambda$-frame.

Diagram 2.1.1 shows how they compare to one another.

Before introducing the notion of a w-good frame, we will introduce a notion of weak density.

Definition 2.3.6. A $(\mathbf{K}, \downfree, \mathbf{gS}^{bs})$ pre-$[\lambda, \mu)$-frame has *weak density for basic types* when: if $M \in \mathbf{K}_\lambda$ and $M <_{\mathbf{K}} N \in \mathbf{K}_{[\lambda,\mu)}$ then there are $a \in |N| \backslash |M|$ and $M' <_{\mathbf{K}} N'$ with $M' \in \mathbf{K}_\lambda, N' \in \mathbf{K}_{[\lambda,\mu)}$ such that $a \in |N'| \backslash |M'|$, $\mathbf{tp}(a/M', N') \in \mathbf{gS}^{bs}(M')$ and $(a, M, N) \leq (a, M', N')$.

Observe that if a pre-frame has density for basic types then it has weak density for basic types. We do not know if under the other axioms of a good λ-frames the conditions are equivalent (but we suspect it is not the case).[5]

We are ready to introduce the notion of a w-good frame.

Definition 2.3.7. Let $\lambda < \mu$ where λ is an infinite cardinal and μ is an infinite cardinal or infinity. A *w-good $[\lambda, \mu)$-frame* is triple $(\mathbf{K}, \downfree, \mathbf{gS}^{bs})$ where the following properties hold:

1. $(\mathbf{K}, \downfree, \mathbf{gS}^{bs})$ is a pre-$[\lambda, \mu)$-frame
2. $\mathbf{K}_{[\lambda,\mu)}$ has amalgamation, joint embedding and no maximal models.

$(4)^-$ <u>Weak density</u>

5. <u>Existence of non-forking extension</u>
6. <u>Uniqueness</u>
9. <u>Continuity</u>

Using the notation introduced in 2.3.3, a w-good $[\lambda, \mu)$-frame is a $good^{-(St,D,S,Lc)}\lambda$-frame with the additional property that it satisfies weak density.

[5] Shelah shows in [Sh:h, §VI.7.4] that under additional hypothesis weak density implies density.

Remark 2.3.8.

- As in [Sh:h, §II.2.18] one can show that in a w-good frame transitivity of non-forking holds.
- As we can see by comparing Definition 2.3.5 and Definition 2.3.6, a w-good frame is weaker than all the notions presented in Definition 2.3.5.

It is natural to ask which of the notions introduced in Definition 2.3.5 and Definition 2.3.6 are strictly stronger. In [JaSh13, §2.2] Jarden and Shelah showed that good λ-frames are strictly stronger than semi-good λ-frames. Adapting an example of [JaSh13, §2], we show that w-good λ-frames are strictly stronger than pre-λ-frames.

Example 2.3.9. Let $L(\mathbf{K}) = \{<\}$, where $<$ is a binary relation, and $\mathbf{K} = (Mod(T_{LO}), \subseteq)$, where T_{LO} is the first-order theory of linear orders. Let $\mathfrak{s} = (\mathbf{K}, \mathbf{gS}^{na}, \mathop{\smile\!\!\!\!\!\!\;|}\,)$ where for $M_0, M_1, N \in \mathbf{K}_\lambda$: $a \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}}\, M_1$ if and only if $M_0 \subseteq M_1 \subseteq N$ and $a \in |N| \backslash |M_1|$. It is trivial to check that $\mathfrak{s}$ is a pre-λ-frame. Moreover, the uniqueness property fails so $\mathfrak{s}$ is not a w-good λ-frame.

The following example shows that *good*$^-\lambda$-frames are strictly stronger than w-good λ-frames. This example appears in a different context in other papers ([Sh:h, II.6.4], [Adl09, 6.6] and [BGKV16, 4.15]).

Example 2.3.10. Let $L(\mathbf{K}) = \{E\}$, where E is a binary relation, and $\mathbf{K} = (Mod(T_{ind}), \preceq)$, where T_{ind} is the first-order theory of the random graph. Let $\mathfrak{s} = (\mathbf{K}, \mathbf{gS}^{na}, \mathop{\smile\!\!\!\!\!\!\;|}\,)$ where for $M_0, M_1, N \in \mathbf{K}_\lambda$: $a \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}}\, M_1$ if and only if $M_0 \preceq M_1 \preceq N$, $a \in |N| \backslash |M_1|$ and there are no edges between a and $|M_1| \backslash |M_0|$.

It is easy to check that $\mathfrak{s}$ is a w-good λ-frame, we show that $\mathfrak{s}$ does not have local character. Build $\{M_i : i < \omega\} \subseteq \mathbf{K}_\lambda$ strictly increasing and continuous. Let $M_\omega = \bigcup_{i<\omega} M_i$ and let $N \in \mathbf{K}_\lambda$ such that $M_\omega \preceq N$ and there is $a \in |N| \backslash |M_\omega|$ such that for every $b \in M_\omega$ there is an edge between a and b. Observe that $tp(a/M_\omega, N)$ does not fork over M_ω, but for any $i < \omega$ $tp(a/M_\omega, N)$ forks over M_i.

Therefore, $\mathfrak{s}$ is a w-good λ-frame and it is not a *good*$^-\lambda$-frame.

Adapting another example of [JaSh13, §2], we show that semi-good λ-frames and *good*$^{-(S)}\lambda$-frames are strictly stronger than *good*$^-\lambda$-frames. Moreover, the example also exhibits that almost-good λ-frames are strictly stronger than w-good λ-frames.

Example 2.3.11. Suppose that $2^\lambda \geq \lambda^{++}$. Let $L(\mathbf{K}) = \{R_\alpha : \alpha < \lambda\}$, where each R_α is a unary predicate, and $\mathbf{K} = (L(\mathbf{K})\text{-structures }, \subseteq)$. Let $\mathfrak{s} = (\mathbf{K}, \mathbf{gS}^{na}, \mathop{\smile\!\!\!\!\!\!\;|}\,)$ where for $M_0, M_1, N \in \mathbf{K}_\lambda$: $a \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\;|}}}\, M_1$ if and only if $M_0 \subseteq M_1 \subseteq N$ and $a \in |N| \backslash |M_1|$.

$\mathbf{K}_\lambda$ has amalgamation, joint embedding and no maximal models. Moreover, for every $M_0, M_1, N_0, N_1 \in \mathbf{K}_\lambda$, $a_0 \in |N_0| \backslash |M_1|$ and $a_1 \in |N_1| \backslash |M_1|$ it follows that:

$$\mathbf{tp}(a_0/M_0, N_0) = \mathbf{tp}(a_1/M_1, N_1) \text{ if and only if } \{\alpha < \lambda : a_0 \in R_\alpha^{N_0}\} = \{\alpha < \lambda : a_1 \in R_\alpha^{N_1}\}.$$

Using this property it is easy to show that all the conditions of a *good*$^-\lambda$-frame are satisfied and that for any $M \in \mathbf{K}_\lambda(|\mathbf{gS}^{bs}(M)| = 2^\lambda)$. Since $2^\lambda \geq \lambda^{++}$ it follows that $\mathfrak{s}$ is neither a semi-good λ-frame or a *good*$^{-(S)}\lambda$-frame. Observe that the hypothesis that $2^\lambda \geq \lambda^{++}$ is only used to show that $\mathfrak{s}$ is not a semi-good λ-frame.

For the moreover part, it is clear that $\mathfrak{s}$ is a w-good λ-frames, but not an almost-good λ-frame.

Below we revise the diagram of the introduction, we write "s" above those arrows for which it is known that the source frame is strictly stronger than the target frame and we write "s*" above those arrows for which it is known that the source frame is strictly stronger than the target frame but under some set-theoretic hypothesis.

$$\begin{tikzcd}
 & \boxed{\text{semi-good } \lambda\text{-frame}} \arrow[r, "\text{s*}"] & \boxed{\textit{good}^-\lambda\text{-frame}} \arrow[d, "\text{s}"] & \\
\boxed{\text{good } \lambda\text{-frame}} \arrow[ur, "\text{s}"] \arrow[r] \arrow[dr] & \boxed{\textit{good}^{-(S)}\lambda\text{-frame}} \arrow[ur, "\text{s}"] & \boxed{\text{w-good } \lambda\text{-frame}} \arrow[r, "\text{s}"] & \boxed{\text{pre-}\lambda\text{-frame}} \\
 & \boxed{\text{almost-good } \lambda\text{-frame}} \arrow[ur, "\text{s}"] & &
\end{tikzcd} \tag{2.3.1}$$

Question 2.3.12. Are any of the notions introduced above the same? Are all the notions introduced above the same under some additional hypothesis on $\mathbf{K}$?

Question 2.3.13. Let T be a first-order theory. It is easy to show that if T is λ-stable then T has a w-good λ-frame. Example 2.3.10 shows that simple theories might have a w-good λ-frame. So the question is: under what hypothesis does T have a w-good λ-frame?

Another interesting question in this neighborhood is the following: is there a w-good λ-frame on a λ-stable theory T different from first-order non-forking?

2.3.2 Inside a w-good $[\lambda, \mu)$-frame

Let us recall the definition of a coherent sequence of types. This were already implicit in the work of Grossberg and VanDieren [GrVan06b], but did not appear in print until [Bal09].

Definition 2.3.14. Given $\{M_i : i < \delta\}$ an increasing continuous chain and $\{p_i \in \mathbf{gS}^{na}(M_i) : i < \delta\}$ an increasing sequence of types, the sequence is a *coherent sequence of types* if and only if there are $\{(a_i, N_i) : i < \delta\}$ and $\{f_{j,i} : j < i < \delta\}$ such that:

1. $f_{j,i} : N_j \to N_i$.
2. For all $k < j < i$, we have $f_{k,i} = f_{j,i} \circ f_{k,j}$.
3. $\mathbf{tp}(a_i/M_i, N_i) = p_i$.
4. $f_{j,i} \restriction_{M_j} = \mathrm{id}_{M_j}$.
5. $f_{j,i}(a_j) = a_i$.

The following lemma is straightforward but due to its importance in what follows we will sketch the proof.

Lemma 2.3.15. *If $\{p_i \in \mathbf{gS}^{na}(M_i) : i < \delta\}$ is a coherent sequence of types, then there is $p \in \mathbf{gS}^{na}(M_\delta)$ upper bound for the sequence of types, i.e., for every $i < \delta(p$ extends $p_i)$.*

Proof. Let $(N, \{f_i : N_i \to N : i < \delta\})$ be the direct limit of the sequence such that $M_\delta \leq_{\mathbf{K}} N$ and $f_i \restriction_{M_i} = \mathrm{id}_{M_i}$. Let $a := f_0(a_0)$ and $p := \mathbf{tp}(a/M_\delta, N)$. Observe that $\mathbf{tp}(a/M_\delta, N) \in \mathbf{gS}^{na}(M_\delta)$, if $a \in M_\delta$ then there is $i < \delta$ such that $a \in M_i$, then using that $f_i \circ f_{0,i} = f_0$, $f_{0,i}(a_0) = a_i$ and $f_i \restriction_{M_i} = \mathrm{id}_{M_i}$ it follows that $a_i \in M_i$, which contradicts the fact that p_i is nonalgebraic. It is easy to show that $\mathbf{tp}(a/M_\delta, N)$ is an upper bound for the sequence of types. $\square$

Lemma 2.3.16. *Let $\mathfrak{s}$ be a w-good $[\lambda, \mu)$-frame. Let $\{M_i \in \mathbf{K}_{[\lambda,\mu)} : i < \delta\}$ an increasing continuous chain such that $\delta \leq \mu$. If $\{p_i \in \mathbf{gS}^{bs}(M_i) : i < \delta\}$ is an increasing sequence of types such that p_i does not fork over M_0 for every $i < \delta$, then $\{p_i : i < \delta\}$ is a coherent sequence of types. Moreover, there is $p_\delta \in \mathbf{gS}^{na}(M_\delta)$ extending all the p_i and if $\delta < \mu$ then $p_\delta \in \mathbf{gS}^{bs}(M_\delta)$ does not fork over M_0.*

Proof. The exact same proof of [Bon14, 5.2] works, since the only properties of good frames that are used in [Bon14, 5.2] are amalgamation, uniqueness and continuity. $\square$

Lemma 2.3.17. *If $\mathfrak{s}$ is a w-good $[\lambda, \mu)$-frame without the assumption that $\mathbf{K}_{(\lambda,\mu)}$ has no maximal models, then $\mathbf{K}_{[\lambda,\mu]}$ has no maximal models.*

Proof. We show that for every $\kappa \in [\lambda, \mu]$ $\mathbf{K}_\kappa$ has no maximal models. The case when $\kappa = \lambda$ follows directly from the definition of a w-good $[\lambda, \mu)$-frame and the assumption. So suppose $\kappa \in (\lambda, \mu]$ and $M \in \mathbf{K}_\kappa$ is a maximal model. Let $R <_{\mathbf{K}} S \leq_{\mathbf{K}} M$ such that $R, S \in \mathbf{K}_\lambda$. By weak density, $[\lambda, \mu)$-amalgamation property and using the fact that M is maximal, there are $R' <_{\mathbf{K}} S' \leq_{\mathbf{K}} M$ both in $\mathbf{K}_\lambda$ and $a \in |S| \backslash |R|$ such that $\mathbf{tp}(a/R', S') \in \mathbf{gS}^{bs}(R')$.

We build $\{M_i : i < \kappa\} \subseteq \mathbf{K}_{<\kappa}$ an increasing and continuous resolution of M such that $M_0 := R'$. We build $\{p_i : i < \kappa\}$ such that:

1. $p_0 = \mathbf{tp}(a/R', S')$.
2. For all $i < \kappa$, $p_i \in \mathbf{gS}^{bs}(M_i)$.
3. For all $i < \kappa$, p_i does not fork over M_0.
4. If $j < i$, then $p_j \leq p_i$.

Enough: By Lemma 2.3.16 there is $p \in \mathbf{gS}^{na}(\bigcup_{i<\kappa} M_i)$. Observe that $\bigcup_{i<\kappa} M_i = M$ and since the type is nonalgebraic there is $N \in \mathbf{K}_\kappa$ and $a \in |N| \backslash |M|$ such that $p = \mathbf{tp}(a/M, N)$. Hence $M <_{\mathbf{K}} N$, this contradicts the fact that M is maximal.

Construction: The base step is (1) and if i is limit we apply Lemma 2.3.16. So the only interesting case is when $i = j + 1$. By construction we have $p_j \in \mathbf{gS}^{bs}(M_j)$ that does not fork over M_0. Since $M_j <_{\mathbf{K}} M_{j+1}$ and both models are in $\mathbf{K}_{[\lambda,\kappa)}$, by the extension property there is $p_{j+1} \in \mathbf{gS}^{bs}(M_{j+1})$ such that $p_j \leq p_{j+1}$ and p_{j+1} does not fork over M_j. Then by transitivity p_{j+1} does not fork over M_0

□

Theorem 2.3.18. *If $\mathfrak{s}$ is a w-good $[\lambda, \mu)$-frame, then $\mathbf{K}_\kappa \neq \emptyset$ for all $\kappa \in [\lambda, \mu^+]$.*

Proof. It follows from the fact that $\mathbf{K}_\lambda \neq \emptyset$ and Lemma 2.3.17. □

The following corollary has a long history. First, Shelah proved it for good λ-frames in [Sh:h, §II.4.13], then Jarden and Shelah proved it for *good*$^{-(St,Lc)}$ λ-frames[6] in [JaSh13, 3.1.9] . Later Vasey proved it for *good*$^{-(S)}$ λ-frames in [Vas16a, 8.9]. Below we prove it for w-good λ-frames.

Corollary 2.3.19. *If $\mathfrak{s}$ is a w-good λ-frame, then $\mathbf{K}_{\lambda^+} \neq \emptyset$ and $\mathbf{K}_{\lambda^{++}} \neq \emptyset$.*

Proof. Observe that $\mathfrak{s}$ is a w-good $[\lambda, \lambda^+)$-frame and use Theorem 2.3.18. □

[6] It is clear that a *good*$^{-(St,Lc)}$ λ-frame is stronger than a w-good λ-frame. It is suspected that symmetry does not follow from the other axioms of a good λ-frame, so we suspect that *good*$^{-(St,Lc)}$ λ-frames are strictly stronger than w-good λ-frames. The reason we do not mention *good*$^{-(St,Lc)}$ λ-frames until this point is because they are simply a technical tool developed in [JaSh13] to encompass both semi-good frames and almost-good frames.

2.3.3 Extending w-good λ-frames

Similarly to [Bon14][7], one can show that under the amalgamation property and tameness one can extend a w-good λ-frame to a w-good $[\lambda, \infty)$-frame. We will only sketch the proof since all the proofs of [Bon14] work for our weaker setting, except the proof of weak density and of no maximal models.

The following definition is a local version of $(\geq \mathfrak{s})$ which appears in [Sh:h, §II.2.4] for good λ-frames.

Definition 2.3.20. Let $LS(\mathbf{K}) \leq \lambda < \mu$ where λ is an infinite cardinal and μ is an infinite cardinal or infinity. Given $\mathfrak{s}$ a w-good λ-frame we define:

- $K^{3,bs}_{\mathfrak{s}_{[\lambda,\mu)}} = \{(a, M, N) \in K^3_{[\lambda,\mu)}$: there is $M' \leq_{\mathbf{K}} M$ in $\mathbf{K}_\lambda$ such that: if $M'' \in \mathbf{K}_\lambda$ with $M' \leq_{\mathbf{K}} M'' \leq_{\mathbf{K}} M$, then $\mathbf{tp}(a/M'', N)$ does not fork over $M'\}$.
- $\mathbf{gS}^{bs}_{\mathfrak{s}_{[\lambda,\mu)}} = \{p \in \mathbf{gS}(M) :$ for some/every $(a, M, N) \in K^{3,bs}_{\mathfrak{s}_{[\lambda,\mu)}}$, $p = \mathbf{tp}(a/M, N)\}$.
- Given $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$ all in $\mathbf{K}_{[\lambda,\mu)}$ and $a \in |N| \backslash |M_1|$: $a \underset{M_0}{\overset{N}{\perp}} M_1$ if and only if there is $R_0 \leq_{\mathbf{K}} M_0$ in $\mathbf{K}_\lambda$ such that for any $R_1, S \in \mathbf{K}_\lambda$ with $R_0 \leq_{\mathbf{K}} R_1 \leq_{\mathbf{K}} M_1$ and $R_1 \cup \{a\} \subseteq S \leq N$ it holds that $a \underset{R_0}{\overset{S}{\perp}} R_1$.

Define $\mathfrak{s}_{[\lambda,\mu)} = (\mathbf{K}, \mathbf{gS}^{bs}_{\mathfrak{s}_{[\lambda,\mu)}}, \underset{\mathfrak{s}_{[\lambda,\mu)}}{\perp})$.

The following is already proven for good λ-frames in [Bon14].

Lemma 2.3.21. *Assume $\mathbf{K}$ is an AEC with the $[\lambda, \mu^+)$-amalgamation property. If $\mathfrak{s}$ is a w-good λ-frame and $\mathbf{K}$ is $(\lambda, \leq \mu)$-tame, then $\mathfrak{s}_{[\lambda,\mu^+)}$ is a pre-$[\lambda, \mu^+)$-frame that satisfies the amalgamation property, the joint embedding property, existence of non-forking extension, uniqueness and continuity.*

Proof. It is trivial to show that $\mathfrak{s}_{[\lambda,\mu^+)}$ is a pre-$[\lambda, \mu^+)$-frame. We have the amalgamation property by hypothesis and the joint embedding property follows from the amalgamation property and the fact that we have the joint embedding property in $\mathbf{K}_\lambda$. The existence of non-forking extension is [Bon14, 5.3], the uniqueness property is [Bon14, 3.2] and continuity is [Sh:h, §II 2.11(6)]. $\square$

Therefore we only need to prove that weak density and no maximal models transfer up.

[7] [Bon14] uses tameness for 2-types to extend symmetry, in [BoVas17a, 6.9] it was established that tameness for 1-types is sufficient. Observe that in this paper the results of [Bon14] are enough since symmetry is not assumed.

Lemma 2.3.22. *Assume* $\mathbf{K}$ *is an AEC with the* $[\lambda, \mu^+)$*-amalgamation property. If* $\mathfrak{s}$ *is a pre-*λ*-frame that has weak density, then* $\mathfrak{s}_{[\lambda,\mu^+)}$ *has weak density.*

Proof. Let $M <_{\mathbf{K}} N$ such that $M \in \mathbf{K}_\lambda$. If $N \in \mathbf{K}_\lambda$ then it follows directly from the fact that $\mathfrak{s}$ satisfies the weak density property. So let us do the case when $\|N\| > \lambda$.

Apply downward Löwenheim-Skolem-Tarski axiom to get $N_0 \in \mathbf{K}_\lambda$ such that $M <_{\mathbf{K}} N_0 \leq_{\mathbf{K}} N$. By weak density in $\mathfrak{s}$ there are $a \in |N_0| \backslash |M|$ and $M' <_{\mathbf{K}} N_0'$ both in $\mathbf{K}_\lambda$ such that $a \in |N_0'| \backslash |M'|$, $\mathbf{tp}(a/M', N_0') \in \mathbf{gS}^{bs}(M')$ and $(a, M, N_0) \leq (a, M', N_0')$. By the amalgamation property there are f and $N' \in \mathbf{K}_{[\lambda,\mu]}$ such that the following diagram commutes:

$$\begin{tikzcd} N_0' \arrow[r, "f"] & N' \\ N_0 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & N \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Observe that $f(a) = a$, $a \in |N'| \backslash |f[M']|$, $(a, M, N) \leq (a, f[M'], N')$ and $\mathbf{tp}(a/f[M'], N') \in \mathbf{gS}^{bs}_{\mathfrak{s}_{[\lambda,\mu^+)}}(f[M'])$. □

The reason we can not simply quote [Bon14, 7.1] to transfer up no maximal models is because Boney's proof uses symmetry, which we are not assuming.

Lemma 2.3.23. *Assume* $\mathbf{K}$ *is an AEC with the* $[\lambda, \mu^+)$*-amalgamation property. If* $\mathfrak{s}$ *is a w-good* λ*-frame and* $\mathbf{K}$ *is* $(\lambda, \leq \mu)$*-tame, then* $\mathbf{K}_{[\lambda,\mu^+)}$ *has no maximal models.*

Proof. By Lemma 2.3.21 and Lemma 2.3.22 $\mathfrak{s}_{[\lambda,\mu^+)}$ is a w-good $[\lambda, \mu^+)$-frame without the property that $\mathbf{K}_{[\lambda,\mu^+)}$ has no maximal models. Since $\mathfrak{s}$ is a w-good λ-frame, $\mathbf{K}_\lambda$ has no maximal models. Therefore, by Lemma 2.3.17 it follows that $\mathbf{K}_{[\lambda,\mu^+)}$ has no maximal models. □

With all the work we have done, we obtain the theorem promised at the beginning of the section.

Theorem 2.3.24. *Assume* $\mathbf{K}$ *is an AEC with the* $[\lambda, \mu^+)$*-amalgamation property. If* $\mathfrak{s}$ *is a w-good* λ*-frame and* $\mathbf{K}$ *is* $(\lambda, \leq \mu)$*-tame, then* $\mathfrak{s}_{[\lambda,\mu^+)}$ *is a w-good* $[\lambda, \mu^+)$*-frame.*

Proof. Follows from Lemma 2.3.21, Lemma 2.3.22 and Lemma 2.3.23. □

In [Vas17c, 4.16] Vasey weakens the hypothesis of the above theorem for good frames from $\mathbf{K}$ has the amalgamation property for that of $\mathbf{K}$ has weak amalgamation. In the proof, it is crucial the density of basic types, therefore we do not know if one can weaken the hypothesis in the above theorem.

2.4 Applications

The following notation will be useful in this section:

Notation 2.4.1. We denote by $(*)_\lambda$ the assertion "$\mathbb{I}(\mathbf{K},\lambda) = \mathbb{I}(\mathbf{K},\lambda^+) = 1 \leq \mathbb{I}(\mathbf{K},\lambda^{++}) < 2^{\lambda^{++}}$".

In this section we will show how w-good frames can be used to prove the following:

Theorem 2.4.2. [8] *Suppose $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $2^{\lambda^+} > \lambda^{++}$. If $(*)_\lambda$ and $\mathbf{K}$ is (λ,λ^+)-tame, then $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$.*

The proof presented here follows the blueprint displayed in [Sh576], unless otherwise noted all the definitions in this section were introduced by Shelah in [Sh576]. We would like to point out that most of what we prove here is already proved by Shelah in [Sh576], but we decided to write down the proofs since some of Shelah's proofs are obscure, in particular those of Section 4.3, and they are central in the study of AECs.

The proof of Theorem 2.4.2 is done by contradiction. We will assume that $\mathbf{K}_{\lambda^{+++}} = \emptyset$ and using this property we will construct an explicit w-good λ-frame. Then using tameness together with Theorem 2.3.24 we will get a contradiction by building a model of size λ^{+++}.

2.4.1 Definition and basic properties

The next definition is crucial.

Definition 2.4.3. $(a, M_0, N_0) \in K^3_\lambda$ is *minimal* when: if $(a, M_0, N_0) \leq_{h_l} (a_l, M_1, N_1^l)$ for $l \in \{1, 2\}$ and $h_1 \restriction_{M_0} = h_2 \restriction_{M_0}$ then $\mathbf{tp}(a_1/M_1, N_1^1) = \mathbf{tp}(a_2/M_1, N_1^2)$.

A type $p \in \mathbf{gS}(M)$ is *minimal* for $M \in \mathbf{K}_\lambda$, if for some a and $N \in \mathbf{K}_\lambda$ we have that $(a, M, N) \in K^3_\lambda$ is minimal and $p = \mathbf{tp}(a/M, N)$.

With this definition we are ready to introduce our candidate for the w-good λ-frame. This frame was introduced in [Sh:h, §VI.8.3].[9]

Definition 2.4.4. We define $\mathfrak{s}_{min} = (\mathbf{K}_{min}, \underset{min}{\perp}, \mathbf{gS}^{bs}_{min})$ as follows:

- $\mathbf{K}_{min} = \mathbf{K}_\lambda$.

[8] As mentioned in the introduction, Shelah claims the same conclusion from fewer assumptions (see Fact 2.1.2 and the two paragraphs above it).

[9] In [Sh:h, §VI.8.3] Shelah shows, under the hypothesis of Fact 2.1.2, that $\mathfrak{s}_{min}$ is an almost good λ-frame. The reason we only show that $\mathfrak{s}_{min}$ is a w-good λ-frame is because by Section 3 this is enough to get a model of size λ^{+++} and because the known proofs of the other properties use the machinery of [Sh:h, §VII] which we avoid.

- $\mathbf{gS}^{bs}_{min} = \{\mathbf{tp}(a/M, N) : (a, M, N) \in K^3_\lambda \text{ minimal}\}$.
- Given $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$ and $a \in |N| \backslash |M_1|$ we define: $a \underset{M_0}{\overset{N}{\downarrow}} M_1$ if and only if $\mathbf{tp}(a/M_1, N) \restriction_{M_0}$ is minimal.

An easy consequence of Fact 2.2.3 is the following.

Remark 2.4.5. Suppose $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. Let $\mathbf{K}$ be an AEC. If $(*)_\lambda$, then $\mathbf{K}_{\{\lambda,\lambda^+\}}$ has the amalgamation property.

Therefore the theorems on this section that assume that $\mathbf{K}$ has λ or λ^+ amalgamation follow from the hypothesis of Theorem 2.4.2.

Definition 2.4.6.

1. $(a_0, M_0, N_0) \in K^3_\lambda$ has the *weak extension property* if there is $(a_1, M_1, N_1) \in K^3_\lambda$ such that $(a_0, M_0, N_0) < (a_1, M_1, N_1)$.
2. K^3_λ has *no maximal pre-type* if every $(a_0, M_0, N_0) \in K^3_\lambda$ has the weak extension property.

As one can see from the definition of minimal pre-type, a pre-type can be minimal if there is no pre-type above it, but we will show that under the hypothesis of Theorem 2.4.2 this can not happen. This appears first as [Sh576, 2.4], but a more straightforward proof is given in [Gro02, 7.11] (in [Gro02] it is assumed that the class is a PC class, but the hypothesis is not necessary).

Fact 2.4.7. Let $\mathbf{K}$ be an AEC. If $\mathbb{I}(\mathbf{K}, \lambda) = \mathbb{I}(\mathbf{K}, \lambda^+) = 1$ and $\mathbf{K}_{\lambda^{++}} \neq \emptyset$, then K^3_λ has no maximal pre-type.

Now that we have that out of the way, we will show some basic properties about minimal pre-types. The following is [Sh576, 2.6]. Although the proofs are easy, we sketch them since they don't appear on [Sh576] and this facts are used throughout the paper.

Lemma 2.4.8.

1. *If $(a, M_0, N_0) \leq (a, M_1, N_1) \in K^3_\lambda$ and (a, M_0, N_0) is minimal, then (a, M_1, N_1) is minimal.*
2. *(λ-amalgamation property is used) (a, M_0, N_0) is minimal if and only if the following holds: if $(a, M_0, N_0) \leq_{h_l} (a_l, M_1, N_1)$ for $l \in \{1, 2\}$ and $h_1 \restriction_{M_0} = h_2 \restriction_{M_0}$ then $\mathbf{tp}(a_1/M_1, N_1) = \mathbf{tp}(a_2/M_1, N_1)$.*

3. *(λ-amalgamation property is used) If $(a, M_0, N_0) \in K^3_\lambda$, $p = \mathbf{tp}(a/M_0, N_0)$ and p is minimal, then (a, M_0, N_0) is minimal.*

4. *(λ-amalgamation property is used) Let $M \leq_{\mathbf{K}} M' \in \mathbf{K}_\lambda$. If $p \in \mathbf{gS}(M)$ minimal and $q \in \mathbf{gS}(M')$ extending p, then q is a minimal type.*

Proof. (1), (3) and (4) are straightforward so let us sketch (2). The forward direction is trivial so let us show the backward one. Suppose $(a, M_0, N_0) \leq_{h_l} (a_l, M_1, N_1^l)$ for $l \in \{1, 2\}$ and $h_1 \restriction_{M_0} = h_2 \restriction_{M_0}$, then apply the amalgamation property to obtain N and j such that the following diagram commutes:

$$\begin{tikzcd} N_2 \arrow[r, "j"] & N \\ N_0 \arrow[u, "h_2"] \arrow[r, "h_1"] & N_1 \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Then simply apply the hypothesis to $h_1' = h_1$, $h_2' = j \circ h_2$ and N.

□

First let us show that $\mathfrak{s}_{\min}$ is a pre-λ-frame. This appears without a proof in [Sh:h, §VI.8.1(1)].

Lemma 2.4.9. *Suppose $2^\lambda < 2^{\lambda^+}$. If $\mathbf{K}$ is λ-categorical and $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathfrak{s}_{min} = (\mathbf{K}_{min}, \underset{min}{\perp}, \mathbf{gS}^{bs}_{min})$ is a pre-λ-frame.*

Proof. It is clear that (1) through (3) of the definition of pre-λ-frame are satisfied, so let us check that (4) through (6) are satisfied:

4. Invariance: It follows from the fact that minimal pre-types are closed under isomorphisms.

5. Monotonicity: It follows from Lemma 2.4.8(4).

6. Non-forking types are basic: By definition.

□

Moreover, we can show the following.

Lemma 2.4.10. *Suppose $2^\lambda < 2^{\lambda^+}$. If $\mathbf{K}$ is λ-categorical and $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathfrak{s}_{min}$ satisfies:*

2. *$\mathbf{K}_\lambda$ has amalgamation, joint embedding and no maximal models.*

6. *Uniqueness.*

9. *Continuity.*

Proof.

2. $\mathbf{K}_\lambda$ has amalgamation, joint embedding and no maximal models: The amalgamation property follows from Remark 2.4.5. Joint embedding follows from λ-categoricity and no maximal models from λ-categoricity and the fact that $\mathbf{K}_{\lambda^+} \neq \emptyset$.

6. Uniqueness: It follows from the definition of minimal type.

9. Continuity: Let $\delta < \lambda^+$, $\{M_i : i < \delta\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain, $\{p_i : i < \delta\}$ with $p_i \in \mathbf{gS}^{bs}_{min}(M_i)$ and for $i < j < \delta$ implies that $p_i = p_j \upharpoonright_{M_i}$ and $p \in \mathbf{gS}^{na}(M_\delta)$ an upper bound. Since $p \upharpoonright_{M_0} = p_0$ and p_0 is minimal by Lemma 2.4.8(4) it follows that p is minimal and hence basic.

   Moreover, if each p_i does not fork over M_0, then by definition $p \upharpoonright_{M_0}$ is minimal. Hence p does not fork over M_0.

□

Therefore to show that $\mathfrak{s}_{min}$ is a w-good λ-frame, we just need to show that it satisfies weak density and existence of non-forking extension. The proofs of these two facts are more complicated and will use all the hypothesis of Theorem 2.4.2 together with the assumption that $\mathbf{K}_{\lambda^{+++}} = \emptyset$. Before we do that there is a useful property that we get by assuming that $\mathbf{K}_{\lambda^{+++}} = \emptyset$.

Definition 2.4.11. Let $M \in \mathbf{K}_\mu$ and $LS(\mathbf{K}) \leq \lambda \leq \mu$ infinite cardinals. M is *universal above* λ if and only if for all $N_0, N_1 \in \mathbf{K}_{[\lambda,\mu]}$ such that $N_1 \geq_{\mathbf{K}} N_0 \leq_{\mathbf{K}} M$ there is $f : N_1 \underset{N_0}{\longrightarrow} M$.

The following is similar to [Sh576, 2.2], but instead of working in λ^{++} we work in λ^{+++}.

Lemma 2.4.12. *If $\mathbf{K}_{\{\lambda,\lambda^+\}}$ has the amalgamation property, $\mathbf{K}_{\lambda^{++}} \neq \emptyset$ and $\mathbf{K}_{\lambda^{+++}} = \emptyset$, then there is $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ universal above λ. Moreover if $\mathbb{I}(\mathbf{K}, \lambda) = \mathbb{I}(\mathbf{K}, \lambda^+) = 1$, for each $N \in \mathbf{K}_{\{\lambda,\lambda^+\}}$ there is $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ universal above λ such that $N \leq_{\mathbf{K}} \mathcal{C}$.*

Proof. Since $\mathbf{K}_{\lambda^{+++}} = \emptyset$ there is $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ maximal. We claim that $\mathcal{C}$ is universal above λ. Let $N_0 \leq_{\mathbf{K}} N_1 \in \mathbf{K}_{\{\lambda,\lambda^+\}}$, then since $\mathbf{K}_{\{\lambda,\lambda^+\}}$ has the amalgamation property, there are $M \in \mathbf{K}_{\lambda^{++}}$ and f such that the following diagram commutes:

$$\begin{tikzcd} N_1 \arrow[r, "f"] & M \\ N_0 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & \mathcal{C} \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Since $\mathcal{C}$ is maximal, we have that $\mathcal{C} = M$. Hence $f : N_1 \underset{N_0}{\longrightarrow} \mathcal{C}$. The moreover part follows from λ-categoricity or λ^+-categoricity copying $\mathcal{C}$.

$\square$

2.4.2 Weak density

The only place where we use the extra cardinal arithmetic hypothesis that $2^{\lambda^+} > \lambda^{++}$ is to prove the following lemma, since we are already assuming that $2^\lambda < 2^{\lambda^+}$ this is a weak hypothesis.

The lemma below is [Sh576, 2.7]. Shelah's proof and our proof are very similar, but we have decided to include it for the sake of completeness.

Lemma 2.4.13. *Suppose $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $2^{\lambda^+} > \lambda^{++}$. If $(*)_\lambda$ and $\mathbf{K}_{\lambda^{+++}} = \emptyset$, then minimal pre-types are dense in K^3_λ, i.e., for every pre-type there is a minimal one above it.*

Proof. We do the proof by contradiction. Let $(a, M, N) \in K^3_\lambda$ with no minimal pre-type above it. We will build $\{(a_\eta, M_\eta, N_\eta) : \eta \in 2^{<\lambda^+}\}$ and $\{h_{\eta,\nu} : \eta < \nu \text{ with } \eta, \nu \in 2^{<\lambda^+}\}$ by induction such that:

1. $(a_{<>}, M_{<>}, N_{<>}) := (a, M, N)$.
2. $(a_\eta, M_\eta, N_\eta) \in K^3_\lambda$ for all $\eta \in 2^{<\lambda^+}$.
3. If $\eta < \nu$, then $(a_\eta, M_\eta, N_\eta) \leq_{h_{\eta,\nu}} (a_\nu, M_\nu, N_\nu)$.
4. If $\eta < \nu < \rho$, then $h_{\eta,\rho} = h_{\nu,\rho} \circ h_{\eta,\nu}$.
5. $M_{\eta^\wedge 0} = M_{\eta^\wedge 1}, N_{\eta^\wedge 0} = N_{\eta^\wedge 1}, h_{\eta,\eta^\wedge 0} \restriction_{M_\eta} = h_{\eta,\eta^\wedge 1} \restriction_{M_\eta}$ for all $\eta \in 2^{<\lambda^+}$.
6. $\mathbf{tp}(a_{\eta^\wedge 0}/M_{\eta^\wedge 0}, N_{\eta^\wedge 0}) \neq \mathbf{tp}(a_{\eta^\wedge 1}/M_{\eta^\wedge 1}, N_{\eta^\wedge 1})$.
7. If $\eta \in 2^\delta$ and $\delta < \lambda^+$ limit then $(M_\eta, \{h_{\eta\restriction_\alpha,\eta}\}_{\alpha<\delta})$, $(N_\eta, \{h_{\eta\restriction_\alpha,\eta}\}_{\alpha<\delta})$ are the direct limits of $(\{M_{\eta\restriction_\alpha} : \alpha < \delta\}, \{h_{\eta\restriction_\alpha,\eta\restriction_\beta} : \alpha < \beta < \delta\})$ and $(\{N_{\eta\restriction_\alpha} : \alpha < \delta\}, \{h_{\eta\restriction_\alpha,\eta\restriction_\beta} : \alpha < \beta < \delta\})$ respectively where $a_\eta = h_{\eta\restriction_1,\eta}(a)$.

<u>Construction:</u> In the base step apply (1). On limits take the direct limits, so the only interesting case is when $\alpha = \beta + 1$. By construction we are given (a_η, M_η, N_η), since $(a_{<>}, M_{<>}, N_{<>}) \leq_{h_{<>,\eta}} (a_\eta, M_\eta, N_\eta)$ it follows that (a_η, M_η, N_η) is not minimal. Applying Lemma 2.4.8(2) we are done.

<u>Enough:</u> By Lemma 2.4.12 there is $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ universal above λ. We build $\{g_\eta : M_\eta \to \mathcal{C} : \eta \in 2^{<\lambda^+}\}$ by induction such that:

1. For every $\nu < \eta$, $g_\nu \circ h_{\eta,\nu} = g_\eta$.

2. $g_{\eta^\wedge 0} = g_{\eta^\wedge 1}$.

<u>Construction:</u> Base: Since $\mathbf{K}$ is λ-categorical there is $g_{<>} : M_{<>} \to \mathcal{C}$.

Induction step: If α is limit using that M_η is a direct limit and the fact that we are constructing a cocone we obtain $g_\eta : M_\eta \to \mathcal{C}$.

If $\alpha = \beta + 1$. Suppose we have $g_\eta : M_\eta \to \mathcal{C}$. By the first construction we have $h_{\eta,\eta^\wedge 0}[M_\eta] \leq_{\mathbf{K}} M_{\eta^\wedge 0}$, copying back the structure we build g and $M'_{\eta^\wedge 0}$ such that:

$$\begin{tikzcd}
M'_{\eta^\wedge 0} \arrow[r, "\cong_g"] & M_{\eta^\wedge 0} \\
M_\eta \arrow[u, "\mathrm{id}"] \arrow[r, "\cong_{h_{\eta,\eta^\wedge 0}}"] & h_{\eta,\eta^\wedge 0}[M_\eta] \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Then copying forward the structure with respect to g_η we build h and $M''_{\eta^\wedge 0}$ such that:

$$\begin{tikzcd}
M'_{\eta^\wedge 0} \arrow[r, "\cong_h"] & M''_{\eta^\wedge 0} \\
M_\eta \arrow[u, "\mathrm{id}"] \arrow[r, "\cong_{g_\eta}"] & g_\eta[M_\eta] \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Then using the universality of $\mathcal{C}$ we get $j : M''_{\eta^\wedge 0} \xrightarrow[g_\eta[M_\eta]]{} \mathcal{C}$. So let $g_{\eta^\wedge 0} := j \circ h \circ g^{-1}$ and $g_{\eta^\wedge 1} := j \circ h \circ g^{-1}$. Since $M_{\eta^\wedge 0} = M_{\eta^\wedge 1}$ it is well-defined.

<u>Enough:</u> For each $\eta \in 2^{\lambda^+}$ let $((a_\eta, M_\eta, N_\eta), \{h_{\nu,\eta} : \nu < \eta\})$ be the direct limit of $(\{M_{\eta\restriction_\alpha} : \alpha < \lambda^+\}, \{h_{\eta\restriction_\alpha,\eta\restriction_\beta} : \alpha < \beta < \lambda^+\})$ and $(\{N_{\eta\restriction_\alpha} : \alpha < \lambda^+\}, \{h_{\eta\restriction_\alpha,\eta\restriction_\beta} : \alpha < \beta < \lambda^+\})$.

By the construction of $\{g_\nu : \nu < \eta\}$ and the definition of direct limits there is $f_\eta : M_\eta \to \mathcal{C}$ such that for any $\nu < \eta (f_\eta \circ h_{\nu,\eta} = g_\nu)$. Using that $\mathcal{C}$ is universal above λ there is $f'_\eta : N_\eta \to \mathcal{C}$ such that $f_\eta \subseteq f'_\eta$.

Observe that for every $\eta \in 2^{\lambda^+}$ we have that $f'_\eta(a_\eta) \in \mathcal{C}$. Since $\|\mathcal{C}\| = \lambda^{++}$ and $2^{\lambda^+} > \lambda^{++}$ we have $\eta \neq \nu \in 2^{\lambda^+}$ such that $f'_\eta(a_\eta) = f'_\nu(a_\nu)$. Let $\alpha < \lambda^+$ least such that $\eta \restriction_\alpha = \nu \restriction_\alpha$ and $\eta(\alpha) \neq \nu(\alpha)$, we may assume without loss of generality that $\eta(\alpha) = 0$ and $\nu(\alpha) = 1$.

$\boxed{\text{Claim}}$ $\mathbf{tp}(a_{\eta\restriction_\alpha^\wedge 0}/M_{\eta\restriction_\alpha^\wedge 0}, N_{\eta\restriction_\alpha^\wedge 0}) = \mathbf{tp}(a_{\eta\restriction_\alpha^\wedge 1}/M_{\eta\restriction_\alpha^\wedge 1}, N_{\eta\restriction_\alpha^\wedge 1})$.

Observe that the following diagram commutes:

$$\begin{tikzcd}
N_{\eta\restriction_\alpha^\wedge 0} \arrow[r, "f'_\eta \circ h_{\eta\restriction_\alpha^\wedge 0,\eta}"] & \mathcal{C} \\
M_{\eta\restriction_\alpha^\wedge 0} \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & N_{\eta\restriction_\alpha^\wedge 1} \arrow[u, "f'_\nu \circ h_{\eta\restriction_\alpha^\wedge 1,\nu}"']
\end{tikzcd}$$

Moreover, since $f'_\eta(a_\eta) = f'_\nu(a_\nu)$ we have that $f'_\eta \circ h_{\eta\restriction_\alpha^\wedge 0,\eta}(a_{\eta\restriction_\alpha^\wedge 0}) = f'_\nu \circ h_{\eta\restriction_\alpha^\wedge 1,\nu}(a_{\eta\restriction_\alpha^\wedge 1})$.
$\dagger_{\text{Claim}}$

Finally observe that this contradicts (6) of the first construction.

□

From the above lemma, the assertion below follows trivially.

Lemma 2.4.14. *Suppose $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $2^{\lambda^+} > \lambda^{++}$. If $(*)_\lambda$ and $\mathbf{K}_{\lambda^{+++}} = \emptyset$, then $\mathfrak{s}_{min}$ has weak density.*

Proof. Let $M <_{\mathbf{K}} N$ both in $\mathbf{K}_\lambda$, then pick $a \in |N| \backslash |M|$. By the previous theorem there is $(a, M', N') \in K^3_\lambda$ such that $(a, M, N) \leq (a, M', N')$ and (a, M', N') is minimal. Hence $\mathbf{tp}(a/M', N') \in \mathbf{gS}^{bs}_{min}(M')$. □

2.4.3 Existence of non-forking extension

Fact 2.4.7 asserts that K^3_λ has the weak extension property, in this section we will deal with the extension property.

Definition 2.4.15.

- $(a_0, M_0, N_0) \in K^3_\lambda$ has the *extension property* if given $M_1 \in \mathbf{K}_\lambda$ and $f : M_0 \to M_1$, there are $N_1 \in \mathbf{K}_\lambda$ and $g : N_0 \to N_1$ such that $(a_0, M_0, N_0) \leq_g (g(a_0), M_1, N_1)$ and $g \supseteq f$.
- $p \in \mathbf{gS}^{na}(M_0)$ has the *extension property* if given $M_1 \in \mathbf{K}_\lambda$ such that $M_0 \leq_{\mathbf{K}} M_1$ there is $q \in \mathbf{gS}^{na}(M_1)$ extending p.

Remark 2.4.16. p has the extension property if and only if there is $(a, M, N) \in K^3_\lambda$ such that $p = \mathbf{tp}(a/M, N)$ and (a, M, N) has the extension property.

The following fact is [Sh576, 2.11], to show it Shelah used the λ-amalgamation property.

Fact 2.4.17. If $(a, M_0, N_0) \leq (a, M_1, N_1) \in K^3_\lambda$ and (a, M_1, N_1) has the extension property, then (a, M_0, N_0) has the extension property.

The proof of the following lemma is similar to [Sh576, 2.9], but our proof is shorter since we assume λ^+-categoricity instead of $1 \leq \mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$ and we assume that $\mathbf{K}_{\lambda^{+++}} = \emptyset$.

Lemma 2.4.18. *Assume $\mathbf{K}_{\{\lambda,\lambda^+\}}$ has the amalgamation property, $\mathbf{K}$ is λ^+-categorical and $\mathbf{K}_{\lambda^{+++}} = \emptyset$. If $(a, M_0, N_0) \in K^3_\lambda$, $M_0 \leq_{\mathbf{K}} R$ and $|\{c \in R : c \text{ realizes } \mathbf{tp}(a/M_0, N_0)\}| \geq \lambda^+$, then (a, M_0, N_0) has the extension property.*

Proof. We may assume $R \in \mathbf{K}_{\lambda^+}$ and by Lemma 2.4.12 there is $\mathcal{C} \geq_{\mathbf{K}} R$ universal above λ. Let $f : M_0 \to M_1$, we may assume that $f = \mathrm{id}_{M_0}$. By universality there is $h : M_1 \underset{M_0}{\longrightarrow} \mathcal{C}$. Since $\|h[M_1]\| = \lambda$ then there are $c \in |R|\backslash|h[M_1]|$ and $R' \leq_{\mathbf{K}} R$ such that $(a, M_0, N_0)E_{at}(c, M_0, R')$. Then by definitions of E_{at} and universality of $\mathcal{C}$, there is $R'' \leq_{\mathbf{K}} \mathcal{C}$ and $g : N_0 \underset{M_0}{\longrightarrow} R''$ such that $g(a) = c$.

Applying downward Löwenheim-Skolem-Tarski axiom to $h[M_1] \cup R''$ inside $\mathcal{C}$ we get $S \leq_{\mathbf{K}} \mathcal{C}$. Let $S^* \geq_{\mathbf{K}} M_1$ and $d : S^* \cong S$ such that $h \subseteq d$. Since $c \notin h[M_1]$ it follows that $(d^{-1}(c), M_1, S^*) \in K^3_\lambda$ and one can show that $(a, M_0, N_0) \leq_{d^{-1}\circ g} (d^{-1}(c), M_1, S^*)$ and $d^{-1} \circ g \supseteq \mathrm{id}_{M_0}$.

□

The next step is to prove the extension property for minimal types, for that we use weak diamond principles. Weak diamonds were introduced (for $\lambda = \aleph_0$) by Devlin and Shelah in [DeSh78].

Definition 2.4.19. Let $S \subseteq \lambda^+$ be a stationary set. $\Phi^2_{\lambda^+}(S)$ holds if and only if $\forall F : (2^\lambda)^{<\lambda^+} \to 2$ $\exists g : \lambda^+ \to 2$ such that $\forall f : \lambda^+ \to 2^\lambda$ the set $\{\alpha \in S : F(f \restriction_\alpha) = g(\alpha)\}$ is stationary.

The following facts will be used in the proof of Lemma 2.4.21 and a proof of them can be found in [Gro2X, §15].

Fact 2.4.20.

1. $2^\lambda < 2^{\lambda^+}$ if and only if $\Phi^2_{\lambda^+}(\lambda^+)$ holds.
2. $\Phi^2_{\lambda^+}(S)$ holds for a stationary set $S \subseteq \lambda^+$ if and only if $\forall F : (2\times 2\times \lambda^+)^{<\lambda^+} \to 2$ $\exists g : \lambda^+ \to 2$ such that $\forall \eta \in 2^{\lambda^+} \forall \nu \in 2^{\lambda^+} \forall h : \lambda^+ \to \lambda^+$ the set $\{\alpha \in S : F(\eta \restriction_\alpha, \nu \restriction_\alpha, h \restriction_\alpha) = g(\alpha)\}$ is stationary.
3. If $\Phi^2_{\lambda^+}(\lambda^+)$ holds, then there exists $\{S_i \subseteq \lambda^+ : i < \lambda^+\}$ pairwise disjoint stationary sets such that $\Phi^2_{\lambda^+}(S_i)$ for each $i < \lambda^+$.

The lemma below is presented precisely in the way it will be used in the proof of Theorem 2.4.24. It is similar to [Sh576, 1.6(1)], but our assumptions and conclusions are weaker.

Lemma 2.4.21. *Suppose $2^{\lambda} < 2^{\lambda^+}$. Let $\{M_\eta : \eta \in 2^{\lambda^+}\}$ such that for each $\eta \in 2^{\lambda^+}$:*

1. *$\{M_{\eta\restriction_\alpha} : \alpha < \lambda^+\}$ strictly increasing and continuous.*
2. *For all $\alpha < \lambda^+ (M_{\eta\restriction_\alpha} \in \mathbf{K}_\lambda)$.*

If for every $\eta \in 2^{\lambda^+}$ and $\alpha < \lambda^+$ $M_{\eta\restriction_\alpha^\wedge 0}$ can not be embedded to M_ν over $M_{\eta\restriction_\alpha}$ when $\eta\restriction_\alpha^\wedge 1 < \nu$ and $\nu \in 2^{<\lambda^+}$, then $\mathbf{K}$ is not λ^+-categorical.

Proof. We may assume that for all $\nu \in 2^{<\lambda^+}$ $(|M_\nu| = \gamma_\eta \in \lambda^+)$, for every $\eta \in 2^{\lambda^+}(\{\gamma_{\eta\restriction_\alpha} : \alpha < \lambda^+\}$ is continuous) and in that case $\forall \eta \in 2^{\lambda^+}(|M_\eta| = \lambda^+)$.

For each $\delta \in \lambda^+$, $\eta \in 2^\delta$, $\nu \in 2^\delta$ and $h : \delta \to \delta$ define:

$$F(\eta, \nu, h) = \begin{cases} 1 & |M_\eta| = |M_\nu| = \delta \text{ and } h : M_\eta \to M_\nu \text{ can be extended to an isomorphism from} \\ & \quad M_{\eta_0} \text{ to } M_{\bar{0}} \text{ where } \eta^\wedge 0 < \eta_0 \text{ and } \nu < \bar{0} \\ 0 & \text{otherwise} \end{cases}$$

Let $\{S_i \subseteq \lambda^+ : i < \lambda^+\}$ pairwise disjoint stationary sets such that $\Phi^2_{\lambda^+}(S_i)$ holds for each $i < \lambda^+$, they exist by the previous fact.

By $\Phi^2_{\lambda^+}(S_i)$ for all $i < \lambda^+$ let $g_i : \lambda^+ \to 2$ such that for any $\eta, \nu \in 2^{\lambda^+}$ and $h : \lambda^+ \to \lambda^+$ the following set is stationary:

$$S_i^* = \{\delta \in S_i : F(\eta\restriction_\delta, \nu\restriction_\delta, h\restriction_\delta) = g_i(\delta)\}$$

Now, given $X \subseteq \lambda^+$ we define $\eta_X : \lambda^+ \to 2$ as follows:

$$\eta_X(\delta) = \begin{cases} g_i(\delta) & \text{if } \exists i \in X(\delta \in S_i) \\ 0 & \text{otherwise} \end{cases}$$

Observe that since $\{S_i : i < \lambda^+\}$ are pairwise disjoint, for each X $\eta_X$is well-defined.

$\boxed{\text{Claim:}}$ If $X \subseteq \lambda^+$ and $X \neq \emptyset$, then $M_{\eta_X} \ncong M_{\eta_\emptyset}$.

Suppose $h : M_{\eta_X} \cong M_{\eta_\emptyset}$. Observe that $\eta_\emptyset = \bar{0}$. Let $i \in X$ and $S_i^* = \{\delta \in S_i : F(\eta_X\restriction_\delta, \bar{0}\restriction_\delta, h\restriction_\delta) = g_i(\delta)\}$ be the stationary set obtained for $\eta_X, \bar{0}$ and h.

Let $C_{\eta_X} = \{\delta < \lambda^+ : |M_{\eta_X\restriction_\delta}| = \delta\}$, $C_{\bar{0}} = \{\delta < \lambda^+ : |M_{\bar{0}\restriction_\delta}| = \delta\}$ and $D = \{\delta < \lambda^+ : h\restriction_\delta : \delta \to \delta\}$. Since they are all clubs we can pick $\delta \in C_{\eta_X} \cap C_{\bar{0}} \cap D \cap S_i^*$. Define $\eta := \eta_X\restriction_\delta$ and $\nu = \bar{0}\restriction_\delta$. There are two cases:

1. <u>Case 1:</u> $\eta_X(\delta) = 1$. Since $\delta \in S_i^*$ we have that $g_i(\delta) = F(\eta, \nu, h\restriction_\delta)$ and since $i \in X$ we have that $\eta_X(\delta) = g_i(\delta)$. Hence $F(\eta, \nu, h\restriction_\delta) = 1$. Then by definition there is $g \supseteq h\restriction_\delta$ and $\eta_0 > \eta^\wedge 0$ such that $g : M_{\eta_0} \cong M_{\bar{0}}$.

By hypothesis $h : M_{\eta_X} \cong M_{\bar{0}}$, so consider $f := h^{-1} \circ g : M_{\eta^\wedge 0} \to M_{\eta_X}$. Since $\{M_{\eta_X \restriction_\alpha} : \alpha < \lambda^+\}$ is strictly increasing and continuous there is $\alpha < \lambda^+$ such that $f[M_{\eta^\wedge 0}] \subseteq M_{\eta_X \restriction_\alpha}$, so $f : M_{\eta^\wedge 0} \to M_{\eta_X \restriction_\alpha}$. Moreover $\eta_X \restriction_\alpha > \eta^\wedge 1$ and M_η is fixed under f; contradicting the hypothesis of the lemma.

2. Case 2: $\eta_X(\delta) = 0$. Then observe that $h \supseteq h \restriction_\delta : M_{\eta_x} \to M_{\bar{0}}$ where $\eta_X > \eta^\wedge 0$. So $F(\eta, \nu, h \restriction_\delta) = 1$. Since $\delta \in S_i^*$ it follows that $\eta_X(\delta) = 1$. A contradiction to the hypothesis of this case. $\dagger_{\text{Claim}}$

Therefore, $\mathbf{K}$ is not λ^+-categorical. □

We recall one last definition before we tackle Theorem 2.4.24.

Definition 2.4.22.

- Given $p = \mathbf{tp}(a/M, N) \in \mathbf{gS}(M)$ and $f : M \cong R$ define $f(p) := \mathbf{tp}(f'(a)/R, f'[N])$ such that $f' : N \cong f'[N]$ and $f' \supseteq f$.
- Let $p = \mathbf{tp}(a/M, N) \in \mathbf{gS}(M)$ and $R \in \mathbf{K}_\lambda$ then $S_p(R) := \{f(p) : f : M \cong R\}$.

Observe that if M and R are not isomorphic then $S_p(R) = \emptyset$, but in this paper when we refer to this notion, we will always assume categoricity in λ. Hence it will always be not empty. We will use the following lemma.

Lemma 2.4.23. *Let $p = \mathbf{tp}(a/M, N) \in \mathbf{gS}^{na}(M)$. If $g : M \cong R$, then $|S_p(M)| = |S_p(R)|$.*

Proof. Define $\Phi : S_p(M) \to S_p(R)$, by $\Phi(\mathbf{tp}(f'(a)/M, f'[N])) = \mathbf{tp}(g_f \circ f'(a)/R, g_f \circ f'[N])$ such that $f' : N \cong f'[N]$ and $f' \supseteq f$ where $f : M \cong M$ and the following square commutes:

$$\begin{tikzcd} f'[N] \arrow[r, "\cong_{g_f}"] & g_f \circ f'[N] \\ M \arrow[u, "\mathrm{id}"] \arrow[r, "\cong_g"] & R \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

It is easy to see that Φ is a bijection. □

The next theorem is [Sh576, 2.13]. Our proof is similar to that of Shelah, but we show that Lemma 2.4.21 is enough.

Theorem 2.4.24. *Suppose $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $2^{\lambda^+} > \lambda^{++}$. Assume $(*)_\lambda$ and $\mathbf{K}_{\lambda^{+++}} = \emptyset$. If $(a, M, N) \in K_\lambda^3$ is minimal, then it has the extension property.*

Since the proof is very long we have divided it into three lemmas.

Lemma 2.4.25. *Under the hypothesis of Theorem 2.4.24. Let* $p = \mathbf{tp}(a/M, N) \in \mathbf{gS}^{na}(M)$ *such that* p *does not have the extension property. If* q *extends* p*, then* q *has less than* λ^+ *realizations.*

Proof. Follows from Lemma 2.4.18. □

Lemma 2.4.26. *Under the hypothesis of Theorem 2.4.24. Let* $p = \mathbf{tp}(a/M, N) \in \mathbf{gS}^{na}(M)$ *such that* (a, M, N) *is reduced, minimal and does not have the extension property. Then there is a reduced pre-type* $(a, M', N') \geq (a, M, N)$ *such that* $|S_{tp(a/M',N')}(M')| \geq \lambda^{++}$.

Proof. Let $M \leq_{\mathbf{K}} \mathcal{C}$ such that $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ universal above λ, this exists by Lemma 2.4.12. We do the proof by contradiction, so suppose it is not the case. We build $\{(a, P_\alpha, Q_\alpha) : \alpha < \lambda^+\}$ such that:

1. $(a, P_0, Q_0) := (a, M, N)$.
2. $\{P_\alpha : \alpha < \lambda^+\}$ and $\{Q_\alpha : \alpha < \lambda^+\}$ are increasing and continuous.
3. $P_\alpha <_{\mathbf{K}} P_{\alpha+1}$.
4. $(a, P_\alpha, Q_\alpha) \in K^3_\lambda$ is reduced for each $\alpha < \lambda^+$.

The construction of the chain is done by combining Fact 2.4.7 and Fact 2.2.6.

We also build $\{R_\alpha : \alpha < \lambda^+\}$ and $\{\Gamma_\alpha : \alpha < \lambda^+\}$ such that:

1. $R_0 := M$
2. $\forall \alpha < \lambda^+ (R_\alpha \leq_{\mathbf{K}} \mathcal{C}$ and $R_\alpha \in \mathbf{K}_\lambda)$.
3. $\{R_\alpha : \alpha < \lambda^+\}$ is increasing and continuous.
4. $\Gamma_\alpha = \{q_i^\alpha : i < \lambda^+\} = \bigcup_{\gamma < \lambda^+} S_{\mathbf{tp}(a/R_\gamma, Q_\gamma)}(R_\alpha)$.
5. $\forall \beta < \lambda^+ \forall q (q = q_i^\alpha$ for $\alpha, i < \beta$ then there is no $N' \in \mathbf{K}_\lambda$ such that $\mathcal{C} \geq_{\mathbf{K}} N' \geq_{\mathbf{K}} R_{\beta+1}$ and $c \in |N'| \backslash |R_{\beta+1}|$ realizing $q)$.

<u>Construction:</u> If $\alpha = 0$ apply (1) and if α is limit one takes unions. So the only interesting case is when $\alpha = \beta + 1$. By hypothesis given $\gamma < \lambda^+$ we have that $|S_{\mathbf{tp}(a/P_\gamma, Q_\gamma)}(P_\gamma)| \leq \lambda^+$, then by λ-categoricity and Lemma 2.4.23 $|S_{\mathbf{tp}(a/P_\gamma, Q_\gamma)}(R_\beta)| \leq \lambda^+$. So let $\{q_i^\beta : i < \lambda^+\}$ an enumeration of $\bigcup_{\gamma < \lambda^+} S_{\mathbf{tp}(a/R_\gamma, Q_\gamma)}(R_\beta)$. Let $\Sigma = \{q_i^\alpha : i, \alpha < \beta\}$, clearly $|\Sigma| \leq \lambda$. Observe that by Lemma 2.4.25 if $u \in \Sigma$ and $A_u = \{c \in \mathcal{C} : c$ realizes $u\}$ then $|A_u| \leq \lambda$. Hence $A = \bigcup_{u \in \Sigma} A_u$ is of size λ and let $R_{\beta+1}$ the structure obtained by applying downward Löwenheim-Skolem-Tarski axiom to $A \cup R_\beta$ in $\mathcal{C}$. $R_{\beta+1}$ works.

Enough: Let $P = \bigcup_{\alpha<\lambda^+} P_\alpha$, $Q = \bigcup_{\alpha<\lambda^+} Q_\alpha$ and $R = \bigcup_{\alpha<\lambda^+} R_\alpha$. By λ^+-categoricty there is $g : P \cong R$. Let $D = \{\delta < \lambda^+ : g : P_\delta \cong R_\delta\}$, by continuity of the chains this is a club. Let $\delta \in D$ and $q = g(\mathbf{tp}(a/P_\delta, Q_\delta)) \in S_{\mathbf{tp}(a/P_\delta,Q_\delta)}(R_\delta)$, by the enumeration there is $i < \lambda^+$ such that $q = q_i^\delta$. Let $g' : Q \cong g'[Q] \leq_{\mathbf{K}} \mathcal{C}$ with $g \subseteq g'$.

Let $\epsilon > \delta, i$. Since $\{P_\alpha : \alpha < \lambda^+\}$ is increasing and continuous there is $\gamma < \lambda^+$ such that $g^{-1}[R_{\epsilon+1}] <_{\mathbf{K}} P_\gamma$. Moreover since $(a, P_\gamma, Q_\gamma) \geq (a, P_\delta, Q_\delta)$, then $g'(a) \in |g'[Q_\gamma]|\backslash|R_{\epsilon+1}|$ and realizes q. Hence $g'[Q_\gamma]$ and $g'(a)$ contradict (5). $\square$

Lemma 2.4.27. *Under the hypothesis of Theorem 2.4.24. Let $p = \mathbf{tp}(a/M, N) \in \mathbf{gS}^{na}(M)$ such that (a, M, N) is reduced, minimal, does not have the extension property and $|S_p(M)| \geq \lambda^{++}$. If $R \in \mathbf{K}_\lambda$, $\Gamma \subseteq \bigcup\{S_p(R') : R' \leq_{\mathbf{K}} R, R' \in \mathbf{K}_\lambda\}$ and $|\Gamma| \leq \lambda^+$ then*

$$\Gamma^* = \{q \in S_p(R) : \exists R^* \in \mathbf{K}_\lambda(R \leq_{\mathbf{K}} R^*, R^* \text{ realizes } q \text{ and there is no } c \in |R^*|\backslash|R| \text{ realizing } u \in \Gamma)\}$$

has size λ^{++}.

Proof. Let $R \leq_{\mathbf{K}} \mathcal{C}$ such that $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ universal above λ, this exists by Lemma 2.4.12. Let $\{C_\alpha : \alpha < \lambda^{++}\}$ be an increasing and continuous resolution of $\mathcal{C}$ such that $C_0 := R$.

Given $q \in S_p(R)$ let $(a_q, R, T_q) \in K^3_\lambda$ such that $(a, M, N) \cong (a_q, R, T_q)$ and $q = \mathbf{tp}(a_q/R, T_q)$. Since (a, M, N) is reduced it follows that (a_q, R, T_q) is reduced. Moreover, from the fact that $\mathcal{C}$ is universal above λ we may assume that $T_q \leq_{\mathbf{K}} \mathcal{C}$.

$\boxed{\text{Claim}}$

1. If $q_1 \neq q_2 \in S_p(R)$, then $a_{q_1} \neq a_{q_2}$.
2. If $a_q \notin C_\alpha$, then $T_q \cap C_\alpha = R$.

The proof of the first claim follows from the fact that $T_{q_1}, T_{q_2} \leq_{\mathbf{K}} \mathcal{C}$. As for the second claim, it is clear that $R \subseteq T_q \cap C_\alpha$, so we will show the other inclusion. Let $b \in T_q \cap C_\alpha$, let R' the structure obtained by applying downward Löwenheim-Skolem-Tarski axiom to $\{b\} \cup R$ in C_α and let T' the structure obtained by applying downward Löwenheim-Skolem-Tarski axiom to $\{a_q\} \cup R'$ in $\mathcal{C}$. Clearly $(a_q, R, T_q) \leq (a_q, R', T') \in K^3_\lambda$ and since (a_q, R, T_q) is reduced $T_q \cap R' = R$. Since $b \in T_q \cap R'$, it follows that $b \in R$. $\dagger_{\text{Claim}}$

For each $u \in \Gamma$, $u \in S_p(R')$ for some $R' \leq_{\mathbf{K}} R$ by definition. Hence by Lemma 2.4.25 if $A_u = \{c \in \mathcal{C} : c \text{ realizes } u\}$, it follows that $|A_u| \leq \lambda$. Since $|\Gamma| = \lambda^+$, $|A| = |\bigcup_{u\in\Gamma} A_u| \leq \lambda^+$.

Pick $\alpha < \lambda^{++}$ such that $A \subseteq C_\alpha$. Let

$$\Sigma = \{q : q = \mathbf{tp}(a_q/R, T_q) \text{ and } a_q \notin C_\alpha\},$$

we will show that $\Sigma \subseteq \Gamma^*$ and $|\Sigma| \geq \lambda^{++}$.

Let $q \in \Sigma$, so $q = \mathbf{tp}(a_q/R, T_q)$ for $a_q \notin C_\alpha$. Suppose there is $c \in |T_q| \backslash |R|$ and $u \in \Gamma$ such that c realizes u. Since $T_q \leq_{\mathbf{K}} \mathcal{C}$, by definition $c \in A_u \subseteq C_\alpha$. Hence by claim (2) $c \in T_q \cap C_\alpha = R$, contradicting the fact that $c \notin R$.

Finally, since $|S_p(M)| \geq \lambda^{++}$, by λ-categoricity and Lemma 2.4.23 we have that $|S_p(R)| \geq \lambda^{++}$. From the fact that $\|C_\alpha\| = \lambda^+$ and Claim (1), it follows that $|\Sigma| \geq \lambda^{++}$. $\square$

Proof of Theorem 2.4.24. Let $\mathcal{C} \in \mathbf{K}_{\lambda^{++}}$ universal above λ, this exists by Lemma 2.4.12.

We do the proof by contradiction, so assume that $p = \mathbf{tp}(a/M, N)$ does not have the extension property.

By λ-categoricity there is $h : N \to \mathcal{C}$ so we may assume that $N \leq_{\mathbf{K}} \mathcal{C}$. Moreover by Fact 2.2.6, Lemma 2.4.26, Lemma 2.4.25 and Lemma 2.4.17 we may assume that (a, M, N) is reduced, minimal and $|S_{\mathbf{tp}(a/M,N)}(M)| \geq \lambda^{++}$.

We build $\{M_\eta : \eta \in 2^{<\lambda^+}\}$ and $\{p^l_\eta : \eta \in 2^{<\lambda^+}, l \in \{0, 1\}\}$ such that:

1. $M_{<>} := M$.
2. $\forall \eta \in 2^{<\lambda^+}(M_\eta \in \mathbf{K}_\lambda$ and $M_\eta \leq_{\mathbf{K}} \mathcal{C})$.
3. If $\eta < \nu$, then $M_\eta <_{\mathbf{K}} M_\nu$.
4. $\forall \eta \in 2^{<\lambda^+} \forall l \in \{0, 1\}(p^l_\eta \in S_p(M_\eta))$.
5. M_η realizes $p^l_{\eta\restriction_\beta}$ if and only if $\beta < lg(\eta)$ and $\eta(\beta) = l$.

Before doing the construction let us show that this is enough.

<u>Enough:</u> Given $\eta \in 2^{\lambda^+}$ let $M_\eta = \bigcup_{\alpha<\lambda^+} M_{\eta\restriction_\alpha}$. Realize that the construction above satisfies the hypothesis of Lemma 2.4.21. In particular, if $\eta \in 2^{\lambda^+}$ and $\alpha < \lambda^+ M_{\eta\restriction^\wedge_\alpha 0}$ can not be embedded to M_ν over $M_{\eta\restriction_\alpha}$ if $\eta \restriction^\wedge_\alpha 1 < \nu$ and $\nu \in 2^{<\lambda^+}$ by condition (5). Hence by the conclusion of Lemma 2.4.21 $\mathbf{K}$ is not λ^+-categorical, contradicting the hypothesis of the theorem.

<u>Construction:</u> For the base step use (1) and in limit stages take unions. So the only interesting case is when $\alpha = \beta + 1$. In that case, we are given by induction hypothesis M_η and need to build $M_{\eta^\wedge 0}$, $M_{\eta^\wedge 1}$ and p^0_η, p^1_η. We build $\{(N^\eta_\delta, a^\eta_\delta) : \delta < \lambda^{++}\}$ such that:

1. $(a^\eta_\delta, M_\eta, N^\eta_\delta) \in K^3_\lambda$ and $N^\eta_\delta \leq_{\mathbf{K}} \mathcal{C}$.
2. $\mathbf{tp}(a^\eta_\delta/M_\eta, N^\eta_\delta) \in S_p(M_\eta)$.
3. N^η_δ omits every $q \in \Gamma_\delta$, where $\Gamma_\delta = \{p^l_{\eta\restriction_\beta} : \beta < lg(\eta), l \neq \eta(\beta)\} \cup \{\mathbf{tp}(b/M_\eta, N^\eta_\gamma) : \gamma < \delta, b \in N^\eta_\gamma$ and $\mathbf{tp}(b/M_\eta, N^\eta_\gamma) \in S_p(M_\eta)\}$.

As before we will first show that this is enough and then we will do the construction.

Enough: For every $\delta < \lambda^{++}$, let $W^{\eta}_{\delta} = \{\gamma < \lambda^{++} : \exists b \in N^{\eta}_{\delta}(\mathbf{tp}(b/M_\eta, N^{\eta}_{\delta}) = \mathbf{tp}(a^{\eta}_{\gamma}/M_\eta, N^{\eta}_{\gamma}))\}$. Observe that for every δ we have that $|W^{\eta}_{\delta}| \leq \lambda$ since N^{η}_{δ} realizes at most λ types and by (3) if $\gamma \neq \gamma'$ then $\mathbf{tp}(a^{\eta}_{\gamma}/M_\eta, N^{\eta}_{\gamma}) \neq \mathbf{tp}(a^{\eta}_{\gamma'}/M_\eta, N^{\eta}_{\gamma'})$.

Therefore, there are $\delta < \epsilon < \lambda^{++}$ such that $\delta \notin W^{\eta}_{\epsilon}$ and $\epsilon \notin W^{\eta}_{\delta}$. Let $M_{\eta^\wedge 0} = N^{\eta}_{\delta}$, $M_{\eta^\wedge 1} = N^{\eta}_{\epsilon}$, $p^0_\eta = \mathbf{tp}(a^{\eta}_{\delta}/M_\eta, N^{\eta}_{\delta})$ and $p^1_\eta = \mathbf{tp}(a^{\eta}_{\epsilon}/M_\eta, N^{\eta}_{\epsilon})$.

By (3) they omit all the restrictions and by $\delta \notin W^{\eta}_{\epsilon}$ and $\epsilon \notin W^{\eta}_{\delta}$ it follows that p^0_η is omitted in $M_{\eta^\wedge 1}$ and p^1_η is omitted in $M_{\eta^\wedge 0}$.

Construction: If $\delta = 0$, then

$$|\Gamma_0| = |\{p^{l}_{\eta\restriction_\beta} : \beta < lg(\eta), l \neq \eta(\beta)\}| \leq \lambda^+.$$

Observe that p, $R = M_\eta$ and $\Gamma = \Gamma_0$ satisfy the hypothesis of Lemma 2.4.27. Therefore, there is (a^{η}_0, N^{η}_0) such that $M_\eta \leq_{\mathbf{K}} N^{\eta}_0$, $\mathbf{tp}(a^{\eta}_0/M_\eta, N^{\eta}_0) \in S_p(M_\eta)$ and no $c \in |N^{\eta}_0| \backslash |M_\eta|$ realizes a type in Γ_0. Moreover, since M_η omits Γ_0 it follows that N^{η}_0 omits Γ_0 and since $\mathcal{C}$ is universal above λ we may assume that $N^{\eta}_0 \leq_{\mathbf{K}} \mathcal{C}$.

If δ is limit or successor, realize that $|\Gamma_\delta| \leq \lambda^+$, then apply Lemma 2.4.27 as we did in the base step.

This finishes the construction and since we got to a contradiction in the first enough statement, we conclude that (a, M, N) has the extension property. □

We are finally able to obtain that $\mathfrak{s}_{min}$ satisfies the existence of non-forking extension property.

Lemma 2.4.28. *Suppose* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and* $2^{\lambda^+} > \lambda^{++}$. *If* $(*)_\lambda$ *and* $\mathbf{K}_{\lambda^{+++}} = \emptyset$, *then* $\mathfrak{s}_{min}$ *satisfies existence of non-forking extension property.*

Proof. Let $p \in \mathbf{gS}^{bs}_{min}(M)$ and $M \leq_{\mathbf{K}} M'$. Since $p \in \mathbf{gS}^{bs}_{min}(M)$, there is (a, M, N) minimal pre-type such that $p = \mathbf{tp}(a/M, N)$. Then by Theorem 2.4.24 there are g and $N' \in \mathbf{K}_\lambda$ such that $(a, M, N) \leq_g (b, M', N') \in K^3_\lambda$ and $g \supseteq \text{id}$. Let $q = \mathbf{tp}(b/M', N')$, it easy to show that g is the witness for $(a, M, N)E_{at}(b, M, N')$, so $p \leq q$. Since $q \restriction_M = p$ is a minimal type, we conclude that q does not fork over M. □

2.4.4 Conclusion

Putting together everything we have done in this section we get:

Theorem 2.4.29. *Suppose* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and* $2^{\lambda^+} > \lambda^{++}$. *If* $(*)_\lambda$ *and* $\mathbf{K}_{\lambda^{+++}} = \emptyset$, *then there is a w-good* λ*-frame.*

Proof. By Lemma 2.4.9 $\mathfrak{s}_{min}$ is a pre-frame. Then by Lemma 2.4.10 $\mathfrak{s}_{min}$ satisfies everything except weak density and existence of non-forking extension. Finally, by Lemma 2.4.14 $\mathfrak{s}_{min}$ satisfies weak density and by Lemma 2.4.28 $\mathfrak{s}_{min}$ satisfies existence of non-forking extension. $\square$

Now, using the ideas from Section 3 together with the above theorem we are able to prove Theorem 2.4.2. We repeat the statement of the theorem for the convenience of the reader.

Theorem 2.4.2. Suppose $2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$ and $2^{\lambda^+} > \lambda^{++}$. If $\mathbb{I}(\mathbf{K}, \lambda) = \mathbb{I}(\mathbf{K}, \lambda^+) = 1 \leq \mathbb{I}(\mathbf{K}, \lambda^{++}) < 2^{\lambda^{++}}$ and $\mathbf{K}$ is (λ, λ^+)-tame, then $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$.

Proof. Suppose for the sake of contradiction that $\mathbf{K}_{\lambda^{+++}} = \emptyset$. Then by Lemma 2.4.29 $\mathfrak{s}_{min}$ is a w-good λ-frame. Since $\mathbf{K}$ is (λ, λ^+)-tame and $\mathbf{K}_{\{\lambda,\lambda^+\}}$ has the amalgamation property (by Remark 2.4.5), it follows from Theorem 2.3.24 that $\mathfrak{s}_{min,\{\lambda,\lambda^+\}}$ is a w-good $[\lambda, \lambda^{++})$-frame. Hence by Theorem 2.3.18 $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$, which contradicts the hypothesis. $\square$

Lastly, let us show how we can apply Theorem 2.4.2 to universal classes. In [Sh300] Shelah introduced the concept of universal classes in the non-elementary setting.

Definition 2.4.30. A class of structures K is a *universal class* if:

1. K is a class of τ-structures, for some fixed vocabulary $\tau = \tau(K)$.
2. K is closed under isomorphisms.
3. K is closed under $\subseteq$-increasing chains.
4. If $M \in K$ and $N \subseteq M$, then $N \in K$.

Observe that if K is a universal class then $\mathbf{K} = (K, \subseteq)$ is an AEC with $\text{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$. We identify K and $\mathbf{K}$.

When $\mathbf{K}$ is a universal class, without any additional hypothesis, Will Boney proved that $\mathbf{K}$ is $(< \aleph_0)$-tame. It appears in print in [Vas17c, 3.7].

Fact 2.4.31. If $\mathbf{K}$ is a universal class, then $\mathbf{K}$ is $(< \aleph_0)$-tame. In particular, $\mathbf{K}$ is λ-tame for every $\lambda \geq LS(\mathbf{K})$.

Putting together this fact with Theorem 2.4.2 we get the following.

Theorem 2.4.32. *Suppose* $2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$ *and* $2^{\lambda^+} > \lambda^{++}$. *Assume* $\mathbf{K}$ *is a universal class. If* $(*)_\lambda$, *then* $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$.[10]

[10]Similarly to Theorem 2.4.2, this is not the best known result for universal classes, stronger results are obtained in [Ch. 3].

Proof. By Fact 3.2.7 $\mathbf{K}$ is (λ, λ^+)-tame and by Theorem 2.4.2 $\mathbf{K}_{\lambda^{+++}} \neq \emptyset$. $\square$

Observe that the proof of Theorem 2.4.32 is around 30 pages long (we cited a couple of facts in this paper), while Shelah's original proof is around 250 pages long, making the above proof for universal classes 200 pages shorter. We use the additional hypothesis that $2^{\lambda^+} > \lambda^{++}$, but as mentioned in Section 4.2 this is a weak hypothesis.

Chapter 3

Universal classes near $\aleph_1$

This chapter is based on [Ch. 3] and is joint work with Sebastien Vasey. In this chapter the *first author* is Marcos Mazari-Armida and the *second author* is Sebastien Vasey.

Abstract

Shelah has provided sufficient conditions for an $\mathbb{L}_{\omega_1,\omega}$-sentence ψ to have arbitrarily large models and for a Morley-like theorem to hold of ψ. These conditions involve structural and set-theoretic assumptions on all the $\aleph_n$'s. Using tools of Boney, Shelah, and the second author, we give assumptions on $\aleph_0$ and $\aleph_1$ which suffice when ψ is restricted to be universal:

Theorem. Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$-sentence.

1. If ψ is categorical in $\aleph_0$ and $1 \leq \mathbb{I}(\psi, \aleph_1) < 2^{\aleph_1}$, then ψ has arbitrarily large models and categoricity of ψ in some uncountable cardinal implies categoricity of ψ in all uncountable cardinals.

2. If ψ is categorical in $\aleph_1$, then ψ is categorical in all uncountable cardinals.

The theorem generalizes to the framework of $\mathbb{L}_{\omega_1,\omega}$-definable tame abstract elementary classes with primes.

3.1 Introduction

In a milestone paper, Shelah [Sh87a, Sh87b] gives the following classification-theoretic analysis of $\mathbb{L}_{\omega_1,\omega}$-sentences:

Fact 3.1.1. Assume that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. Let $\psi \in \mathbb{L}_{\omega_1,\omega}$ be a complete sentence. Assume that ψ has an uncountable model and for all $n > 0$, $\mathbb{I}(\psi, \aleph_n) < \mu_{\text{wd}}(\aleph_n)$.[1] Then ψ has arbitrarily large models and categoricity of ψ in *some* uncountable cardinal implies categoricity of ψ in *all* uncountable cardinals.

It is provably necessary to make hypotheses on all the $\aleph_n$'s: a family of examples of Hart and Shelah [HaSh90] (analyzed in detail by Baldwin and Kolesnikov [BaKo09]) gives for each $n < \omega$ an $\mathbb{L}_{\omega_1,\omega}$-sentence ψ_n which is categorical in $\aleph_0$, $\aleph_1$, ..., $\aleph_n$ but not in any cardinal above $\aleph_n$.

In the present paper, we show that if we restrict the complexity of the sentence, then it suffices to make model-theoretic and set-theoretic assumptions on $\aleph_0$ and $\aleph_1$. More precisely:

Theorem 3.3.3. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence (i.e. ψ is of the form $\forall \mathbf{x} \phi(\mathbf{x})$, where ϕ is quantifier-free). If ψ is categorical in $\aleph_0$ and $1 \leq \mathbb{I}(\psi, \aleph_1) < 2^{\aleph_1}$, then:*

1. *ψ has arbitrarily large models.*
2. *If ψ is categorical in* some *uncountable cardinal then ψ is categorical in* all *uncountable cardinals.*

We more generally prove Theorem 3.3.3 for universal classes (classes of models closed under isomorphisms, substructures, and unions of $\subseteq$-increasing chains, see Definition 3.2.1 and Fact 3.2.2) in a countable vocabulary. The assumption of categoricity in $\aleph_0$ can be removed if we instead assume categoricity in $\aleph_1$. In this case, we obtain the following upward categoricity transfer:

Theorem 3.3.5. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence. If ψ is categorical in $\aleph_1$, then it is categorical in all uncountable cardinals.*

The statements of Theorems 3.3.3 and 3.3.5 should be compared to the second author's eventual categoricity theorem for universal classes[Vas17d].

Fact 3.1.2. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$-sentence. If ψ is categorical in *some* $\mu \geq \beth_{\beth_{\omega_1}}$, then ψ is categorical in *all* $\mu' \geq \beth_{\beth_{\omega_1}}$.

Fact 3.1.2 is a ZFC theorem while the results of this paper use $2^{\aleph_0} < 2^{\aleph_1}$. However, Fact 3.1.2 is an eventual statement, valid for "big" cardinals (in fact there is

[1] See [Sh:h, VII.0.4] for a definition of μ_{wd} and [Sh:h, VII.0.5] for some of its properties. We always have that $2^{\aleph_n} \leq \mu_{\text{wd}}(\aleph_{n+1})$.

a generalization to any universal class, not necessarily in a countable vocabulary), while the focus of this paper is on structural properties holding in $\aleph_0$ and $\aleph_1$.

The reader may wonder: are there any interesting examples of eventually categorical universal classes? After the initial submission of this paper, Hyttinen and Kangas [HyKa18] showed that the answer is no: in any universal class categorical in a high-enough regular cardinal, any big-enough model will eventually look like either a set or a vector space (the methods are geometric in nature and also eventual, hence completely different from the tools used in this paper). Thus a reader wanting a non-trivial example illustrating e.g. Theorem 3.3.5 is out of luck: the statement of Theorem 3.3.5 combined with the Hyttinen-Kangas result implies that *any* example will eventually look like a class of vector spaces or a class of sets![2] One can think of this result as saying that an eventual version of Zilber's trichotomy holds for universal classes (but since algebraically closed fields are not universal, it is really a dichotomy).

Nevertheless, we still believe that the theorems of this paper are important for several reasons. First, the fact that there are no nontrivial examples is itself not obvious and Theorem 3.3.5 helps establish it. Second, it has many times been asked whether Morley's categoricity theorem can be applied to any interesting examples, and so far none has been found: the point is that the *methods* used to prove Morley's theorem are important. Similarly, we believe that the methods to prove the theorems here (good frames and tameness) are important to develop a classification theory of AECs – the statements of Theorems 3.3.3 and 3.3.5 here are showcases for the methods. Third, while Hyttinen and Kangas' proofs seem to only work for universal classes, our result can be generalized[3] as follows:

Theorem 3.4.4. Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let $\mathbf{K}$ be an AEC with $\mathrm{LS}(\mathbf{K}) = \aleph_0$. Assume that $\mathbf{K}$ has primes, is $\aleph_0$-tame, and is $\mathrm{PC}_{\aleph_0}$ (see [Sh:h, I.1.4], this is essentially the class of reducts of models of an $\mathbb{L}_{\omega_1,\omega}$-sentence).

1. If $\mathbf{K}$ is categorical in $\aleph_0$ and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then $\mathbf{K}$ has arbitrarily large models and categoricity in *some* uncountable cardinal implies categoricity in *all* uncountable cardinals.

2. If $\mathbf{K}$ is categorical in $\aleph_1$, then $\mathbf{K}$ is categorical in all uncountable cardinals.

The hypotheses of Theorem 3.4.4 are very general: they encompass for example the class of models of any $\aleph_0$-stable first-order theory (the setup of Morley's theorem) as well as any quasiminimal pregeometry class [Kir10] (see e.g. [ShVas18, 4.2] for why they are $\mathrm{PC}_{\aleph_0}$). There are many such classes (e.g. the pseudoexponential fields [Zil05]) which are not sets or vector spaces.

[2] It is however possible to add some noise in the low cardinals, see Example 3.4.1 here.

[3] Similarly, Fact 3.1.2 can be generalized. See for example the recent result of Ackerman, Boney, and the second author on multiuniversal classes [ABV19].

It is worth noting, that the results of this paper are direct consequences of putting together general facts about AECs (many only recently discovered): Shelah's construction of good frames [Sh:h, §II.3], Boney's proof of tameness in universal classes, and the second author's proof of the eventual categoricity conjecture in tame AECs with primes [Vas17c, Vas17b]. We decided to publish them because it is not completely obvious how to use these tools, and also because we believe that it is worth demonstrating how they can be used to solve such test questions.

Let us outline the proof of Theorem 3.3.3 in more details. We start with K, the class of models of our universal $\mathbb{L}_{\omega_1,\omega}$ sentence ψ. This is a universal class (see Definition 3.2.1). We are further assuming that ψ is categorical in $\aleph_0$ and has one but not too many models in $\aleph_1$. The first step is to show that K is well-behaved in $\aleph_0$: we use machinery of Shelah (Fact 3.2.13) to build a *good* $\aleph_0$*-frame*. The second step is to observe that universal classes have a locality property for Galois-types called *tameness* (see Definition 3.2.6): in fact Galois-types are determined by their finite restrictions (this is due to Will Boney, see Fact 3.2.7). The third step is the easy observation that in universal classes there is a prime model over every set (see Definition 3.2.8): take the closure of the set under the functions of an ambient model. The fourth and final step is to use the second author's results on AECs that have a good frame, are tame, and have primes [Vas17c, Vas17b]: any such class has arbitrarily large models and further Morley's categoricity theorem holds of such classes.

Note that the above argument only used the structural assumption on the class in the first step (to get the good frame). Once we have a good frame, the result follows because any universal class is tame and has primes. Moreover, the argument to get the good frame works in a much more general setup than universal classes. This is the reason our main theorem can be generalized to Theorem 3.4.4.

To sum up, any tame AEC with primes which has good behavior in the "low" cardinals ($\aleph_0$ and $\aleph_1$) will have good behavior everywhere. If on the other hand it is not clear that the AEC is tame or has primes, Shelah's results [Sh87a, Sh87b] and the Hart-Shelah example [HaSh90, BaKo09] tell us that one will need to use higher cardinals (the $\aleph_n$'s) to sort out whether the AEC is well-behaved past $\aleph_\omega$.

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3.2 Preliminaries

We assume that the reader has some familiarity with the basics of abstract elementary classes, as presented in for example [Bal09, §4-8]. In this section, we recall the main notions that we will use.

The notion of a universal class was studied already in Tarski's [Tar54]. Shelah [Sh300] was the first to develop classification theory for non-elementary universal classes.

Definition 3.2.1. A class of structures K is a universal class if:

1. K is a class of τ-structures, for some fixed vocabulary $\tau = \tau(K)$.
2. K is closed under isomorphisms.
3. K is closed under $\subseteq$-increasing chains.
4. If $M \in K$ and $N \subseteq M$, then $N \in K$.

The following basic characterization of universal classes is essentially due to Tarski [Tar54] (he proved it for finite vocabulary, but the proof generalizes). This will not be used in the present paper.

Fact 3.2.2 (Tarski's presentation theorem)**.** Let K be a class of structures. The following are equivalent:

1. K is a universal class.
2. K is the class of models of a universal $\mathbb{L}_{\infty,\omega}$ theory.

In this paper we will use tools of the more general framework of abstract elementary classes:

Definition 3.2.3 (Definition 1.2 in [Sh88])**.** An *abstract elementary class* (AEC for short) is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

1. K is a class of τ-structures, for some fixed vocabulary $\tau = \tau(\mathbf{K})$.
2. $\leq_{\mathbf{K}}$ is a partial order (that is, a reflexive and transitive relation) on K.
3. $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.
4. If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.

5. Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$;

6. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\langle M_i \in K : i < \delta \rangle$ is an increasing chain. Then:

   (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_0 \leq_{\mathbf{K}} M_\delta$.

   (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.

7. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

Remark 3.2.4.

1. When we write $M \leq_{\mathbf{K}} N$, it is assumed that $M, N \in K$.

2. We write $\mathbf{K}$ for the pair $(K, \leq_{\mathbf{K}})$, and K (no boldface) for the actual class. However we may abuse notation and write for example $M \in \mathbf{K}$ instead of $M \in K$ when there is no danger of confusion.

3. Given $[\lambda, \mu)$ an interval of cardinals (we allow $\mu = \infty$), let $\mathbf{K}_{[\lambda,\mu)} = \{M \in \mathbf{K} : \|M\| \in [\lambda, \mu)\}$. We write $\mathbf{K}_\lambda$ for $\mathbf{K}_{\{\lambda\}}$ and $\mathbf{K}_{\geq\lambda}$ for $\mathbf{K}_{[\lambda,\infty)}$.

4. If K is a universal class, then $\mathbf{K} := (K, \subseteq)$ is an AEC with $\mathrm{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$. Throughout this paper, we think of K as the AEC $\mathbf{K}$ and may write "$\mathbf{K}$ is a universal class" instead of "K is a universal class".

In any AEC $\mathbf{K}$, we can define a semantic notion of type, called Galois or orbital type in the literature (such types were introduced by Shelah in [Sh300] but we use the definition from [Vas16d, 2.16]).

Definition 3.2.5. Let $\mathbf{K}$ be an AEC.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.

2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E_{\mathrm{at}} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \xrightarrow[A]{} N$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$.

3. Note that E_{at} is a symmetric and reflexive relation on $\mathbf{K}^3$. We let E be the transitive closure of E_{at}.

4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$. We call such an equivalence class a *Galois-type* (or just a *type*). Usually, $\mathbf{K}$ will be clear from context and we will omit it.

Note that Galois-types are defined as the finest notion of type respecting $\mathbf{K}$-embeddings. When $\mathbf{K}$ is an elementary class, $\mathbf{tp}(\mathbf{b}/A; M)$ contains the same information as the usual notion of $\mathbb{L}_{\omega,\omega}$-syntactic type, but in general the two notions need not coincide [HaSh90, BaKo09]. We will see shortly (Fact 3.2.7) that in universal classes the Galois-types coincide with the quantifier-free types.

The *length* of $\mathbf{tp}(\mathbf{b}/A; M)$ is the length of $\mathbf{b}$. For $M \in \mathbf{K}$ and α a cardinal, p is a type over M of length α if there is $N \geq_{\mathbf{K}} M$ and $\mathbf{b} \in N^\alpha$ such that $p = \mathbf{tp}(\mathbf{b}/M, N)$. We write $\mathbf{gS}^\alpha_{\mathbf{K}}(M) = \mathbf{gS}^\alpha(M) = \{\mathbf{tp}(\mathbf{b}/M; N) : \mathbf{b} \in {}^\alpha N, M \leq_{\mathbf{K}} N\}$ for the set of types over M of length α. When $\alpha = 1$, we just write $\mathbf{gS}(M)$. We define naturally what it means for a type to be realized inside a model, to extend another type, and to take the image of a type by a $\mathbf{K}$-embedding. We call an AEC $\mathbf{K}$ λ-*stable* if $|\mathbf{gS}(M)| \leq \lambda$ for every $M \in \mathbf{K}$ of cardinality λ.

The notion of a good λ-frame is introduced in [Sh:h, §II.2]. As an approximation, the reader can think of the statement "$\mathbf{K}$ has a good λ-frame" as saying "$\mathbf{K}$ has a model of cardinality λ, amalgamation in λ, no maximal models in λ, joint embedding in λ, is stable in λ, and has a superstable-like nonforking notion for types over models of cardinality λ" (for a full definition see [Ch. 2, 3.2])[4].

Tameness is a locality property of Galois-types (which may or may not hold), first isolated by Grossberg and VanDieren in [GrVan06]:

Definition 3.2.6. We say an AEC $\mathbf{K}$ is $(< \kappa)$-*tame* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}(M)$, there is $A \subseteq |M|$ such that $|A| < \kappa$ and $p \upharpoonright A \neq q \upharpoonright A$. By κ-*tame* we mean $(< \kappa^+)$-*tame*. If we write $(< \kappa, \lambda)$-*tame* we restrict to $M \in \mathbf{K}_\lambda$. We may also talk of tameness for types of *finite length*, which means that we allow p, q above to be in $\mathbf{gS}^{<\omega}(M)$ rather than just in $\mathbf{gS}(M)$ (i.e. they could be types of finite sequences rather than types of singletons).

The following important fact is due to Will Boney. It appears in print as [Vas17c, 3.7].

Fact 3.2.7. If $\mathbf{K}$ is a universal class, then $\mathbf{K}$ is $(< \aleph_0)$-tame for types of finite length (in fact for types of all lengths). Moreover, Galois-types are the same as quantifier-free types.

The final main concept use in this paper is that of prime models (here over sets of the form $M \cup \{a\}$). The appropriate definition was introduced to AECs by Shelah in [Sh:h, III.3.2]. The definition is what the reader would expect when working inside

[4]In this paper our frames will always be *type-full*.

a fixed monster model, but here we may not have amalgamation, so we have to use Galois-types to describe the embedding of the base set.

Definition 3.2.8. Let $\mathbf{K}$ be an AEC.

- A *prime triple* is (a, M, N) such that $M \leq_{\mathbf{K}} N$, $a \in |N|$ and for every $N' \in \mathbf{K}$ and $a' \in |N'|$ such that $\mathbf{tp}(a/M, N) = \mathbf{tp}(a'/M, N')$, there exists $f : N \underset{M}{\longrightarrow} N'$ so that $f(a) = a'$.
- We say that $\mathbf{K}$ *has primes* if for any $M \in \mathbf{K}$ and every $p \in \mathbf{gS}(M)$, there is a prime triple (a, M, N) such that $p = \mathbf{tp}(a/M, N)$.

By taking the closure of $M \cup \{a\}$ under the functions of an ambient model, we obtain [Vas17c, 5.3]:

Fact 3.2.9. If $\mathbf{K}$ is a universal class, then $\mathbf{K}$ has primes.

The past two facts show that universal classes are tame and have primes. The next facts show that if we have a good frame in addition to that, then the structure of the frame transfers upward and in fact categoricity can be transferred.

We first give an approximation, due to Boney and the second author [BoVas17a, 6.9], which assumes amalgamation instead of primes (an earlier result is [Bon14, 1.1], which assumes tameness for types of length two instead of just length one).

Fact 3.2.10. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is λ-tame, $\mathbf{K}$ has amalgamation and $\mathbf{K}$ has a type-full good λ-frame, then $\mathbf{K}$ has a type-full good $[\lambda, \infty)$-frame.

The second author showed that one could replace amalgamation by primes (in fact a weak version of amalgamation suffices) [Vas17c, 4.16]:

Fact 3.2.11. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is λ-tame, has primes, and $\mathbf{K}$ has a type-full good λ-frame, then $\mathbf{K}_{\geq \lambda}$ has amalgamation. Hence a type-full good $[\lambda, \infty)$-frame by Fact 3.2.10.

Finally, the second author used Fact 3.2.11 together with the orthogonality calculus of good frames to prove the following categoricity transfer [Vas17b, 2.8]:

Fact 3.2.12. Let $\mathbf{K}$ be an AEC and let $\lambda \geq \mathrm{LS}(\mathbf{K})$. Assume that $\mathbf{K}$ is λ-tame, has primes, is categorical in λ, and $\mathbf{K}$ has a type-full good λ-frame. If $\mathbf{K}$ is categorical in *some* $\mu > \lambda$, then $\mathbf{K}$ is categorical in *all* $\mu' > \lambda$.

To get the good frame, we will use the following result from the study of AECs axiomatized by $\mathbb{L}_{\omega_1, \omega}$. It is due to Shelah and already present in some form in [Sh48, Sh87a] (see also [Sh:h, II.3.4]), but we cite from other sources and sketch some details here for the convenience of the reader.

Fact 3.2.13. Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be an $\mathbb{L}_{\omega_1,\omega}$-sentence. If $1 \leq \mathbb{I}(\psi, \aleph_1) < 2^{\aleph_1}$, then there exists an AEC $\mathbf{K}$ such that:

1. $\tau(\mathbf{K}) = \tau(\psi)$.
2. Any model in $\mathbf{K}$ satisfies ψ.
3. For $M, N \in \mathbf{K}$, $M \leq_{\mathbf{K}} N$ if and only if $M \preceq_{\mathbb{L}_{\infty,\omega}} N$.
4. $\mathbf{K}$ is categorical in $\aleph_0$ and has only infinite models.
5. $\mathbf{K}$ has a type-full good $\aleph_0$-frame.

One key of the proof is the following classical consequence of Keisler's omitting type theorem [Kei70, 5.10].

Fact 3.2.14. Let ψ be an $\mathbb{L}_{\omega_1,\omega}$-sentence and L^* be a countable fragment of $\mathbb{L}_{\omega_1,\omega}$. If there is a model M of ψ realizing uncountably-many L^*-types over the empty set, then $\mathbb{I}(\psi, \aleph_1) = 2^{\aleph_1}$.

Another crucial result of Shelah will be used to obtain amalgamation from few models. See [Sh:h, I.3.8] or [Gro02, 4.3] for a proof.

Fact 3.2.15. Assume $2^{\lambda} < 2^{\lambda^+}$. Let $\mathbf{K}$ be an AEC. If $\mathbf{K}$ is categorical λ and $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$, then $\mathbf{K}$ has amalgamation in λ.

We will also use the following fact from [Sh:h, IV.1.12] (there it is assumed that $\lambda, \mu > \mathrm{LS}(\mathbf{K})$ but the proof goes through without this hypothesis).

Fact 3.2.16. Let $\mathbf{K}$ be an AEC, let $\lambda \geq \mathrm{LS}(\mathbf{K})$, and let μ be an infinite cardinal. If $\mathbf{K}$ is categorical in λ and $\lambda = \lambda^{<\mu}$, then for any $M, N \in \mathbf{K}_{\geq \lambda}$, $M \leq_{\mathbf{K}} N$ implies $M \preceq_{\mathbb{L}_{\infty,\mu}} N$.

Finally, we will use [ShVas18, 5.8]:

Fact 3.2.17. If $\mathbf{K}$ is categorical in $\aleph_0$, has amalgamation and no maximal models in $\aleph_0$, is $(< \aleph_0, \aleph_0)$-tame and is stable in $\aleph_0$, then $\mathbf{K}$ has a type-full good $\aleph_0$-frame.

Proof sketch for Fact 3.2.13. By [Bal09, 6.3.2], there is a *complete* $\mathbb{L}_{\omega_1,\omega}$ sentence ψ_0 that implies ψ and has a model of cardinality $\aleph_1$. Let L^* be a countable fragment containing ψ_0 and let $\mathbf{K} := (\mathrm{Mod}(\psi_0), \preceq_{L^*})$.

Note that $\mathbf{K}$ is an AEC with $\mathrm{LS}(\mathbf{K}) = \aleph_0$, which by completeness of ψ_0 is categorical in $\aleph_0$ and has only infinite models. Hence it has joint embedding in $\aleph_0$. Since it has a model of cardinality $\aleph_1$ by assumption, $\mathbf{K}$ also has no maximal models in $\aleph_0$.

Moreover, $\mathbf{K}$ has amalgamation in $\aleph_0$ by Fact 3.2.15. Finally, by Fact 3.2.16 with $\lambda = \mu = \aleph_0$ and since $\mathbf{K}$ has only infinite models, $M \leq_{\mathbf{K}} N$ if and only if $M \preceq_{\mathbb{L}_{\infty,\omega}} N$.

It remains to show that $\mathbf{K}$ has a type-full good $\aleph_0$-frame. We first show:

Claim: Let $M \in \mathbf{K}_{\aleph_0}$. If $\langle p_i : i < \omega_1 \rangle$ are Galois-types over M, then there exists $i < j < \omega_1$ such that $p_i \upharpoonright A = p_j \upharpoonright A$ for all finite $A \subseteq |M|$.

Proof of Claim: By amalgamation in $\aleph_0$, we can find an uncountable model N extending M such that all the p_i's are realized inside N. Say $p_i = \mathbf{tp}(\mathbf{a}_i/M; N)$. For $A \subseteq |M|$, let τ_A denote $\tau(\mathbf{K}) \cup \{c_a \mid a \in A\}$, where the c_a's are new constant symbols. Whenever $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$, let N'_A denote the expansion of N' to τ_A with $c_a^{N'} = a$. Observe that whenever $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ and $A \subseteq |M|$ is finite, then, since $\leq_{\mathbf{K}} = \preceq_{\mathbb{L}_{\infty,\omega}}$, we have that $M_A \preceq_{\mathbb{L}_{\infty,\omega}(\tau_A)} N'_A \preceq_{\mathbb{L}_{\infty,\omega}(\tau_A)} N_A$.

Let L^{**} be a countable fragment of $\mathbb{L}_{\omega_1,\omega}$ extending L^* and containing Scott sentences of M_A for all $A \subseteq |M|$ finite. We now apply Fact 3.2.14 to the following sentence:

$$\bigwedge_{n\in\omega} \{\phi(c_{a_0}, \ldots c_{a_{n-1}}) : \phi \in L^{**}, a_0, \ldots, a_{n-1} \in |M|, M \models \phi[a_0, \ldots, a_{n-1}]\}$$

Note that the models of this sentence are essentially the extensions of M. Moreover $2^{\aleph_0} < 2^{\aleph_1}$ implies that the sentence still has few models in $\aleph_1$. Thus Fact 3.2.14 indeed applies and we get in particular that there must exist $i < j$ such that $tp_{L^{**}}(\mathbf{a}_i/\emptyset; N_{|M|}) = tp_{L^{**}}(\mathbf{a}_j/\emptyset; N_{|M|})$. Now fix $N' \leq_{\mathbf{K}} N$ countable containing M and $\mathbf{a}_i\mathbf{a}_j$. Also fix $A \subseteq |M|$ finite. Since $M_A \preceq_{\mathbb{L}_{\infty,\omega}(\tau_A)} N'_A$, there exists an isomorphism $f : N' \cong_A M$. Let $\mathbf{b}_i := f(\mathbf{a}_i)$, $\mathbf{b}_j := f(\mathbf{a}_j)$. By equality of the types $(N'_A, \{c_{b^i_k}^{N'_A} = a^i_k\}_{k<n}) \equiv_{L^{**}\upharpoonright\tau_{A\mathbf{b}_i}} (N'_A, \{c_{b^i_k}^{N'_A} = a^j_k\}_{k<n})$, hence $(M_A, \{c_{b^i_k}^{M_A} = b^i_k\}_{k<n}) \equiv_{L^{**}\upharpoonright\tau_{A\mathbf{b}_i}} (M_A, \{c_{b^i_k}^{M_A} = b^j_k\}_{k<n})$. Since L^{**} includes all the relevant Scott sentences, this means that there exists an automorphism g of M sending $\mathbf{b}_i$ to $\mathbf{b}_j$ and fixing A. Composing maps, we obtain an automorphism of N' fixing A and sending $\mathbf{a}_i$ to $\mathbf{a}_j$. Thus $p_i \upharpoonright A = p_j \upharpoonright A$, as desired. $\dagger_{\text{Claim}}$

Combining the Claim with [HyKe06, 3.12], we get that $\mathbf{K}$ is stable in $\aleph_0$ and is $(< \aleph_0, \aleph_0)$-tame for types of finite length. Therefore by Fact 3.2.17, $\mathbf{K}$ has a type-full good $\aleph_0$-frame. $\square$

3.3 Main results

In this section we prove the main theorems of this paper. We start by applying Fact 3.2.16 to a universal class categorical in $\aleph_0$:

Lemma 3.3.1. *Let $\mathbf{K}$ be a universal class in a countable vocabulary. If $\mathbf{K}$ is categorical in $\aleph_0$, then for $M, N \in \mathbf{K}_{\geq \aleph_0}$, $M \subseteq N$ if and only if $M \preceq_{\mathbb{L}_{\infty,\omega}} N$. Moreover, $\mathbf{K}_{\geq \aleph_0}$ is the class of models of an $\mathbb{L}_{\omega_1,\omega}$-sentence.*

Proof. Use Fact 3.2.16 with $\lambda = \mu = \aleph_0$ and recall that $\leq_{\mathbf{K}}$ is just the substructure relation. For the moreover part, take the Scott sentence of a countable model. $\square$

Applying Fact 3.2.13, we get directly:

Corollary 3.3.2. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let $\mathbf{K}$ be a universal class in a countable vocabulary. If $\mathbf{K}$ is categorical in $\aleph_0$ and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then $\mathbf{K}$ has a type-full good $\aleph_0$-frame.*

Proof. By Lemma 3.3.1, $\mathbf{K}_{\geq \aleph_0}$ is axiomatized by an $\mathbb{L}_{\omega_1,\omega}$ sentence and the ordering on $\mathbf{K}_{\geq \aleph_0}$ coincides with $\preceq_{\mathbb{L}_{\infty,\omega}}$. Since $\mathbf{K}$ is already categorical, $\mathbf{K}_{\geq \aleph_0}$ is equal to the class given by Fact 3.2.13, so $\mathbf{K}$ has a type-full good $\aleph_0$-frame. $\square$

We obtain one of our main theorems:

Theorem 3.3.3. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let $\mathbf{K}$ be a universal class in a countable vocabulary. If $\mathbf{K}$ is categorical in $\aleph_0$ and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then:*

1. *$\mathbf{K}$ has arbitrarily large models.*
2. *If $\mathbf{K}$ is categorical in some uncountable cardinal then $\mathbf{K}$ is categorical in all uncountable cardinals.*

Proof. By Corollary 3.3.2, $\mathbf{K}$ has a type-full good $\aleph_0$-frame. By facts 3.2.7 and 3.2.9, $\mathbf{K}$ is $\aleph_0$-tame and has primes. Therefore Fact 3.2.11 yields (1) and Fact 3.2.12 yields (2). $\square$

Observe that the only place where we used the hypotheses "$2^{\aleph_0} < 2^{\aleph_1}$ and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$" was to derive amalgamation and stability. Thus the conclusion of Theorem 3.3.3 also holds in ZFC if we assume that $\mathbf{K}$ is universal, $\aleph_0$-categorical, has amalgamation and no maximal models in $\aleph_0$, and is stable in $\aleph_0$ (using Fact 3.2.17 to get the good frame).

We can also replace the assumption of categoricity in $\aleph_0$ by categoricity in $\aleph_1$. To see this, we will use the following local version of Facts 3.2.10, 3.2.11, 3.2.12.

Fact 3.3.4. Let $\mathbf{K}$ be an AEC and $\lambda \geq \text{LS}(\mathbf{K})$. If $\mathbf{K}$ has a type-full good λ-frame, is categorical in λ and λ^+, is (λ, λ^+)-tame, and $\mathbf{K}_{\lambda^+}$ has primes, then $\mathbf{K}$ has a type-full good λ^+-frame and is categorical in λ^{++}.

Proof. The proof of Fact 3.2.11 is local, so $\mathbf{K}$ has a good λ^+-frame. That $\mathbf{K}$ is λ^{++}-categorical follows from [Vas17a, 6.14]. $\square$

Theorem 3.3.5. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence. If ψ is categorical in $\aleph_1$, then it is categorical in all uncountable cardinals.*

Proof. Let $\mathbf{K}$ be the class of models of ψ. Let $\mathbf{K}^*$ be the class obtained in Fact 3.2.13. Note that since $\mathbf{K}^*_{\aleph_1} \neq \emptyset$ (by the existence of the good $\aleph_0$-frame), $\mathbf{K}^* \subseteq \mathbf{K}$ and $\mathbf{K}$ is categorical in $\aleph_1$, $\mathbf{K}^*$ is also categorical in $\aleph_1$. Moreover $\mathbf{K}^*_{\aleph_1} = \mathbf{K}_{\aleph_1}$, because by Fact 3.2.16 with $\lambda = \aleph_1$ and $\mu = \aleph_0$ for $M, N \in \mathbf{K}_{\aleph_1}$, $M \subseteq N$ if and only if $M \preceq_{\mathbb{L}_{\infty,\omega}} N$. Since the behavior of an AEC is determined by its behavior in the Löwenheim-Skolem-Tarski number, $\mathbf{K}^*_{\geq\aleph_1} = \mathbf{K}_{\geq\aleph_1}$.

Now $\mathbf{K}^*$ has a type-full good $\aleph_0$-frame and since $\mathbf{K}$ is a universal class, $\mathbf{K}$ is $(\aleph_0, \aleph_1)$-tame. Since $\mathbf{K}^*_{\geq\aleph_1} = \mathbf{K}_{\geq\aleph_1}$, one can check that $\mathbf{K}^*$ is also $(\aleph_0, \aleph_1)$-tame. Furthermore, since $\mathbf{K}_{\aleph_1}$ has primes and $\mathbf{K}^*_{\aleph_1} = \mathbf{K}_{\aleph_1}$, $\mathbf{K}^*_{\aleph_1}$ also has primes. By Fact 3.3.4, $\mathbf{K}^*$ has a type-full good $\aleph_1$-frame and is categorical in $\aleph_2$. But this means that $\mathbf{K}_{\geq\aleph_1}$ has a type-full good $\aleph_1$-frame and is categorical in $\aleph_2$, so we can now apply Fact 3.2.12, to $\mathbf{K}_{\geq\aleph_1}$ to get the result. $\square$

3.4 Open questions and generalizations

The following variation on an example of Morley shows that for every countable ordinal α there are universal classes with models only up to size $\beth_\alpha$.

Example 3.4.1. Fix $\alpha < \omega_1$. Let τ be a vocabulary consisting of unary predicates $\langle P_i : i \leq \alpha \rangle$, a binary relation E and a binary function f. Let K be the class of τ-structures M such that:

1. $P_i^M \subseteq P_j^M$ for all $i < j < \alpha$.
2. $P_0^M = \emptyset$.
3. $|M| = P_\alpha^M$.
4. $P_i^M = \bigcup_{j<i} P_j^M$ for i limit.
5. xE^My implies $x \in P_i^M$ and $y \in P_j^M$ for some $i < j < \alpha$.
6. For any $i < \alpha$ and any two distinct $y_1, y_2 \in P_i^M$, $x := f(y_1, y_2)$ satisfies:

$$(xEy_1 \wedge \neg(xEy_2)) \vee (\neg(xEy_1) \wedge xEy_2)$$

Then K is a universal class in a countable vocabulary with amalgamation, joint embedding, and a model of cardinality $\beth_\alpha(0)$ but no models of cardinality $\beth_\alpha(0)^+$.[5] Taking the disjoint union of K with the class of $\mathbb{Q}$-vector spaces, we obtain (when $\alpha \geq \omega$) a universal class in a countable vocabulary which is categorical in an infinite cardinal λ exactly when $\lambda > \beth_\alpha(0)$.

This shows that some conditions on the class are necessary to derive arbitrarily large models. However it is not clear to us that Theorem 3.3.3 is optimal. Indeed it is not clear to us that the hypotheses on $\aleph_1$ are necessary (see Baldwin-Lachlan [BaLa73] for a positive result when K is axiomatized by a Horn theory):

Question 3.4.2. If K is a universal class categorical in $\aleph_0$ with a model in $\aleph_1$, must it be categorical in $\aleph_1$?

It would also be really nice to have a proof of Theorem 3.3.3 in ZFC, so it is natural to ask the following question.

Question 3.4.3. Can we drop the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ from Theorem 3.3.3? Can it be dropped if we add more categoricity assumptions?

Shelah [Sh:h, §I.6] has given an example of an analytic AEC which under Martin's axiom is categorical in $\aleph_0$ and $\aleph_1$ yet does not have amalgamation in $\aleph_0$. It seems however plausible that there are no such examples which are universal classes.

We end this paper with a generalization of Theorem 3.3.3. The key is that we have not used the full strength of the universal assumption: all we used was tameness, having primes, and some definability. Using harder results of Shelah, Theorems 3.3.3 and 3.3.5 generalize to:

Theorem 3.4.4. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Let* $\mathbf{K}$ *be an AEC with* $\mathrm{LS}(\mathbf{K}) = \aleph_0$. *Assume that* $\mathbf{K}$ *has primes, is* $\aleph_0$*-tame, and is* $\mathrm{PC}_{\aleph_0}$ *(see [Sh:h, I.1.4]).*

1. *If* $\mathbf{K}$ *is categorical in* $\aleph_0$ *and* $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, *then* $\mathbf{K}$ *has arbitrarily large models and categoricity in* some *uncountable cardinal implies categoricity in* all *uncountable cardinals.*
2. *If* $\mathbf{K}$ *is categorical in* $\aleph_1$, *then* $\mathbf{K}$ *is categorical in all uncountable cardinals.*

Proof. As in the proof of Theorems 3.3.3, 3.3.5 but using [Sh:h, I.3.10] to derive an $\aleph_0$-categorical subclass and [Sh:h, II.3.4] to derive the good $\aleph_0$-frame (actually in this case we only obtain a semi-good $\aleph_0$-frame with conjugation, see [JaSh13, 2.3.10], but this suffices for the proof). $\square$

[5] For a (possibly finite) cardinal μ and an ordinal α, $\beth_\alpha(\mu)$ is defined inductively by $\beth_0(\mu) = \mu$, $\beth_{\beta+1}(\mu) = 2^{\beth_\beta(\mu)}$, and $\beth_\delta(\mu) = \sup_{\beta<\delta} \beth_\beta(\mu)$ for δ limit.

Chapter 4

Simple-like independence relations in abstract elementary classes

This chapter is based on [Ch. 4] and is joint work with Rami Grossberg. In this chapter the *first author* is Rami Grossberg and the *second author* is Marcos Mazari-Armida.

Abstract

We introduce and study simple and supersimple independence relations in the context of AECs with a monster model.

Theorem 4.0.1. *Let* $\mathbf{K}$ *be an AEC with a monster model.*

- *If* $\mathbf{K}$ *has a simple independence relation, then* $\mathbf{K}$ *does not have the* 2*-tree property.*
- *If* $\mathbf{K}$ *has a simple independence relation with the* $(< \aleph_0)$*-witness property for singletons, then* $\mathbf{K}$ *does not have the tree property.*

The proof of both facts is done by finding cardinal bounds to classes of small Galois-types over a fixed model that are inconsistent for large subsets. We think that this finer way of counting types is an interesting notion in itself.

We characterize supersimple independence relations by finiteness of the Lascar rank under locality assumptions on the independence relation.

4.1 Introduction

Simple theories were discovered by Shelah in the mid seventies, an early characterization from his 1978 book [Sh:a] is Theorem III.7.7. Originally they were named *theories without the tree property*, Shelah's first paper on them was published in 1980 [Sh93]. Simple theories were ignored for more than a decade. In 1991 Hrushovski circulated [Hru02] (which was published in 2002), there he discovered that the first-order theory of an ultraproduct of finite fields while unstable is simple in the sense of Shelah and established an early version of the *type-amalgamation theorem* (also known as the independence theorem). This work was extended later by Chatzidakis and Hrushovski in the mid nineties, eventually published as [ChHr99]. Influenced by these papers, Kim in [Kim98] and with Pillay in [KiPi97] managed to adapt the type-amalgamation theorem from the algebraic context to complete first-order theories and solved a technical difficulty Shelah had with forking. We recommend [GIL02] for some of the basic results, history (approved by Shelah) as well as some technical simplifications and the chain condition. The subject of simple theories and more generally studying various variants of forking-like relations for unstable first-order theories got much attention in the last 20 years as witnessed by three books dedicated to the subject: [Wag00], [Cas11], and [Kim14].

In 1976 and 1977 Shelah circulated preprints of [Sh87a], [Sh87b] and [Sh88] starting the far reaching program of extending his classification theory of first-order theories to several non-elementary classes. First classes axiomatizable by a theory in $L_{\omega_1,\omega}(\mathbf{Q})$ and later to the more general syntax-free context of *Abstract Elementary Classes (AECs for short)*. An elementary introduction to the theory of AECs can be found in [Gro02]. A more in depth introduction is the two volume book by Shelah [Sh:h]. Another book is Baldwin's [Bal09]. For many years Shelah was the only person who managed to make progress in the field. Much of the early work was motivated by Shelah's categoricity conjecture (a generalization of Morley's categoricity theorem). Naturally the work was closely related to generalizing first-order $\aleph_0$-stability and superstability.

There is a very extensive literature about attempts to develop analogues to $\aleph_0$-stability, superstability and stability for various classes of AECs. Always under some extra assumptions on the AEC. This massive effort occupies thousands of pages and is impossible to summarize in this paper. A start can be found in the above mentioned books by Baldwin and Shelah, however in the last decade much was added. See in particular in the PhD theses of Boney [Bon14a] and Vasey [Vas17e].

The goal of this paper is to begin exploring analogues of simplicity in the context of AECs. A-priori it is unclear that there is a natural property (for AECs) that correspond directly to simplicity. It is plausible that there are several such properties. We introduce *simple* and *supersimple* independence relations. The main difference between stable independence relations and the relations that we introduce is that we do not assume uniqueness of non-forking extensions and instead assume the type-amalgamation property. Although this may seem like a minor change, based on our knowledge of forking in first-order theories this is actually a significant one.

Simplicity in first-order theories can be approached from several points of view: using ranks, tree-property, axiomatic properties of forking (or independence properties in general), and counting families of types. In this paper we too approach simplicity-like properties of AECs from various different directions.

We introduce the function $NT(\mu, \lambda, \kappa)$ to connect the existence of a simple-like independence relation with structural properties of the AEC. Our function generalizes $NT(\mu, \lambda)$ of [Cas99]. The function $NT(\mu, \lambda, \kappa)$ assigns to each $\mu \leq \lambda$ and κ cardinals the supremum of $|\Gamma|$ such that Γ is a subset of Galois-types over models of size less than μ which are contained in a fixed model of size λ and such that any subset of Γ of cardinality greater than κ is inconsistent. Intuitively this function let us count types in a finer way than just calculating the number of types over a fixed model.

We find the following bounds for the different kinds of independence relations studied in this paper.

Theorem. Let $\mathbf{K}$ be an AEC with a monster model.

1. (Theorem 4.4.2) If $\overline{\downarrow}$ is a stable independence relation, then

$$NT(\mu, \lambda, \kappa) \leq \lambda^{\kappa_1(\overline{\downarrow})} + \kappa^-.$$

2. (Theorem 4.5.12) If $\overline{\downarrow}$ is a simple independence relation, $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then

$$NT(\mu, \lambda, \aleph_0) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^\mu.$$

3. (Theorem 4.6.2, 4.7.6) If $\overline{\downarrow}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons or a supersimple independence relation, $\kappa(\overline{\downarrow}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\downarrow})} = \mu$, then

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\kappa(\overline{\downarrow})} + 2^\mu.$$

We show that these bounds are useful as they imply that the AEC is stable or the failure of the tree property. The extension of the tree property to AECs is another of the contributions of the paper and is the based on the the idea that small types play the role of formulas (see Definition 4.3.5).

Corollary. Let $\mathbf{K}$ be an AEC with a monster model.

1. (Corollaries 4.4.3, 4.4.4) If $\overline{\downarrow}$ is a stable independence relation independence relation, then $\mathbf{K}$ is stable and does not have the tree property.

2. (Corollary 4.5.14) If $\mathop{\smile\!\!\!\!\!\!\,|}$ is a simple independence relation, then $\mathbf{K}$ does not have the 2-tree property.

3. (Corollaries 4.6.3, 4.7.6) If $\mathop{\smile\!\!\!\!\!\!\,|}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons or a supersimple independence relation, then $\mathbf{K}$ does not have the tree property.

Moreover, using similar ideas to those used to prove the previous corollary, we obtain a new characterization of stable first-order theories assuming simplicity. We show that if first-order non-forking is contained in non-splitting and T is simple, then T is stable (Lemma 4.4.16).

In a different direction, we characterize supersimple independence relations via the Lascar rank (extended to AECs in [BoGr17]) under the $(< \aleph_0)$-witness property for singletons. This extends [Kim14, 2.5.16] to the AEC context.

Theorem 4.7.12. Assume $\mathbf{K}$ has a monster model. Let $\mathop{\smile\!\!\!\!\!\!\,|}$ be a simple independence relation with the $(< \aleph_0)$-witness property for singletons. The following are equivalent.

1. $\mathop{\smile\!\!\!\!\!\!\,|}$ is a supersimple independence relation.

2. If $M \in \mathbf{K}$ and $p \in \mathbf{gS}(M)$, then $U(p) < \infty$.

A natural question whenever encountering work in pure model theory is about applications. In this paper we do not deal with applications, we believe that it is premature to focus in applications as even for first-order simple theories the first significant applications were found more than 15 years after the basic results were discovered. Only recently some early applications were discovered of the much better understood theory of stable and superstable AECs. For this we refer the interested reader to recent results of the second author on classes of modules, among them: [Ch. 7], [Ch. 8], [Ch. 6], [Ch. 9], and [Ch. 11].

It is worth mentioning that there have been some efforts to extend the notion of simplicity to non-elementary settings. Buechler and Lessman introduced a notion of simplicity for a strongly homogeneous structure in [BuLe03], Ben-Yaccov introduced a notion of simplicity for compact abstract theories in [Ben03], Hyttinen and Kesälä introduced a notion of simplicity for $\aleph_0$-stable finitary AECs with disjoint amalgamation and a prime model in [HyKe06], and Shelah and Vasey introduced a notion of supersimplicity for $\aleph_0$-nicely stable AECs in [ShVas18]. One major difference between our context and that of [BuLe03] is that in their context types can be identified with sets of first-order formulas. As for [Ben03], types in his setting have a strong finitary character built in. While in our context types are orbits of the monster model $\mathfrak{C}$ under the action of $\mathrm{Aut}_A(\mathfrak{C})$. As for [HyKe06] and [ShVas18], a major difference is that we do not assume any trace of stability.

On March 3rd, 2020, two days before posting this paper in the arXiv, Kamsma paper [Kam20] was posted in the arXiv. In it, he introduced simple independence relations in AECats. Our papers study different aspects of simplicity in non-elementary classes. An important difference is that simple independence relations in his sense have finite character (called *union* in his paper), while in ours they do not have it. Kamsma answers partially Question 4.8.1 of this paper (see Remark 4.8.2).

The paper is organized as follows. Section 2 presents necessary background. Section 3 introduces the function $NT(\text{-},\text{-},\text{-})$, which is the main technical device of the paper, and a tree property. Section 4 deals with stable independence relations, a bound for $NT(\mu, \lambda, \kappa)$ is found, and it is shown that it implies stability and the failure of the tree property. We also study the consequences of weakening the uniqueness property by inclusion of the relation in explicitly non-splitting. Section 5 introduces simple independence relations, a bound for $NT(\mu, \lambda, \aleph_0)$ is found and it is shown that it implies the failure of the 2-tree property. Section 6 studies simple independence relations with locality assumptions. A bound for $NT(\mu, \lambda, (2^\mu)^+)$ is found and it is shown that it implies the failure of the tree property. Section 7 introduces supersimple independence relations and characterizes them by the Lascar rank. It is also shown that the existence of a supersimple independence relation in a class that admits intersections implies the $(< \aleph_0)$-witness property for singletons.

This paper was written while the second author was working on a Ph.D. under the direction of the first author at Carnegie Mellon University and the second author would like to thank the first author for his guidance and assistance in his research in general and in this work in particular. We thank Hanif Cheung for helpful conversations. We would also like to thank Mark Kamsma, Samson Leung, Sebastien Vasey, and a couple of referees for comments that helped improve the paper.

4.2 Preliminaries

We assume the reader has some familiarity with abstract elementary classes as presented for example in [Bal09, §4 - 8] and [Gro2X, §2, §4.4]. Familiarity with [LRV19] would be useful, but it is not required as we will recall the notions from [LRV19] that are used in this paper. We begin by quickly introducing the basic notions of AECs that we will use in this paper.

Since the main results of the paper assume joint embedding, amalgamation and no maximal models, we will assume these since the beginning.[1]

Hypothesis 4.2.1. Let **K** be an AEC with joint embedding, amalgamation and no maximal models.

[1]Some of the definitions presented here make sense without these hypothesis.

4.2.1 Basic concepts

We begin by introducing some model theoretic notation.

Notation 4.2.2.

- If $M \in \mathbf{K}$, $|M|$ is the underlying set of M and $\|M\|$ is the cardinality of M.
- If λ is a cardinal, $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$ and $\mathbf{K}_{<\lambda} = \{M \in \mathbf{K} : \|M\| < \lambda\}$.
- If $M \in \mathbf{K}$ and $\lambda \leq \|M\|$, $[M]^\lambda = \{N : N \leq_{\mathbf{K}} M\} \cap \mathbf{K}_\lambda$ and $[M]^{<\lambda} = \{N : N \leq_{\mathbf{K}} M\} \cap \mathbf{K}_{<\lambda}$.
- Let $M, N \in \mathbf{K}$. If we write "$f : M \to N$" we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$.

We will also use the next set theoretic notation.

Notation 4.2.3.

- For κ a cardinal, we define $\kappa^- = \theta$ if $\kappa = \theta^+$ and $\kappa^- = \kappa$ otherwise.
- For κ a cardinal and $\kappa \leq |A|$, let $\mathcal{P}_{<\kappa}(A) = \{B \subseteq A : |B| < \kappa\}$.

Recall the following definitions due to Shelah.

Definition 4.2.4. Let $M \in \mathbf{K}$.

1. M is λ-universal if for every $N \in \mathbf{K}_{<\lambda}$, there exists $f : N \to M$.
2. M is λ-model homogeneous if for every $M_0 \leq_{\mathbf{K}} N_0$ both in $\mathbf{K}_{<\lambda}$, if $M_0 \leq_{\mathbf{K}} M$ then there exists $f : N_0 \xrightarrow[M_0]{} M$.

Remark 4.2.5. Since $\mathbf{K}$ has joint embedding, amalgamation and no maximal models, we work inside a monster model $\mathcal{C}$ (as in complete first-order theories). A monster model $\mathcal{C}$ is large compared to all the models we consider and is universal and model homogeneous for small cardinals. As usual, we assume that all the elements and sets we consider are contained in the monster model $\mathcal{C}$. Further details are given in [Vas, §7].

Shelah introduced a notion of semantic type in [Sh300]. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We use the terminology of [Ch. 3, 2.5] and introduce Galois-types without using the monster model.

Definition 4.2.6.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.

2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1)E(\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \underset{A}{\rightarrow} N$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$.

3. Note that E is an equivalence relation on $\mathbf{K}^3$. It is transitive because $\mathbf{K}$ has amalgamation.

4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$. We call such an equivalence class a *Galois-type*. If $N = \mathcal{C}$ (where $\mathcal{C}$ is a monster model) we write $\mathbf{tp}(\mathbf{a}/A)$ instead of $\mathbf{tp}(\mathbf{a}/A; \mathcal{C})$.

5. For $N \in \mathbf{K}$, $A \subseteq N$ and I a non-empty set, $\mathbf{gS}^I(A; N) = \{\mathbf{tp}(\mathbf{b}/A; N) : \mathbf{b} = \langle b_i \in N : i \in I\rangle\}$. Let $\mathbf{gS}(M) := \mathbf{gS}^1(M)$ and $\mathbf{gS}^{<\infty}(M) := \bigcup_{\alpha<\infty} \mathbf{gS}^\alpha(M)$.

6. An AEC is λ-*stable* if for any $M \in \mathbf{K}_\lambda$ it holds that $|\mathbf{gS}(M)| \leq \lambda$. An AEC is *stable* if there is $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is λ-stable.

7. For $p = \mathbf{tp}_{\mathbf{K}}((b_i)_{i\in I}/A; N) \in \mathbf{gS}^I(A; N)$, $A' \subseteq A$ and $I_0 \subseteq I$, $p^{I_0} \restriction_{A'} := [((b_i)_{i\in I_0}, A', N)]_E$.

The following fact shows that in the presence of a monster model, the Galois-type of $\mathbf{b}$ over a set A is simply the orbit of $\mathbf{b}$ under the action of the automorphisms of $\mathcal{C}$ fixing A.

Fact 4.2.7. $\mathbf{tp}(\mathbf{b}_1/A; \mathcal{C}) = \mathbf{tp}(\mathbf{b}_2/A; \mathcal{C})$ if and only if there exists $f \in Aut_A(\mathcal{C})$ with $f(\mathbf{b}_1) = \mathbf{b}_2$.

The notion of tameness was isolated by the first author and VanDieren in [GrVan06] and type-shortness by Boney in [Bon14b].

Definition 4.2.8.

- $\mathbf{K}$ is $(<\kappa)$-*tame* for θ-types if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}^I(M)$ with $|I| = \theta$, there is $A \in \mathcal{P}_{<\kappa}(M)$ such that $p \restriction_A \neq q \restriction_A$.

- $\mathbf{K}$ is κ-*tame* for θ-types if it is $(<\kappa^+)$-*tame* for θ-types.

- $\mathbf{K}$ is *fully* $(<\kappa)$-*tame* if for every θ ordinal, $\mathbf{K}$ is $(<\kappa)$-tame for θ-types.

- $\mathbf{K}$ is *fully* $(<\kappa)$-*tame and -type-short* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}^I(M)$, there is $A \in \mathcal{P}_{<\kappa}(M)$ and $I_0 \in \mathcal{P}_{<\kappa}(I)$ such that $p^{I_0} \restriction_A \neq q^{I_0} \restriction_A$.

4.2.2 Independence relations and the witness property

Global independence relations in the context of AECs and μ-AECs have been extensively studied in the last few years, see for example [BoGr17], [Vas16a], and [LRV19]. Below we introduce a weak independence notion. Our notation and choice of axioms is inspired by [LRV19] and the specific independence relations that we will study in this paper.

Definition 4.2.9. $\mathop{\smile\!\!\!\!\!\!|}$ is an *independence relation* in an AEC $\mathbf{K}$ if the following properties hold:

1. $\mathop{\smile\!\!\!\!\!\!|} \subseteq \{(M, A, B) : M \leq_{\mathbf{K}} \mathcal{C}$ and $A, B \subseteq \mathcal{C}\}$. We say that $\mathbf{tp}(\bar{a}/B)$ does not fork over M if $ran(\bar{a})\mathop{\smile\!\!\!\!\!\!|}_M B$. This is well-defined by the next three properties.

2. (Preservation under $\mathbf{K}$-embeddings) Given $M \leq_{\mathbf{K}} \mathcal{C}$, $A, B \subseteq \mathcal{C}$ and $f \in Aut(\mathcal{C})$, we have that $A\mathop{\smile\!\!\!\!\!\!|}_M B$ if and only if $f[A]\mathop{\smile\!\!\!\!\!\!|}_{f[M]} f[B]$.

3. (Monotonicity) If $A\mathop{\smile\!\!\!\!\!\!|}_M B$ and $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0\mathop{\smile\!\!\!\!\!\!|}_M B_0$.

4. (Normality) $A\mathop{\smile\!\!\!\!\!\!|}_M B$ if and only if $A \cup M\mathop{\smile\!\!\!\!\!\!|}_M B \cup M$.

5. (Base monotonicity) If $A\mathop{\smile\!\!\!\!\!\!|}_M B$, $M \leq_{\mathbf{K}} N \leq_{\mathbf{K}} \mathcal{C}$ and $|N| \subseteq B$, then $A\mathop{\smile\!\!\!\!\!\!|}_N B$.

6. (Existence) If $M \leq_{\mathbf{K}} N$ and $p \in \mathbf{gS}^{<\infty}(M)$, then there exists $q \in \mathbf{gS}^{<\infty}(N)$ extending p such that q does not fork over M.

7. (Transitivity) If $M \leq_{\mathbf{K}} N$, $A\mathop{\smile\!\!\!\!\!\!|}_M N$ and $A\mathop{\smile\!\!\!\!\!\!|}_N B$, then $A\mathop{\smile\!\!\!\!\!\!|}_M B$.

Let us introduce some notation.

Notation 4.2.10. Given $\mathop{\smile\!\!\!\!\!\!|}$ an independence relation:

- For α a cardinal, let $\kappa_\alpha(\mathop{\smile\!\!\!\!\!\!|})$ be the minimum λ (or ∞) such that: If $p \in \mathbf{gS}^\alpha(M)$, then there exists $M_0 \leq_{\mathbf{K}} M$ with $||M_0|| \leq \lambda$ and p does not fork over M_0.

- Let $(\kappa(\mathop{\smile\!\!\!\!\!\!|}), \ell(\mathop{\smile\!\!\!\!\!\!|}))$ be the minimum pair (λ, θ) of cardinals[2] (or (∞, ∞)) such that: If $p \in \mathbf{gS}^\alpha(M)$, there exists $M_0 \in \mathbf{K}$ with $M_0 \leq_{\mathbf{K}} M$, $\|M_0\| \leq \lambda + \alpha^{<\theta}$ and p does not fork over M_0.

The following notion is a locality notion for independence relations.

Definition 4.2.11 ([Vas16a, 3.12.(9)])**.** Let $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation. $\mathop{\smile\!\!\!\!\!\!|}$ has the $(< \theta)$*-witness property* of length α if for all $M \leq_{\mathbf{K}} N$ and $\mathbf{b} \in \mathcal{C}^\alpha$: $\mathbf{b}\mathop{\smile\!\!\!\!\!\!|}_M N$ if and only if $\mathbf{b}\mathop{\smile\!\!\!\!\!\!|}_M A$ for every $A \in \mathcal{P}_{<\theta}(N)$. We say that $\mathop{\smile\!\!\!\!\!\!|}$ has the $(< \theta)$-witness property if and only if $\mathop{\smile\!\!\!\!\!\!|}$ has the $(< \theta)$-witness property of length α for all α.

[2] λ is an infinite cardinal, but θ might be a finite cardinal. The minimum is taken with respect to the canonical ordering in pairs of ordinals.

Observe that since first-order non-forking has finite character, first-order non-forking has the $(< \aleph_0)$-witness property. This might not be the case for independence relations as the next example shows. This example was first considered in [Adl09, 1.43].

Example 4.2.12. Let $L(\mathbf{K}) = \emptyset$ and $\mathbf{K} = (Sets, \subseteq)$. Given $M, A, B \in \mathbf{K}$ let:

$$A \underset{M}{\overline{\smile}} B \text{ if and only if } |(A \cap B) \backslash M| \leq \aleph_0$$

It is easy to show that $\overline{\smile}$ is an independence relation. $\overline{\smile}$ has the $(< \aleph_0)$-witness property of length α for α countable, but not for α uncountable. Hence $\overline{\smile}$ does not have the $(< \aleph_0)$-witness property.

In a few places in the paper we will assume that the independence relation under consideration has the witness property in order to be able to carry out some of the proofs (see for example Lemma 4.6.1 and Theorem 4.7.12).

The next lemma gives a natural condition that implies the witness property. It fixes a small gap in [Vas16a, 4.3]; the argument in [Vas16a, 4.3] seems to only work for M of cardinality less than or equal to $\kappa_\alpha(\overline{\smile})$ as we need $M \leq_{\mathbf{K}} N$ in order to apply transitivity.

Lemma 4.2.13. *Let $\overline{\smile}$ be an independence relation. If $\kappa_\alpha(\overline{\smile}) = \lambda$, then $\overline{\smile}$ has the $(< \lambda^+)$-witness property of length α.*

Proof. The proof is divided into two cases:

Case 1: Assume that $\|M\| \leq \lambda$. Let $M \leq_{\mathbf{K}} N$ and $\mathbf{a} \in \mathcal{C}^\alpha$, by $\kappa_\alpha(\overline{\smile}) = \lambda$ there is $N' \in [N]^\lambda$ such that $\mathbf{a} \underset{N'}{\overline{\smile}} N$. Since $\|M\| \leq \lambda$ and $M \leq_{\mathbf{K}} N$, we may assume without lost of generality that $M \leq_{\mathbf{K}} N'$. Moreover, as $N' \in \mathcal{P}_{\leq\lambda}(N)$, we have that $\mathbf{a} \underset{M}{\overline{\smile}} N'$. Then by transitivity we conclude that $\mathbf{a} \underset{M}{\overline{\smile}} N$.

Case 2: Assume that $\|M\| > \lambda$. Let $M \leq_{\mathbf{K}} N$ and $\mathbf{a} \in \mathcal{C}^\alpha$. Since $\kappa_\alpha(\overline{\smile}) = \lambda$ there is $M' \in [M]^\lambda$ such that $\mathbf{a} \underset{M'}{\overline{\smile}} M$. Using that $\forall B \in \mathcal{P}_{\leq\lambda}(N)(\mathbf{a} \underset{M}{\overline{\smile}} B)$ and transitivity, it follows that $\forall B \in \mathcal{P}_{\leq\lambda}(N)(\mathbf{a} \underset{M'}{\overline{\smile}} B)$. Then by the first case we have that $\mathbf{a} \underset{M'}{\overline{\smile}} N$. Hence $\mathbf{a} \underset{M}{\overline{\smile}} N$ by base monotonicity. $\square$

We will give a few other natural conditions that imply the witness property, see for example Fact 4.5.7 and Corollary 4.7.16.

4.3 The basic notions

In this section we introduce a way of counting Galois-types over small submodels and generalize the tree property to AECs. We think that this finer way of counting types is an interesting notion in itself. As mentioned in the preliminaries we are assuming Hypothesis 4.2.1.

In this paper Galois-types over submodels will play a central role.

Definition 4.3.1. Let $M \in \mathbf{K}$ and $\mu \leq \|M\|$:

$$\mathbf{gS}(M, \leq \mu) = \{\mathbf{tp}(a/N) : N \leq_{\mathbf{K}} M \text{ and } \|N\| \leq \mu\}$$

Definition 4.3.2. Let Γ be a set of Galois-types. Γ is *consistent* if there is $a \in \mathcal{C}$ such that a realizes every Galois-type in Γ, i.e., $\mathbf{tp}(a/dom(p)) = p$ for each $p \in \Gamma$. If such an $a \in \mathcal{C}$ does not exist we say that Γ is *inconsistent*.

The following notion generalizes [Cas99, 2.3] to the AEC setting.

Definition 4.3.3. Let $\mu, \lambda \in [\mathrm{LS}(\mathbf{K}), \infty)$ such that $\mu \leq \lambda$ and κ a cardinal (possibly finite). We define the following:

$$NT(\mu, \lambda, \kappa) = sup\{|\Gamma| : \exists M \in \mathbf{K}_\lambda(\Gamma \subseteq \mathbf{gS}(M, \leq \mu) \text{ and } \forall \Delta \subseteq \Gamma(|\Delta| \geq \kappa \to \Delta \text{ is inconsistent}))\}$$

If $\kappa = 2$ instead of writing $NT(\mu, \lambda, 2)$, we write $NT(\mu, \lambda)$ as in [Cas99].[3]

The following bounds are easy to calculate and hold in general. In what follows, see Theorems 4.4.2, 4.4.13, 4.5.12 and 4.6.2, we will find sharper bounds which will be the key to show stability or the failure of the tree property under additional hypothesis.

Proposition 4.3.4.

1. *If $M \in \mathbf{K}_\lambda$, then $|\mathbf{gS}(M)| \leq NT(\lambda, \lambda, 2)$.*
2. *If $\mu_1 \leq \mu_2$, $\lambda_1 \leq \lambda_2$ and $\kappa_1 \leq \kappa_2$ then $NT(\mu_1, \lambda_1, \kappa_1) \leq NT(\mu_2, \lambda_2, \kappa_2)$.*
3. *If $\mu \leq \lambda$, then the value of $NT(\mu, \lambda, \text{-})$ is bounded as follows:*
   (a) *If $\kappa \in [2, (\lambda^\mu)^+]$, then $NT(\mu, \lambda, \kappa) \leq \lambda^\mu$.*
   (b) *If $\kappa \in ((\lambda^\mu)^+, (2^\lambda)^+]$, then $NT(\mu, \lambda, \kappa) \leq 2^\lambda$.*
   (c) *If $\kappa \in ((2^\lambda)^+, 2^{\lambda^\mu}]$, then $NT(\mu, \lambda, \kappa) \leq 2^{\lambda^\mu}$.*
4. *$\mathbf{K}$ is λ-stable if and only if $NT(\mu, \lambda, \kappa) \leq \lambda$ for every $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda]$ and $\kappa \in [2, \lambda^+]$.*

[3]The definition given here does not fully match the definition of [Cas99] when $\mathbf{K} = (Mod(T), \preceq)$ for a complete first-order theory T, since the bound μ on [Cas99] refers to the cardinality of the type (the number of formulas in it) while in our definition it refers to the cardinality of the domain of the type.

Proof.

1. Let $\chi = |\mathbf{gS}(M)|$ and $\{p_\alpha : \alpha < \chi\}$ an enumeration without repetitions of $\mathbf{gS}(M)$. Observe $\{p_\alpha : \alpha < \chi\} \subseteq \mathbf{gS}(M, \leq \lambda)$ and any set $\{p_\alpha, p_\beta\}$ is inconsistent if $\alpha \neq \beta$. Therefore, $|\mathbf{gS}(M)| = \chi \leq NT(\lambda, \lambda, 2)$.

2. Follows from the fact that if $\Gamma \subseteq \mathbf{gS}(M, \leq \mu_1)$ for $M \in \mathbf{K}_{\lambda_1}$ and each subset of size greater or equal to κ_1 is inconsistent, then there is $M^* \in \mathbf{K}_{\lambda_2}$ with $M \leq_{\mathbf{K}} M^*$ and $\Gamma \subseteq \mathbf{gS}(M^*, \leq \mu_2)$ such that any subset of size greater or equal to κ_2 is inconsistent.

3. (a) Let $\kappa \in [2, (\lambda^\mu)^+]$, $\chi := \lambda^\mu$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$. Let $\Phi : \chi^+ \to [M]^{\leq \mu}$ be defined as $\Phi(\alpha) = dom(p_\alpha)$, since $|[M]^{\leq \mu}| = \lambda^\mu$ by the pigeonhole principle there is $S \subseteq \chi^+$ of size χ^+ and $N \in [M]^{\leq \mu}$ such that $dom(p_\alpha) = N$ for each $\alpha \in S$. Let $\Psi : S \to \mathbf{gS}(N)$ be defined as $\Psi(\alpha) = p_\alpha$, since $|\mathbf{gS}(N)| \leq 2^\mu$ by the pigeonhole principle there is $S' \subseteq S$ of size χ^+ and $q \in \mathbf{gS}(N)$ such that $p_\alpha = q$ for each $\alpha \in S'$. In particular $\{p_\alpha : \alpha \in S'\}$ is a consistent set of size χ^+. Hence $NT(\mu, \lambda, \kappa) \leq \lambda^\mu$.

   (b) Let $\kappa \in ((\lambda^\mu)^+, (2^\lambda)^+]$, $\chi := 2^\lambda$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.
   Given $\alpha < \chi^+$, let $q_\alpha \in \mathbf{gS}(M)$ such that $q_\alpha \geq p_\alpha$, it exists because we assumed that $\mathbf{K}$ has amalgamation. Let $\Phi : \chi^+ \to \mathbf{gS}(M)$ be defined as $\Phi(\alpha) = q_\alpha$, since $|\mathbf{gS}(M)| \leq 2^\lambda$ by the pigeonhole principle there is $S \subseteq \chi^+$ of size χ^+ and $q \in \mathbf{gS}(M)$ with $q_\alpha = q$ for every $\alpha \in S$. In particular $\{p_\alpha : \alpha \in S'\}$ is a consistent set of size χ^+. Hence $NT(\mu, \lambda, \kappa) \leq 2^\lambda$.

   (c) Similar to (b).

4. The forward direction is similar to (3).(a) but using that for every $M \in \mathbf{K}_\lambda$ we have that $|\mathbf{gS}(M)| \leq \lambda$ instead of only $|\mathbf{gS}(M)| \leq 2^\lambda$. The backward direction follows from (1).

$\square$

The next concept extends the tree property to the AEC context. The main idea is that Galois-types over *small sets* in AECs play a similar role as that of formulas in first-order theories. This correspondence is explored in [Vas16b].

Definition 4.3.5. Let $\mu, \lambda \in [\mathrm{LS}(\mathbf{K}), \infty)$ and $k < \omega$. $\mathbf{K}$ has the (μ, λ, k)-tree property if there is $\{(a_\eta, B_\eta) : \eta \in {}^{<\mu}\lambda\}$[4] such that:

1. $\forall \eta \in {}^{<\mu}\lambda(|B_\eta| < \mathrm{LS}(\mathbf{K}))$.

[4]As always we assume that $\forall \eta (a_\eta \in \mathcal{C}$ and $B_\eta \subseteq \mathcal{C})$.

2. $\forall \nu \in {}^{\mu}\lambda(\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ is consistent).

3. $\forall \eta \in {}^{<\mu}\lambda(\{\mathbf{tp}(a_{\eta^\wedge\alpha}/B_{\eta^\wedge\alpha}) : \alpha < \lambda\}$ is k-contradictory).

We say that $\mathbf{K}$ has the k-tree property if for all $\mu, \lambda \in [\mathrm{LS}(\mathbf{K}), \infty)$ $\mathbf{K}$ has the (μ, λ, k)-tree property and $\mathbf{K}$ has the tree property if there is a $k < \omega$ such that $\mathbf{K}$ has the k-tree property.

The following lemma relates the two concepts we just introduced. A similar construction in the first-order context appears in[Cas99, 2.3].

Lemma 4.3.6. *Assume $\lambda^{<\mu} = \lambda$ and $\mathrm{LS}(\mathbf{K}) \leq \mu \leq \lambda$. If $\mathbf{K}$ has the $(\mu, \lambda, 2)$-tree property, then $NT(\mu, \lambda, 2) = \lambda^\mu$. Moreover, $NT(\mu, \lambda, \kappa) \geq \lambda^\mu$ for all $\kappa \geq 2$.*[5]

Proof. By the definition of the tree property we have $\{(a_\eta, B_\eta) : \eta \in {}^{<\mu}\lambda\}$ such that:

1. $\forall \eta \in {}^{<\mu}\lambda(|B_\eta| < \mathrm{LS}(\mathbf{K}))$.

2. $\forall \nu \in {}^{\mu}\lambda(\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ is consistent).

3. $\forall \eta \in {}^{<\mu}\lambda(\{\mathbf{tp}(a_{\eta^\wedge\alpha}/B_{\eta^\wedge\alpha}) : \alpha < \lambda\}$ is 2-contradictory).

Let $A = \bigcup_{\eta \in {}^{<\mu}\lambda} B_\eta$. Since $\lambda^{<\mu} = \lambda$ and each B_η has cardinality less than $\mathrm{LS}(\mathbf{K})$, we have that $|A| \leq \lambda$. So applying downward Löwenheim-Skolem in $\mathcal{C}$ we obtain $M \in \mathbf{K}_\lambda$ such that $\forall \eta \in {}^{<\mu}\lambda(B_\eta \subseteq |M|)$.

For each $\nu \in {}^{\mu}\lambda$, pick $a_\nu \in \mathcal{C}$ realizing $\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ and apply downward Löwenheim-Skolem to $\bigcup_{\alpha<\mu} B_{\nu\restriction_\alpha}$ in M to get $M_\nu \in [M]^{\leq\mu}$. Then define $p_\nu := \mathbf{tp}(a_\nu/M_\nu)$.

Observe that $\{p_\nu : \nu \in {}^{\mu}\lambda\} \subseteq \mathbf{gS}(M, \leq \mu)$ and using part (3) of the definition of the tree property it is easy to show that: if $\nu_1 \neq \nu_2$, then $p_{\nu_1} \neq p_{\nu_2}$. Therefore $|\{p_\nu : \nu \in {}^{\mu}\lambda\}| = \lambda^\mu$. Moreover, using part (3) of the definition of the tree property it follows that any pair of types is inconsistent. Hence $NT(\mu, \lambda, 2) \geq \lambda^\mu$.

The equality and moreover part follow from Proposition 4.3.4. □

As we will see later, if we only know that $\mathbf{K}$ has the tree property it becomes more complicated to obtain a lower bound on NT(-, -, -).

4.4 Stable independence relations

In this section we deal with stable independence relations. The definition given here for a stable independence relation is similar to the one given in [LRV19]. The properties given here are obtained by taking the "closure" of a stable independence relation

[5] As usual we assume that λ, μ are cardinals way below the size of the monster model.

in the sense of [LRV19]; this is formalized in [LRV19, 8.2]. An important difference with [LRV19] is that we do not assume the witness property.

Definition 4.4.1 ([LRV19, 8.4, 8.5, 8.6])**.** $\mathop{\smile\!\!\!\!\!\!\!\overline{}}$ is a *stable independence relation* in $\mathbf{K}$ if the following properties hold:

1. $\mathop{\smile\!\!\!\!\!\!\!\overline{}}$ is an independence relation.
2. (Symmetry) $A \mathop{\smile\!\!\!\!\!\!\!\overline{}}_M B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\!\overline{}}_M A$.
3. (Uniqueness) Let $p, q \in \mathbf{gS}^{<\infty}(B; N)$ with $M \leq_{\mathbf{K}} N$ and $|M| \subseteq B \subseteq |N|$. If $p \restriction_M = q \restriction_M$ and p, q do not fork over M, then $p = q$.
4. (Local character) For each cardinal α there exists a cardinal λ (depending on α) such that: If $p \in \mathbf{gS}^{\alpha}(M)$, then there exists $M_0 \leq_{\mathbf{K}} M$ with $\|M_0\| \leq \lambda$ and p does not fork over M_0.

We begin by bounding NT(-, -, -).

Theorem 4.4.2. *If $\mathop{\smile\!\!\!\!\!\!\!\overline{}}$ is a stable independence relation, then*

$$NT(\mu, \lambda, \kappa) \leq \lambda^{\kappa_1(\mathop{\smile\!\!\!\!\!\!\!\overline{}})} + \kappa^{-}.$$

In particular, we get that $NT(\mu, \lambda) \leq \lambda^{\kappa_1(\mathop{\smile\!\!\!\!\!\!\!\overline{}})}$.

Proof. Let $\lambda_0 = \kappa_1(\mathop{\smile\!\!\!\!\!\!\!\overline{}})$, $\chi = \lambda^{\lambda_0} + \kappa^{-}$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.

By local character for every $\alpha < \chi^+$ there is $R_\alpha \in [M]^{\lambda_0}$ such that p_α does not fork over R_α. We define $\Phi : \chi^+ \to [M]^{\lambda_0}$ as $\Phi(\alpha) = R_\alpha$. Then by the pigeonhole principle there is $R \in [M]^{\lambda_0}$ and $S \subseteq \chi^+$ of cardinality χ^+ such that p_α does not fork over R for every $\alpha \in S$. Now define $\Psi : S \to \mathbf{gS}(R)$ as $\Psi(\alpha) = p_\alpha \restriction_R$, since $|\mathbf{gS}(R)| \leq 2^{\lambda_0}$, by the pigeonhole principle there is $p \in \mathbf{gS}(R)$ and $S' \subseteq S$ of size χ^+ such that $p_\alpha \restriction_R = p$ for every $\alpha \in S'$. Observe that $p_\alpha \geq p$ and p_α does not fork over R for every $\alpha \in S'$.

By the extension property and transitivity for each $\alpha \in S'$, there is $q_\alpha \in \mathbf{gS}(M)$ extending p_α such that q_α does not fork over R. Then by uniqueness, using that for all $\alpha, \beta \in S'$ we have that $q_\alpha \restriction_R = p_\alpha \restriction_R = p = p_\beta \restriction_R = q_\beta \restriction_R$ and that both q_α, q_β do not fork over R, we conclude that there is $q \in \mathbf{gS}(M)$ such that $q_\alpha = q$ for every $\alpha \in S'$. In particular, $\{p_\alpha : \alpha \in S'\}$ is consistent and $|S'| \geq \kappa$. Hence $NT(\mu, \lambda, \kappa) \leq \lambda^{\lambda_0} + \kappa^{-}$.

□

The next corollary follows directly from Proposition 4.3.4 and the above theorem. A version of it already appears in [BGKV16, 5.17] and [LRV19, 8.15].

Corollary 4.4.3. *If $\mathop{\smile\!\!\!\!\!\!\perp}$ is a stable independence relation, then $\mathbf{K}$ is λ-stable for every λ such that $\lambda^{\kappa_1(\mathop{\smile\!\!\!\!\!\!\perp})} = \lambda$.*

We show that the existence of a stable independence relation implies the failure of the tree property.

Lemma 4.4.4. *If $\mathbf{K}$ has $\mathop{\smile\!\!\!\!\!\!\perp}$ a stable independence relation, then $\mathbf{K}$ does not have the tree property.*

Proof. Let $\kappa_1(\mathop{\smile\!\!\!\!\!\!\perp}) = \lambda_0$ and $k < \omega$ such that $\mathbf{K}$ has the k-tree property. Let $\mu = \lambda_0^+$ and $\lambda = \beth_\mu(\mu)$. By the definition of the (μ, λ, k)-tree property there are $\{(a_\eta, B_\eta) : \eta \in {}^{<\mu}\lambda\}$ such that:

1. $\forall \eta \in {}^{<\mu}\lambda(\|B_\eta\| < \mathrm{LS}(\mathbf{K}))$.
2. $\forall \nu \in {}^{\mu}\lambda(\{\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) : \alpha < \mu\}$ is consistent $)$.
3. $\forall \eta \in {}^{<\mu}\lambda(\{\mathbf{tp}(a_{\eta^\wedge\alpha}/B_{\eta^\wedge\alpha}) : \alpha < \lambda\}$ is k-contradictory$)$.

Realize that $\lambda^{<\mu} = \lambda$, so doing a similar construction to that of Lemma 4.3.6 we have $M \in \mathbf{K}_\lambda$ and for each $\nu \in \lambda^\mu$ we fix $p_\nu = \mathbf{tp}(a_\nu/M_\nu)$ such that $M_\nu \in [M]^{\leq\mu}$ and $\forall \alpha < \mu(\mathbf{tp}(a_{\nu\restriction_\alpha}/B_{\nu\restriction_\alpha}) \leq p_\nu)$.

Observe that if $A \subseteq {}^{\mu}\lambda$ and $\{p_\nu : \nu \in A\}$ is consistent then the tree $\{\nu\restriction_\alpha : \alpha < \mu, \nu \in A\}$ is finitely branching by condition (3) of the tree property, hence $|A| \leq 2^\mu$. Therefore we can conclude that for all $\Delta \subseteq \{p_\nu : \nu \in \lambda^\mu\}$, if $|\Delta| \geq (2^\mu)^+$, then Δ is inconsistent.

Since $cf(\lambda) = \mu$, by König Lemma, we have that $\lambda^\mu = \beth_\mu(\mu)^\mu \geq \beth_\mu(\mu)^+ = \lambda^+$. We claim that $|\{p_\nu : \nu \in \lambda^\mu\}| \geq \lambda^+$. If it was not the case, then there would be $S \subseteq \lambda^\mu$ with $|S| = \lambda^+$ and $\{p_\mu : \nu \in S\}$ consistent; but this would contradict the previous paragraph since $(2^\mu)^+ < \beth_\mu(\mu)^+ = \lambda^+$. Hence

$$\lambda^+ \leq NT(\mu, \lambda, (2^\mu)^+). \tag{4.4.1}$$

On the other hand, by Theorem 4.4.2, we have that $NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\lambda_0} + 2^\mu$. Moreover, one can show that $\lambda^{\lambda_0} = \lambda$ and that $2^\mu \leq \lambda$, hence

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda. \tag{4.4.2}$$

The last two equations give us a contradiction. $\square$

The above proof can also be carried out in Shelah's context of good frames, see [Sh:h, §II] or [Ch. 2, §3] for the definition.

Corollary 4.4.5. *Let $\mathbf{K}$ be an AEC. If $\mathbf{K}$ has a type-full good $[\lambda_0, \infty)$-frame, then $\mathbf{K}$ does not have the tree property.*

Proof sketch. Using local character (in the sense of a good frame) it is easy to show by induction on $\|M\|$ that for every $p \in \mathbf{gS}(M)$ there is $N \in [M]^{\lambda_0}$ such that p does not fork over N. Using this fact together with the properties of type-full good $[\lambda_0, \infty)$-frame one can show that the proofs of Theorem 4.4.2 and Lemma 4.4.4 go through. $\square$

Remark 4.4.6. The above corollary goes through in the weaker setting of a type-full *good*$^-[\lambda_0, \infty)$-frame (see [Ch. 2, 3.5.(4)]). We do not know if it still goes through in the even weaker setting of w-good frames (see [Ch. 2, 3.7]).

4.4.1 Almost-stable independence relations

In this small subsection, we study what happens if instead of assuming uniqueness of extensions one assumes that the independence relation is contained in non-splitting. We show that this weaker assumption still implies stability of the AEC and the existence of a subμ-AEC with a stable independence relation. Moreover, the results in this subsection are used to obtain a new characterization of stable first-order theories assuming simplicity. A similar notion is studied in [ShVas18, §6] under stability assumptions.

A generalization of non-splitting to AECs was introduced in [BGKV16].

Definition 4.4.7 ([BGKV16, 3.14])**.** We say that A *does not explicitly split* from B over M, denoted by $A \overset{(nes)}{\downfree}_M B$, if and only if for every $B_1, B_2 \subseteq B$, if $\mathbf{tp}(B_1/M) = \mathbf{tp}(B_2/M)$ then $\mathbf{tp}(AB_1/M) = \mathbf{tp}(AB_2/M)$.

To ease the reference to stable independence relations without uniqueness but contained in explicitly non-splitting, we introduce the following notion.

Definition 4.4.8. $\downfree$ is an *almost-stable independence relation* in $\mathbf{K}$ if the following hold:

1. $\downfree$ is an independence relation.
2. (Symmetry) $A \downfree_M B$ if and only if $B \downfree_M A$.
3. (Local character) For each cardinal α there exists a cardinal λ (depending on α) such that: If $p \in \mathbf{gS}^\alpha(M)$, then there exists $M_0 \leq_{\mathbf{K}} M$ with $||M_0|| \leq \lambda$ and p does not fork over M_0. Recall that $\kappa_\alpha(\downfree)$ is the least λ given a fixed cardinal α.
4. $\downfree \subseteq \overset{(nes)}{\downfree}$.

Remark 4.4.9. It follows from [BGKV16, 4.2] that if $\overline{\downarrow}$ is a stable independence relation, then $\overline{\downarrow} \subseteq \overline{\overset{(nes)}{\downarrow}}$. Hence, a stable independence relation is an almost-independence relation.

We begin by showing that a class with an almost-stable independence relation is tame. This extends [LRV19, 8.16] as they prove it for stable independence relations.

Lemma 4.4.10. *If $\overline{\downarrow}$ is an almost-stable independence relation, then $\mathbf{K}$ is $\kappa_{2\alpha}(\overline{\downarrow})$-tame for types of length α.*

Proof. Let $N \in \mathbf{K}$ and $p, q \in \mathbf{gS}^\alpha(N)$ such that $p \restriction_D = q \restriction_D$ for every $D \in \mathcal{P}_{\leq \kappa_{2\alpha}(\overline{\downarrow})}(N)$. Assume that $p = \mathbf{tp}(\mathbf{a}/N)$ and $q = \mathbf{tp}(\mathbf{b}/N)$ for $\mathbf{a}, \mathbf{b} \in \mathcal{C}^\alpha$.

Consider $\mathbf{tp}(\mathbf{ab}/N)$, then by local character there is $N_0 \leq_{\mathbf{K}} N$ such that $\mathbf{tp}(\mathbf{ab}/N)$ does not fork over N_0 and $\|N_0\| \leq \kappa_{2\alpha}(\overline{\downarrow})$. By symmetry and the hypothesis that $\overline{\downarrow} \subseteq \overline{\overset{(nes)}{\downarrow}}$ we have that:

$$N \overline{\overset{(nes)}{\downarrow}}_{N_0} \mathbf{ab}.$$

Since $\mathbf{tp}(\mathbf{a}/N_0) = p \restriction_{N_0} = q \restriction_{N_0} = \mathbf{tp}(\mathbf{b}/N_0)$ because N_0 is small, we have by the definition of explicitly non-splitting that $\mathbf{tp}(\mathbf{a}N/N_0) = \mathbf{tp}(\mathbf{b}N/N_0)$. Hence $p = q$. $\square$

The next result is the key result for many of the arguments given in this subsection. The idea of the proof is similar to that of the proof of the weak uniqueness property given in [Van06, Theorem I.4.12].

Lemma 4.4.11. *Let μ, κ be infinite cardinals. Assume $\overline{\downarrow}$ is an almost-stable independence relation and $\mu \geq \kappa_\kappa(\overline{\downarrow})$. If M is μ^+-model homogeneous, $M \leq_{\mathbf{K}} N$, $p, q \in \mathbf{gS}^{<\infty}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p^{I_0} \restriction_A = q^{I_0} \restriction_A$ for every $A \in \mathcal{P}_{<\mu^+}(N)$ and $I_0 \in \mathcal{P}_{<\kappa}(|p|)$.*

Proof. Let A, I_0 be as required and assume that $p = \mathbf{tp}(\mathbf{a}/N)$, $q = \mathbf{tp}(\mathbf{b}/N)$ for $\mathbf{a}, \mathbf{b} \in \mathcal{C}^\alpha$ and α is an ordinal.

Consider $p^{I_0} \restriction_M$ and $q^{I_0} \restriction_M$ then by local character, base monotonicity and using that $|I_0| < \kappa$ there is $L \leq_{\mathbf{K}} M$ such that $p^{I_0} \restriction_M, q^{I_0} \restriction_M$ do not fork over L and $\|L\| \leq \kappa_\kappa(\overline{\downarrow}) \leq \mu$.

Let L' be the structure obtained by applying downward Löwenheim-Skolem to $L \cup A$ in N, observe that $\|L'\| \leq \mu$. Since M is μ^+-model homogeneous, there is $f : L' \underset{L}{\rightarrow} M$.

Then by monotonicity, transitivity and the fact that $\overline{\downarrow} \subseteq \overline{\overset{(nes)}{\downarrow}}$, we obtain that:

$$\mathbf{a}\restriction_{I_0} \overset{(nes)}{\overline{\downarrow}}_L N \text{ and } \mathbf{b}\restriction_{I_0} \overset{(nes)}{\overline{\downarrow}}_L N.$$

Let $C_1 = L'$ and $C_2 = f[L']$. Realize that $L \leq_{\mathbf{K}} C_1, C_2 \leq_{\mathbf{K}} N$ and $\mathbf{tp}(C_1/L) = \mathbf{tp}(C_2/L)$, then by the above equations, the definition of explicitly non-splitting and the choice of C_1, C_2 we obtain that:

$$\mathbf{tp}(\mathbf{a}\restriction_{I_0} L'/L) = \mathbf{tp}(\mathbf{a}\restriction_{I_0} f[L']/L) \text{ and } \mathbf{tp}(\mathbf{b}\restriction_{I_0} L'/L) = \mathbf{tp}(\mathbf{b}\restriction_{I_0} f[L']/L).$$

Since by hypothesis $p\restriction_M = q\restriction_M$ and $f[L'] \leq_{\mathbf{K}} M$, we have that $\mathbf{tp}(\mathbf{a}\restriction_{I_0} /f[L']) = \mathbf{tp}(\mathbf{b}\restriction_{I_0} /f[L'])$. Then it follows that $\mathbf{tp}(\mathbf{a}\restriction_{I_0} f[L']/L) = \mathbf{tp}(\mathbf{b}\restriction_{I_0} f[L']/L)$. Hence $\mathbf{tp}(\mathbf{a}\restriction_{I_0} L'/L) = \mathbf{tp}(\mathbf{b}\restriction_{I_0} L'/L)$. Therefore, as $A \subseteq L'$, we conclude that $p^{I_0}\restriction_A = q^{I_0}\restriction_A$. $\square$

Remark 4.4.12. For $\mathbf{K}$ an AEC with joint embedding, amalgamation and no maximal models, one can show, like in first-order, that if $\lambda \geq \kappa > \text{LS}(\mathbf{K})$, $M \in \mathbf{K}_{\leq\lambda}$ and $\lambda^{<\kappa} = \lambda$, then there is $N \in \mathbf{K}_\lambda$ such that N is κ-Galois-saturated extending M. Moreover, N is κ-model homogeneous as Shelah showed the equivalence between saturation and model homogeneity for AECs in [Sh:h, §II.1.14].

We obtain a bound for almost-stable independence relations.

Theorem 4.4.13. *If $\overline{\downarrow}$ is an almost-stable independence relation, then*

$$NT(\mu, \lambda, \kappa) \leq \lambda^{(2^{\kappa_2(\overline{\downarrow})})} + \kappa^-.$$

Proof. Let $\lambda_0 = \kappa_2(\overline{\downarrow})$, $\chi = \lambda^{2^{\lambda_0}} + \kappa^-$ and $\{p_\alpha : \alpha < \chi^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ for $M \in \mathbf{K}_\lambda$.

Observe that by the above remark there is M' extending M such that M' is $(2^{\lambda_0})^+$-model homogeneous and $\|M'\| = \lambda^{2^{\lambda_0}}$. For each $\alpha < \chi^+$, fix $q_\alpha \in \mathbf{gS}(M')$ such that $p_\alpha \leq q_\alpha$, this exist by amalgamation. Moreover, given $\alpha < \chi^+$, by local character there is $N \in K_{\lambda_0}$ such that q_α does not fork over N. Since $(2^{\lambda_0})^{\lambda_0} = 2^{\lambda_0}$, by the remark above there is N' extending N such that N' is (λ_0^+)-model homogeneous and $\|N'\| = 2^{\lambda_0}$. Since M' is $(2^{\lambda_0})^+$-model homogeneous, there is $f : N' \underset{N}{\rightarrow} M'$. So fix $N_\alpha = f[N']$, realize $N_\alpha \in \mathbf{K}_{2^{\lambda_0}}$, N_α is (λ_0^+)-model homogeneous and q_α does not fork over N_α by base monotonicity.

Define $\Phi : \chi^+ \to [M']^{2^{\lambda_0}}$ as $\Phi(\alpha) = N_\alpha$. Then by the pigeonhole principle there is $N^* \in [M']^{2^{\lambda_0}}$ and $S \subseteq \chi^+$ of cardinality χ^+ such that $N_\alpha = N^*$ for every $\alpha \in S$. Now define $\Psi : S \to \mathbf{gS}(N^*)$ as $\Psi(\alpha) = q_\alpha\restriction_{N^*}$, since $|\mathbf{gS}(N^*)| \leq 2^{2^{\lambda_0}}$, by the pigeonhole principle there is $q \in \mathbf{gS}(N^*)$ and $S' \subseteq S$ of size χ^+ such that $q_\alpha\restriction_{N^*} = q$ for every $\alpha \in S'$

Observe that $q_\alpha \geq q$ and q_α does not fork over N^* for every $\alpha \in S'$. Then since N^* is (λ_0^+)-model homogeneous and $\mathbf{K}$ is λ_0-tame (by Lemma 4.4.10), it follows from Lemma 4.4.11 that $q_\alpha = q_\beta$ for every $\alpha, \beta \in S'$. In particular, $\{p_\alpha : \alpha \in S'\}$ is consistent and $|S'| \geq \kappa$. Hence $NT(\mu, \lambda, \kappa) \leq \lambda^{2^{\lambda_0}} + \kappa^-$. □

The next results show that having an almost-stable independence relation implies that $\mathbf{K}$ is stable and that $\mathbf{K}$ does not have the tree property.

Corollary 4.4.14. *If $\overline{\smile\!\!\!\!\!\!\perp}$ is an almost-stable independence relation, then $\mathbf{K}$ is stable and $\mathbf{K}$ does not have the tree property.*

Proof. We show that $\mathbf{K}$ does not have the tree property by contradiction, the proof that $\mathbf{K}$ is stable is straightforward. Let $\mu = (2^{\kappa_2(\overline{\smile\!\!\!\!\!\!\perp})})^+$ and $\lambda = \beth_\mu(\mu)$. Since $\lambda^{<\mu} = \lambda$, the same construction as that of Lemma 4.4.4 gives us that:

$$\lambda^+ \leq NT(\mu, \lambda, (2^\mu)^+).$$

On the other hand, by the previous theorem we have that:

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{2^{\kappa_2(\overline{\smile\!\!\!\!\!\!\perp})}} + 2^\mu = \lambda.$$

Putting together the last two equation we get a contradiction. □

The next result shows that an almost-stable independence relation is close to being a stable independence relation. Recall that $\mathbf{K}^{\mu^+\text{-mh}}$ is the μ^+-AEC (see [BGLRV16]) which models are the μ^+-model homogeneous models of $\mathbf{K}$ and which order is the same as that of $\mathbf{K}$.

Lemma 4.4.15. *Assume $\mathbf{K}$ is fully $(<\kappa)$-tame and -type-short. If $\overline{\smile\!\!\!\!\!\!\perp}$ is an almost-stable independence relation and $\mu \geq \kappa_\kappa(\overline{\smile\!\!\!\!\!\!\perp}) + \kappa$, then $\mathbf{K}^{\mu^+\text{-mh}}$ has a stable independence relation. This is precisely the restriction of $\overline{\smile\!\!\!\!\!\!\perp}$ to μ^+-model homogeneous models.*

Proof. A big monster model of $\mathbf{K}$ is a monster model of $\mathbf{K}^{\mu^+\text{-mh}}$. For $M \in \mathbf{K}^{\mu^+\text{-mh}}$, $A, B \subseteq \mathcal{C}$ define:

$$A \overset{(*)}{\overline{\smile\!\!\!\!\!\!\perp}}_M B \text{ if and only if } A \overline{\smile\!\!\!\!\!\!\perp}_M B.$$

We claim that $\overset{(*)}{\overline{\smile\!\!\!\!\!\!\perp}}$ is a stable independence relation in $\mathbf{K}^{\mu^+\text{-mh}}$. It is straightforward to show that it is an independence relation that satisfies symmetry. Uniqueness follows from Lemma 4.4.11. As for local character, we have that given α and $p \in \mathbf{gS}^\alpha(M)$ with $M \in \mathbf{K}^{\mu^+\text{-mh}}$ there is $N \in \mathbf{K}^{\mu^+\text{-mh}}$ such that p does not $\overset{(*)}{\overline{\smile\!\!\!\!\!\!\perp}}$-forks over N and $\|N\| \leq \kappa_\alpha(\overline{\smile\!\!\!\!\!\!\perp}) + \mathrm{LS}(\mathbf{K})^\mu$. □

We finish this section by showing that the results in this subsection can be used to obtain a new characterization of stability assuming simplicity for first-order theories. In order to present it, let us recall the notion of non-splitting for first-order theories. A complete type p in $\bar{x}$ does not split over A a subset of the monster model if and only if for every $\bar{a}, \bar{b} \in Dom(p)$ and $\phi(\bar{x}, \bar{y})$ first-order formula, if $tp(\bar{a}/A) = tp(\bar{b}/A)$, then $\phi(\bar{x}, \bar{a}) \in p$ if and only if $\phi(\bar{x}, \bar{b}) \in p$. This notion was introduced by Shelah in Definition 2.2 of [Sh3].

Lemma 4.4.16. *Let T be a simple complete first-order theory. The following are equivalent.*

1. $\underset{M}{\mathop{\smile}\limits^{|}} \subseteq \underset{M}{\overset{(ns)}{\mathop{\smile}\limits^{|}}}$ *for every M model of T, where $\mathop{\smile}\limits^{|}$ denotes first-order non-forking and $\overset{(ns)}{\mathop{\smile}\limits^{|}}$ denotes first-order non-splitting.*

2. *T is stable.*

Proof. $\Rightarrow$: Lemma 4.4.11, Theorem 4.4.13 and Corollary 4.4.14 can be carried out if one replaces explicitly non-splitting for non-splitting in complete first-order theories.

$\Leftarrow$: Since T is stable, non-forking has uniqueness (stationarity) over models. Under this hypothesis it is easy to show that $\underset{M}{\mathop{\smile}\limits^{|}} \subseteq \underset{M}{\overset{(ns)}{\mathop{\smile}\limits^{|}}}$ for every M model of T (a proof is given in [BGKV16, 4.2]). $\square$

4.5 Simple independence relations

We introduce simple independence relations and begin their study. We bound the possible values of $NT(\text{-},\text{-},\text{-})$ under the existence of a simple independence relation and as a corollary we are able to show the failure of the 2-tree property. As in the previous section we are assuming Hypothesis 4.2.1.

Definition 4.5.1. $\overline{\mathop{\smile}\limits^{|}}$ is a *simple independence relation* in $\mathbf{K}$ if the following properties hold:

1. $\overline{\mathop{\smile}\limits^{|}}$ is an independence relation.

2. (Symmetry) $A \overline{\mathop{\smile}\limits^{|}}_M B$ if and only if $B \overline{\mathop{\smile}\limits^{|}}_M A$.

3. (Type-amalgamation) If $p \in \mathbf{gS}^{<\infty}(M)$, $M \subseteq A, B \subseteq \mathcal{C}$ and $A \overline{\mathop{\smile}\limits^{|}}_M B$, then for all $q_1 \in \mathbf{gS}^{<\infty}(A; \mathcal{C}), q_2 \in \mathbf{gS}^{<\infty}(B; \mathcal{C})$ and $N^* \supseteq A, B$ such that $q_1, q_2 \geq p$ and q_1, q_2 do not fork over M, there exists $q \in \mathbf{gS}^{<\infty}(N^*)$ such that $q \geq q_1, q_2$ and q does not fork over M.

4. (Uniform local character) There exists θ and λ cardinals such that: If $p \in \mathbf{gS}^{\alpha}(M)$, then there exists $M_0 \leq_{\mathbf{K}} M$ with $||M_0|| \leq \lambda + \alpha^{<\theta}$ and p does not fork over M_0. Recall that $(\kappa(\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}), \ell(\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}))$ are the least (λ, θ) with such a property.

Remark 4.5.2. Let T be a complete first-order theory. If T is simple and $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ is first-order non-forking, then $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ is a simple independence relation.

Remark 4.5.3. The only difference between stable independence relations and simple independence relations are conditions (3) and (4). As for (3), while we assume uniqueness in stable independence relations, we only assume type-amalgamation in simple independence relations. Although this may seem like a minor change, based on our knowledge of forking in first-order theories this is actually a significant one. As for (4), this is a minor change and we give natural conditions under which local character implies uniform local character (see Fact 4.5.5 and Corollary 4.5.8).

The next strengthening of the witness property is the key property to show that stable independence relations are simple independence relations if the AEC is tame and type-short.

Definition 4.5.4. Let $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ be an independence relation. $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ has the $(< \theta)$-*strong witness property* if for all $M \leq_{\mathbf{K}} N$, α ordinals, and $\mathbf{b} \in \mathcal{C}^{\alpha}$: $\mathbf{b} \mathop{\smile\!\!\!\!\!\!\!\!\:\mid}_M N$ if and only if $\mathbf{b} \restriction_I \mathop{\smile\!\!\!\!\!\!\!\!\:\mid}_M A$ for every $A \in \mathcal{P}_{<\theta}(N)$ and $I \in \mathcal{P}_{<\theta}(\alpha)$.

The proof of the following fact is the same as that of [LRV19, 8.10], since the hypothesis are slightly different and the proof is short we repeat the argument for the convenience of the reader.

Fact 4.5.5. Let $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ be an independence relation. If $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ has local character and the $(< \theta)$-strong witness property, then $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ has uniform local character.

Proof. Since $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ has local character, for each $\alpha < \theta$ we have that $\kappa_{\alpha}(\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}) < \infty$. Let $\lambda_0 = sup\{\kappa_{\alpha}(\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}) : \alpha < \theta\}$. We show that the pair (λ_0, θ) is a witness for uniform local character.

Let $M \in \mathbf{K}$ and $p = \mathbf{tp}(\mathbf{b}/M) \in \mathbf{gS}^{\beta}(M)$. For each $I \subseteq \beta$ with $|I| < \theta$, let $M_I \in [M]^{\lambda_0}$ such that $\mathbf{b} \restriction_I \mathop{\smile\!\!\!\!\!\!\!\!\:\mid}_{M_I} M$, this exists by the choice of λ_0. Let $A = \bigcup_{I \subseteq \beta, |I| < \theta} M_I$ and M_0 be the structure obtained by applying downward Löwenheim-Skolem in M to A. Observe that $\|M_0\| \leq \lambda_0 + \beta^{<\theta}$ and the $(< \theta)$-strong witness property together with monotonicity imply that $\mathbf{b} \mathop{\smile\!\!\!\!\!\!\!\!\:\mid}_{M_0} M$. $\square$

The next lemma gives a condition under which a stable independence relation is a simple independence relation.

Lemma 4.5.6. *If $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ is a stable independence relation that has the $(< \theta)$-strong witness property, then $\mathop{\smile\!\!\!\!\!\!\!\!\:\mid}$ is a simple independence relation.*

Proof. We only need to check properties (3) and (4). As for (4), this follows from Fact 4.5.5. So we only need to show the type-amalgamation property.

Let $p \in \mathbf{gS}^{<\infty}(M)$, $M \subseteq A, B \subseteq \mathcal{C}$, $A \mathop{\smile\!\!\!\!\!\!\mid\,}_M B$, $q_1 \in \mathbf{gS}^{<\infty}(A; \mathcal{C})$ and $q_2 \in \mathbf{gS}^{<\infty}(B; \mathcal{C})$ and $N^* \supseteq A, B$ such that $q_1, q_2 \geq p$ and q_1, q_2 do not fork over M. Since $p \in \mathbf{gS}^{<\infty}(M)$ and $M \leq_{\mathbf{K}} N^*$, by the extension property there is $q \in \mathbf{gS}^{<\infty}(N^*)$ such that $q \geq p$ and q does not fork over M.

Observe that $q \restriction_A, q_1 \in \mathbf{gS}^{<\infty}(A, \mathcal{C})$, $q \restriction_A, q_1$ do not fork over M and $(q \restriction_A) \restriction_M = p = q_1 \restriction_M$, then by the uniqueness property ((3) of Definition 4.4.1) we have that $q \restriction_A = q_1$. Hence $q_1 \leq q$. One can similarly show that $q \restriction_B = q_2$.

Therefore, $q \geq q_1, q_2$ and q does not fork over M. $\square$

The next assertion gives a natural assumption on $\mathbf{K}$ that implies the $(< \theta)$-strong witness property. The proof is similar to that of [LRV19, 8.8], but we obtain a stronger result.

Fact 4.5.7. If $\mathbf{K}$ is fully $(< \theta)$-tame and -type-short and $\mathop{\smile\!\!\!\!\!\!\mid\,}$ is a stable independence relation, then $\mathop{\smile\!\!\!\!\!\!\mid\,}$ has the $(< \theta)$-strong witness property.

Proof. Let $M \leq_{\mathbf{K}} N$ and $\mathbf{b} \in \mathcal{C}^\alpha$ such that $\mathbf{b} \restriction_I \mathop{\smile\!\!\!\!\!\!\mid\,}_M A$ for every $A \in \mathcal{P}_{<\theta}(N)$ and $I \in \mathcal{P}_{<\theta}(\alpha)$. Let $p = \mathbf{tp}(\mathbf{b}/N)$ and $q \in \mathbf{gS}(N)$ such that q does not fork over M and q extends $p \restriction_M$, q exists because of the extension property. Using that $\mathbf{K}$ is fully $(< \theta)$-tame and -type-short together with the uniqueness property one can show that $p = q$. As q does not fork over M by construction, it follows that $\mathbf{b} \mathop{\smile\!\!\!\!\!\!\mid\,}_M N$. $\square$

Corollary 4.5.8. *If $\mathbf{K}$ is fully $(< \theta)$-tame and -type-short and $\mathop{\smile\!\!\!\!\!\!\mid\,}$ is a stable independence relation, then $\mathop{\smile\!\!\!\!\!\!\mid\,}$ is a simple independence relation.*

The next technical proposition is important as it shows that even when we are considering independence relations over sets in some sense models are ubiquitous

Proposition 4.5.9. *Let $\mathop{\smile\!\!\!\!\!\!\mid\,}$ be a simple independence relation. If $A \mathop{\smile\!\!\!\!\!\!\mid\,}_M B$, then there is $M^* \in \mathbf{K}$ with $B \cup M \subseteq M^*$ and $A \mathop{\smile\!\!\!\!\!\!\mid\,}_M M^*$.*

Proof. Assume $A \mathop{\smile\!\!\!\!\!\!\mid\,}_M B$. By normality and monotonicity we can conclude that $A \mathop{\smile\!\!\!\!\!\!\mid\,}_M B \cup M$. Let $M' \in \mathbf{K}$ the structure obtained by applying downward Löwenheim-Skolem in $\mathcal{C}$ to $M \cup B \subseteq M'$.

Consider $p = \mathbf{tp}(A/M)$, $q_1 = \mathbf{tp}(A/M \cup B)$ and $q_2 = \mathbf{tp}(A/M)$. Observe that $p \leq q_1, q_2$, $q_1 \in \mathbf{gS}^{<\infty}(M \cup B; \mathcal{C})$ does not fork over M, $q_2 \in \mathbf{gS}^{<\infty}(M)$ does not fork over M, $M \subseteq M \cup B, M \subseteq M'$ and $M \cup B \mathop{\smile\!\!\!\!\!\!\mid\,}_M M$. Recognize that p, q_1, q_2 and $M \subseteq M, M \cup B \subseteq M'$ satisfy the hypothesis of the type-amalgamation property, then there is $r \in \mathbf{gS}^{<\infty}(M') \geq q_1, q_2$ such that r does not fork over M.

Suppose that $r = \mathbf{tp}(A'/M')$, since $r \geq q_1$ there is $f \in Aut_{M \cup B}(\mathcal{C})$ such that $f[A'] = A$. Since r does not fork over M, we have that $A' \mathop{\smile\!\!\!\!\!\!\mid\,}_M M'$. Then by invariance $f[A'] \mathop{\smile\!\!\!\!\!\!\mid\,}_{f[M]} f[M']$. Observe $f[A'] = A$, $f[M] = M$, so $A \mathop{\smile\!\!\!\!\!\!\mid\,}_M f[M']$. Finally, realize that $M \cup B \subseteq f[M']$, hence $M^* := f[M']$ satisfies what is needed. $\square$

The following notion generalizes the chain condition introduced in [Les00, 2.3].

Definition 4.5.10. Let ι be an infinite cardinal. We say $\overline{\downarrow}$ has the ι-bound condition if: $\forall \lambda \in [\kappa(\overline{\downarrow}), \infty)\forall M \in \mathbf{K}_\lambda \forall \kappa \in [\mathrm{LS}(\mathbf{K}), \lambda] \forall p \in \mathbf{gS}(M, \kappa) \forall \mu \in [\kappa(\overline{\downarrow}) + \kappa, \lambda]$(If $\mu^{<\ell(\overline{\downarrow})} = \mu$ and $\{p_\alpha : \alpha < (2^\mu)^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ are such that p_α is a non-forking extension of p for every $\alpha < (2^\mu)^+$, then there are $A \subseteq (2^\mu)^+$ and q a type such that $|A| = \iota$ and q is an extension of p_α for every $\alpha \in A$). Moreover, we say that $\overline{\downarrow}$ has the strong ι-bound condition if the type q is a non-forking extension of p.

The following is a generalization of [Les00, 2.4], which is based on an argument of Shelah which appeared in [GIL02, 4.9]. Compared to [Les00, 2.4], instead of showing that two types are comparable we show that countably many types are comparable, [Les00, 2.5] mentions that this can be done in the first-order case. We have decided to present the argument to show that it does come through in this more general setting and because we will extend it in Lemma 4.6.1.

Lemma 4.5.11. *If $\overline{\downarrow}$ is a simple independence relation, then $\overline{\downarrow}$ has the $\aleph_0$-bound condition.*[6]

Proof. Let $\lambda, \mu, \kappa \in Car$, $M \in \mathbf{K}_\lambda$, $R \in [M]^\kappa$, $p \in \mathbf{gS}(R)$ and $\{p_\alpha \in \mathbf{gS}(N_\alpha) : \alpha < (2^\mu)^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ be as in the definition of the $\aleph_0$-bound condition. By extension and transitivity, we may assume that all N_α have size μ.

We build $\{M_\alpha : \alpha < (2^\mu)^+\}$ strictly increasing and continuous chain such that:

1. $\forall \alpha \in (2^\mu)^+ (M_\alpha \in \mathbf{K}_{2^\mu})$.
2. $R \leq_{\mathbf{K}} M_0$.
3. $\forall \alpha \in (2^\mu)^+ (N_\alpha \leq_{\mathbf{K}} M_{\alpha+1})$

Let $S = \{\alpha < (2^\mu)^+ : cf(\alpha) = \mu^+\}$ and $\Phi : S \to (2^\mu)^+$ be defined as $\Phi(\alpha) = min\{\beta : \mathbf{tp}(N_\alpha / M_\alpha)$ does not fork over $M_\beta\}$. Observe that Φ is regressive by local character and the fact that $\mu^{<\ell(\overline{\downarrow})} = \mu$. Then by Fodor's lemma there is $S^* \subseteq S$ stationary and $\alpha^* < (2^\mu)^+$ such that $\forall \alpha \in S^* (\mathbf{tp}(N_\alpha / M_\alpha)$ does not fork over $M_{\alpha^*})$. We may assume without loss of generality that $S = S^*$ and $\alpha^* = 0$. Hence,

$$\forall \alpha \in S(\mathbf{tp}(N_\alpha / M_\alpha) \text{ does not fork over } M_0). \tag{4.5.1}$$

By local character and using again that $\mu^{<\ell(\overline{\downarrow})} = \mu$ we have that for all $\alpha \in S$ there is $R_\alpha \in [M_0]^\mu$ such that $\mathbf{tp}(N_\alpha / M_\alpha) \restriction_{M_0}$ does not fork over R_α. Define $\Psi : S \to [M_0]^\mu$ as $\Psi(\alpha) = R_\alpha$. Then by the pigeonhole principle, since $|[M_0]^\mu| = 2^\mu$, we may assume that there is a $R^* \in [M_0]^\mu$ such that:

[6] Symmetry is not used to obtain this result.

$$\forall \alpha \in S(\mathbf{tp}(N_\alpha/M_\alpha)\restriction_{M_0} \text{ does not fork over } R^*). \tag{4.5.2}$$

By base monotonicity we may further assume that $R \leq_{\mathbf{K}} R^*$. Then applying transitivity to the previous two equations we obtain that:

$$\forall \alpha \in S(N_\alpha \mathop{\smile\!\!\!\!\!\!\;|}\nolimits_{R^*} M_\alpha). \tag{4.5.3}$$

Moreover, given $\alpha \in S$ $p_\alpha \in \mathbf{gS}(N_\alpha)$ does not fork over R and $N_\alpha \leq_{\mathbf{K}} M_{\alpha+1}$. Applying extension and transitivity, there is $q_\alpha \in \mathbf{gS}(M_{\alpha+1})$ extending p_α and q_α does not fork over R. By base monotonicity, since $R \leq_{\mathbf{K}} R^* \leq_{\mathbf{K}} M_{\alpha+1}$, we also have that q_α does not fork over R^*.

Let $\Upsilon : S \to \mathbf{gS}(R^*)$ be defined as $\Upsilon(\alpha) = q_\alpha \restriction_{R^*}$, by the pigeonhole principle we may assume that there is $q \in \mathbf{gS}(R^*)$ such that:

$$\forall \alpha \in S(q_\alpha \geq q \text{ and } q_\alpha \text{ does not fork over } R^*). \tag{4.5.4}$$

Let $\{\alpha_n : n \in \omega\} \subseteq S$ be an increasing set of ordinals. We build $\{r_n : n \in \omega\}$ such that:

1. $r_0 = q_{\alpha_0}$.
2. $r_{n+1} \geq r_n, p_{\alpha_{n+1}}$.
3. $r_n \in \mathbf{gS}(M_{\alpha_n+1})$.
4. r_n does not fork over R.

The base step is given so let us do the induction step. By equation (4.5.3) $N_{\alpha_{n+1}} \mathop{\smile\!\!\!\!\!\!\;|}\nolimits_{R^*} M_{\alpha_{n+1}}$. Since $\alpha_n + 1 \leq \alpha_{n+1} \in S$, we have that $M_{\alpha_n+1} \leq_{\mathbf{K}} M_{\alpha_{n+1}}$, so by monotonicity $N_{\alpha_{n+1}} \mathop{\smile\!\!\!\!\!\!\;|}\nolimits_{R^*} M_{\alpha_n+1}$ and by normality we have that $N_{\alpha_{n+1}} \cup R^* \mathop{\smile\!\!\!\!\!\!\;|}\nolimits_{R^*} M_{\alpha_n+1}$. Realize that $q \in \mathbf{gS}(R^*)$, $q_{\alpha_{n+1}} \restriction_{N_{\alpha_{n+1}} \cup R^*} \in \mathbf{gS}(N_{\alpha_{n+1}} \cup R^*; \mathcal{C})$, $r_n \in \mathbf{gS}(M_{\alpha_n+1})$ and $M_{\alpha_{n+1}+1}$ substituted by p, q_1, q_2 and N^* satisfy the hypothesis of the type-amalgamation property. Therefore there is $r_{n+1} \in \mathbf{gS}(M_{\alpha_{n+1}+1})$ such that $r_{n+1} \geq q_{\alpha_{n+1}} \restriction_{N_{\alpha_{n+1}} \cup R^*}, r_n$ and r_{n+1} does not fork over R^*.

In particular we have that $r_{n+1} \geq r_n, p_{\alpha_{n+1}}$ (since $q_{\alpha_{n+1}} \geq p_{\alpha_{n+1}}$) and by transitivity (since $r_{n+1} \geq r_n$, $R^* \leq M_{\alpha_n+1}$, and r_n does not fork over R) we have that r_{n+1} does not for over R. This finishes the construction.

Finally $\{r_n \in \mathbf{gS}(M_{\alpha_n+1}) : n \in \omega\}$ is an increasing chain of types so by [Bal09, 11.3], there is $r^* \in \mathbf{gS}(\bigcup_{n\in\omega} M_{\alpha_n+1})$ such that $r^* \geq r_n$ for each $n \in \omega$. In particular, by clause (2) of the construction, we have that r^* extends p_{α_n} for every $n < \omega$, which is precisely what we needed to show. $\square$

The following generalizes [Les00, A] to the AEC context. The proof is similar to that of Theorem 4.4.2, but using the $\aleph_0$-bound condition instead of the uniqueness property.

Theorem 4.5.12. *If $\mathop{\smile\!\!\!\!\!\!\,|}$ is a simple independence relation, $\kappa(\mathop{\smile\!\!\!\!\!\!\,|}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\mathop{\smile\!\!\!\!\!\!\,|})} = \mu$, then*

$$NT(\mu, \lambda, \aleph_0) \leq \lambda^{\kappa(\mathop{\smile\!\!\!\!\!\!\,|})} + 2^{\mu}.$$

In particular, $NT(\mu, \lambda) \leq \lambda^{\kappa(\mathop{\smile\!\!\!\!\!\!\,|})} + 2^{\mu}$

Proof. Let $\lambda_0 = \kappa(\mathop{\smile\!\!\!\!\!\!\,|})$, $\chi = \lambda^{\lambda_0} + 2^{\mu}$ and $\{p_\alpha \in \mathbf{gS}(N_\alpha) : \alpha < \chi^+\} \subseteq \mathbf{gS}(M, \leq \mu)$ where $M \in \mathbf{K}_\lambda$. Observe that by the extension property we may assume that each $N_\alpha \in \mathbf{K}_\mu$. As in the proof of Theorem 4.4.2 there are $S \subseteq \chi^+$ of size χ^+, $R \in [M]^{\lambda_0}$ and $p \in \mathbf{gS}(R)$ such that for every $\alpha \in S$ $p_\alpha \geq p$ and p_α does not fork over R.

By the $\aleph_0$-bound condition, where the cardinal parameters are as in the definition except that $\kappa := \lambda_0$ and all the model theoretic parameters are the same with $\{p_\alpha : \alpha \in S\}$ being the collection of types and $dom(p) = R$, we obtain that there are countable $A \subseteq S$ and q a type such that $q \geq p_\alpha$ for each $\alpha \in A$. In particular $\{p_\alpha : \alpha \in A\}$ is consistent. Hence $NT(\mu, \lambda, \aleph_0) \leq \lambda^{\lambda_0} + 2^{\mu}$. $\square$

Remark 4.5.13. Observe that when $\mathop{\smile\!\!\!\!\!\!\,|}$ is a stable or almost-stable independence relation Theorems 4.4.2 and 4.4.13 give us a better bound. Moreover, Theorems 4.4.2 and 4.4.13 give us a bound for each $\kappa \in Car$ while the above corollary only gives us a bound when κ is countable, as we will see in Theorem 4.6.2 more can be said if we assume the $(< \aleph_0)$-witness property.

The following result shows that we can not have the 2-tree property if $\mathbf{K}$ has a simple independence relation.

Corollary 4.5.14. *If $\mathop{\smile\!\!\!\!\!\!\,|}$ is a simple independent relation, then $\mathbf{K}$ does not have the 2-tree property.*

Proof. Suppose for the sake of contradiction that $\mathbf{K}$ has the 2-tree property.

Let $\lambda_0 = \kappa(\mathop{\smile\!\!\!\!\!\!\,|})$, $\mu = (\beth_{(\aleph_0 + \ell(\mathop{\smile\!\!\!\!\!\!\,|}))^+}(\lambda_0^+))^+$ and $\lambda = \beth_\mu(\mu)$. Observe that the following cardinal arithmetic equalities hold:

1. $\mu^{<\ell(\mathop{\smile\!\!\!\!\!\!\,|})} = \mu$, using that $cf(\beth_{(\aleph_0 + \ell(\mathop{\smile\!\!\!\!\!\!\,|}))^+}(\lambda_0^+)) = (\aleph_0 + \ell(\mathop{\smile\!\!\!\!\!\!\,|}))^+$ and Hausdorff formula.

2. $\lambda^{\lambda_0} + 2^{\mu} = \lambda$, using that $cf(\lambda) = \mu > \lambda_0$ and that $\beth_\mu(\mu) > 2^{\mu}$.

3. $\lambda^{<\mu} = \lambda$, using that $cf(\lambda) = \mu$.

Applying Theorem 4.5.12, this is possible by the first cardinal arithmetic equality, and by the second cardinal arithmetic equality we get that:

$$NT(\mu, \lambda) \leq \lambda^{\lambda_0} + 2^{\mu} = \lambda. \tag{4.5.5}$$

Applying Lemma 4.3.6, this is possible by the third cardinal arithmetic equality, we get that

$$\lambda^{\mu} \leq NT(\mu, \lambda). \tag{4.5.6}$$

So putting inequalities (7) and (8) we obtain that $\lambda^{\mu} \leq \lambda$, but this is a contradiction to König's Lemma since $cf(\lambda) = \mu$. $\square$

Remark 4.5.15. In the result above, instead of showing the failure of the 2-tree property, we would have liked to obtain the failure of the tree property. We will show in Corollary 4.6.3 that this is the case if $\mathop{\smile\!\!\!\!\!\!|}\,$ has the $(< \aleph_0)$-witness property for singletons.

4.6 Simple independent relations with the witness property

In this section we continue the study of simple independence relations under locality assumptions. We begin by showing the failure of the tree property under the existence of a simple independence relation with the $(< \aleph_0)$-witness property. Then we study simple independence relations with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property and obtain some basic results.

4.6.1 Failure of the tree property

The next argument extends the one presented in Lemma 4.5.11.

Lemma 4.6.1. *If $\mathop{\smile\!\!\!\!\!\!|}\,$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons, then $\mathop{\smile\!\!\!\!\!\!|}\,$ has the strong $(2^{\mu})^+$-bound condition.*

Proof sketch . Everything is the same as the proof of Lemma 4.5.11 until equation (4.5.4), but in this case instead of building only countably many $r_n's$ we will build $(2^{\mu})^+$ many of them.

Let $\{\alpha_i : i < (2^{\mu})^+\} \subseteq S$ be an increasing set of ordinals. We build $\{r_i : i < (2^{\mu})^+\}$, $\{a_i : i < (2^{\mu})^+\}$ and $\{f_{j,i} : j < i < (2^{\mu})^+\}$ such that:

1. $r_0 = q_{\alpha_0} = \mathbf{tp}(a_0/M_{\alpha_0+1})$.

2. If $k < j < i < (2^\mu)^+$, then $f_{k,i} = f_{j,i} \circ f_{k,j}$.

3. $\forall j < i(f_{j,i} \restriction_{M_{\alpha_j+1}} = \mathrm{id}_{M_{\alpha_j+1}}, f_{j,i}(a_j) = a_i$ and $f_{j,i} \in Aut(\mathcal{C}))$.

4. $r_i = \mathbf{tp}(a_i/M_{\alpha_i+1})$ does not fork over R.

5. $r_i \geq p_{\alpha_i}$.

6. $\forall j < i(r_j \leq r_i)$.

The construction in the successor step is similar to that of Lemma 4.5.11, so we only show how to do the the step when i is a limit ordinal. Since $\{r_j : j < i\}$, $\{a_j : j < i\}$ and $\{f_{k,j} : k < j < i\}$ is a directed system, by [Bal09, 11.3], there is $p^* = \mathbf{tp}(a^*/\bigcup_{j<i} M_{\alpha_j+1})$ upper bound for $\{r_j : j < i\}$ and $\{f^*_{j,i} : j < i\}$ satisfying (2) and (3) but with a^* substituted for a_i.

Using the $(< \aleph_0)$-witness property, invariance and monotonicity it is easy to show that p^* does not fork over R. Observe that $\bigcup_{j<i} M_{\alpha_j+1} \subseteq M_{\alpha_i}$, $N_{\alpha_i} \overline{\smile}_{R^*} M_{\alpha_i}$ (by equation (4.5.3) of Lemma 4.5.11) and $p^* \geq r_0$. Using these, one can show that $q \in \mathbf{gS}(R^*)$, $q_{\alpha_i} \restriction_{N_{\alpha_i} \cup R^*} \in \mathbf{gS}(N_{\alpha_i} \cup R^*; \mathcal{C})$, $p^* \in \mathbf{gS}(\bigcup_{j<i} M_{\alpha_j+1})$ and M_{α_i+1} substituted for p, q_1, q_2 and N^* satisfy the hypothesis of the type-amalgamation property. Therefore, there is $r_i \in \mathbf{gS}(M_{\alpha_i+1})$ such that $r_i \geq q_{\alpha_i} \restriction_{N_{\alpha_i} \cup R^*}, p^*$ and r_i does not fork over R^*.

Let $r_i := \mathbf{tp}(a_i/M_{\alpha_i+1})$. Since $r_i \restriction_{\bigcup_{j<i} M_{\alpha_j+1}} = p^*$, there is $g \in Aut(\mathcal{C})$ such that $g(a^*) = a_i$ and $g \restriction_{\bigcup_{j<i} M_{\alpha_j+1}} = id_{\bigcup_{j<i} M_{\alpha_j+1}}$. For each $j < i$, let $f_{j,i} := g \circ f^*_{j,i}$. It is easy to show that $r_i, a_i, \{f_{j,i} : j < i\}$ satisfy (1) through (6), for conditions (4)-(6) see the explanation given in Lemma 4.5.11 . This finishes the construction.

We have constructed $\{(r_i, a_i, \{f_{k,j} : k < j < i\}) : i < (2^\mu)^+\}$ a coherent sequence of types, then by [Bal09, 11.3] there is $r^* \in \mathbf{gS}(\bigcup_{i<(2^\mu)^+} M_{\alpha_i+1})$ such that r^* extends r_i for every $i < (2^\mu)^+$. In particular, $p_{\alpha_i} \leq r^*$ for every $i < (2^\mu)^+$, since by condition (5) $p_{\alpha_i} \leq r_i$ for each $i < (2^\mu)^+$. Moreover, using the $(< \aleph_0)$-witness property it follows that r^* does not fork over R. $\square$

Using the above result instead of Lemma 4.5.11 we are able to extend Theorem 4.5.12 to uncountable cardinals. As the proof is similar to that of Theorem 4.5.12 we omit it.

Theorem 4.6.2. *If $\overline{\smile}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons, $\kappa(\overline{\smile}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\overline{\smile})} = \mu$, then*

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\kappa(\overline{\smile})} + 2^\mu.$$

As a corollary we obtain the failure of the tree property.

Corollary 4.6.3. *If $\mathop{\smile}\limits^{\vert}$ is a simple independence relation with the $(< \aleph_0)$-witness property for singletons, then $\mathbf{K}$ does not have the tree property.*

Proof sketch . Let $\lambda_0 = \kappa(\mathop{\smile}\limits^{\vert})$. Let μ and λ be as in Theorem 4.5.14, i.e., $\mu = (\beth_{(\aleph_0+\ell(\mathop{\smile}\limits^{\vert}))^+}(\lambda_0^+))^+$ and $\lambda = \beth_\mu(\mu)$. Then doing a similar construction to that of Lemma 4.4.4 we get that:

$$\lambda^+ \leq NT(\mu, \lambda, (2^\mu)^+). \tag{4.6.1}$$

But by Theorem 4.6.2 we have that $NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\lambda_0} + 2^\mu$, then by choice of μ and λ we have that $\lambda^{\lambda_0} + 2^\mu = \lambda$, so:

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda. \tag{4.6.2}$$

Observe that equations (9) and (10) give us a contradiction.

□

Remark 4.6.4. A trivial example of a simple independence relation with the $(< \aleph_0)$-witness property for singletons is first-order non-forking in T where T is a complete first-order simple theory. This follows from the fact that non-forking has finite character.

4.6.2 Simple independence relations with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property

We continue the study of simple independence relations but with the additional hypothesis of the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. Recall that we have shown that if $\kappa_1(\mathop{\smile}\limits^{\vert}) = \mathrm{LS}(\mathbf{K})$. then $\mathop{\smile}\limits^{\vert}$ has the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons (Lemma 4.2.13).

The following simple proposition will be used to study the Lascar rank in the next section.

Proposition 4.6.5. *Let $\mathop{\smile}\limits^{\vert}$ be a simple independence relation with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. If $M \leq_{\mathbf{K}} N$, $p \in \mathbf{gS}(M)$, $q \in \mathbf{gS}(N)$ and q is a forking extension of p, then there is $M^* \leq_{\mathbf{K}} N$ with $\|M^*\| = \|M\|$, $M \leq_{\mathbf{K}} M^*$ and $q \restriction_{M^*}$ is a forking extension of p.*

Proof. Assume that $q = \mathbf{tp}(b/N)$. Suppose for the sake of contradiction that it is not the case, hence for every $M^* \leq_{\mathbf{K}} N$ with $\|M^*\| = \|M\|$ and $M \leq_{\mathbf{K}} M^*$ it holds that $q \restriction_{M^*}$ does not fork over M. We will show, using the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons, that $b \mathop{\smile}\limits^{\vert}_M N$.

Let $A \subseteq N$ and $|A| \leq \mathrm{LS}(\mathbf{K})$, then apply downward Löwenheim-Skolem to $A \cup M$ inside N to get $M^* \in \mathbf{K}_{\|M\|}$ such that $A \cup M \subseteq M^* \leq_{\mathbf{K}} N$. Then by assumption

$b \underset{M}{\overset{}{\smile}} M^*$. So by monotonicity $b \underset{M}{\smile} A$. Therefore, by the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons, we conclude that $b \underset{M}{\smile} N$, which contradicts the hypothesis that q forks over M. □

The next lemma generalizes [Kim14, 2.3.7].

Lemma 4.6.6. *Let $\smile$ be a simple independence relation that has the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons and without uniform local character. The following are equivalent.*

1. $\kappa_1(\smile) \leq \lambda$.

2. *There are no $\{M_i : i \leq \lambda^+\}$ and $p \in \mathbf{gS}(M_{\lambda^+})$ such that $\{M_i : i \leq \lambda^+\}$ is strictly increasing and continuous chain and p forks over M_i for every $i < \lambda^+$.*[7]

Proof. $\Rightarrow$: Assume for the sake of contradiction that there is $\{M_i : i \leq \lambda^+\}$ a strictly increasing and continuous chain and $p \in \mathbf{gS}(M_{\lambda^+})$ such that p forks over M_i for every $i < \lambda^+$. Then by hypothesis there is $M' \in [M_{\lambda^+}]^{\lambda}$ such that p does not fork over M'. Then by regularity of λ^+ and base monotonicity there is $i < \lambda^+$ such that p_{λ^+} does not fork over M_i. This is a contradiction.

$\Leftarrow$: Assume for the sake of contradiction that $\kappa_1(\smile) > \lambda$, then there is $q = \mathbf{tp}(a/N) \in \mathbf{gS}(N)$ such that q forks over M for every $M \in [N]^{\lambda}$. Realize that $\|N\| \geq \lambda^+$ as q does not fork over N.

We build $\{M_i : i < \lambda^+\}$ strictly increasing and continuous chain such that:

1. For every $i < \lambda^+$, $M_i \in \mathbf{K}_{\lambda}$ and $M_i \leq_{\mathbf{K}} N$.

2. For every $j > i$, $q \restriction_{M_j}$ forks over M_i.

Before we do the construction observe that this is enough by taking $M_{\lambda^+} = \bigcup_{i<\lambda^+} M_i$, $\{M_i : i \leq \lambda^+\}$ and $p = q \restriction_{M_{\lambda^+}}$.

In the base step, just take any $M_0 \in [N]^{\lambda}$. If $i < \lambda^+$ limit take unions and and it works by monotonicity, so the only interesting case is when $i = j + 1$. Then by the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property there is $B \subseteq N$ of size $\mathrm{LS}(\mathbf{K})$ such that $q \restriction_B$ forks over M_j and pick $c \in N \backslash M_j$. Let M_{j+1} be the structure obtained by applying downward Löwenheim-Skolem to $B \cup M_j \cup \{c\}$ in N. This works by the choice of B and monotonicity. □

Realize that even simple assertions as the ones above become very hard to prove or perhaps even false if the independence relation does not have some locality assumptions.

[7]This generalizes the first-order notion of a forking chain.

4.7 Supersimple independence relations and the U-rank

In this section we introduce supersimple independence relations and show that they can be characterized by the Lascar rank under a locality assumption on the independence relation. We also show that the existence of a supersimple independence relation implies the $(< \aleph_0)$-witness property for singletons in classes with intersections.

Let us introduce the notion of a supersimple independence relation.

Definition 4.7.1. $\mathop{\smile}\limits^{\frown}$ is a *supersimple independence relation* if the following properties hold:

1. $\mathop{\smile}\limits^{\frown}$ is a simple independence relation.
2. (Finite local character) For every δ limit ordinal, $\{M_i : i \leq \delta\}$ increasing and continuous chain and $p \in \mathbf{gS}(M_\delta)$, there is $i < \delta$ such that p does not fork over M_i.

Remark 4.7.2. Let T be a complete first-order theory. If T is supersimple and $\mathop{\smile}\limits^{\frown}$ is first-order non-forking, then $\mathop{\smile}\limits^{\frown}$ is a supersimple independence relation.

The following is straightforward but will be useful.

Lemma 4.7.3. *If $\mathop{\smile}\limits^{\frown}$ is a supersimple independence relation, then $\kappa_1(\mathop{\smile}\limits^{\frown}) = \mathrm{LS}(\mathbf{K})$.*

Proof sketch. The proof can be done by induction on the cardinality of the domain of the type. The base step is clear because types do not fork over their domain and for the induction step use that $\mathop{\smile}\limits^{\frown}$ has finite local character. □

The above lemma together with Lemma 4.2.13 can be used to obtain the next result.

Corollary 4.7.4. *If $\mathop{\smile}\limits^{\frown}$ is a supersimple independence relation, then $\mathop{\smile}\limits^{\frown}$ has the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons.*

The next lemma shows that supersimplicity and stability imply superstability.

Lemma 4.7.5. *If $\mathop{\smile}\limits^{\frown}$ is a stable and supersimple independence relation, then $\mathbf{K}$ is stable in a tail of cardinals*[8]*.*

[8]This is equivalent to any notion of superstability in the context of AECs if one assume that the AEC has a monster model and is tame by [GrVas17] and [Vas18].

Proof. Since $\mathop{\smile}\limits^{\frown}$ is a stable independence relation, by Corollary 4.4.3 $\mathbf{K}$ is a stable AEC, so let λ_0 be the first stability cardinal. We show by induction on $\mu \geq \lambda_0$ that $\mathbf{K}$ is μ-stable.

The base step is clear, so let us do the induction step. We proceed by contradiction, let $M \in \mathbf{K}_\mu$ and $\{p_i : i < \mu^+\} \subseteq \mathbf{gS}(M)$ be an enumeration of different Galois-types. Let $\{M_\alpha : \alpha < \mu\} \subseteq \mathbf{K}_{<\mu}$ be an increasing chain of submodels of M such that $\bigcup_{\alpha<\mu} M_\alpha = M$. Then by supersimplicity for every $i < \mu^+$ there is $\alpha_i < \mu$ such that p_i does not fork over M_{α_i}. Then by the pigeonhole principle and using that $\mathop{\smile}\limits^{\frown}$ has uniqueness, one can show (as in Theorem 4.4.2) that there are $i \neq j < \mu^+$ such that $p_i = p_j$. This is clearly a contradiction. Therefore, $\mathbf{K}$ is μ-stable. $\square$

It is worth noticing that Lemma 4.6.1 can be carried out with the finite local character assumption instead of the $(< \aleph_0)$-witness property for singletons. The idea is that by applying finite local character and transitivity in limit stages one can show that the types constructed in the limit stages do not fork over R (where R is the one introduced in condition (4) of Lemma 4.6.1).

Corollary 4.7.6. *If $\mathop{\smile}\limits^{\frown}$ is a supersimple independence relation, then*

- *if $\kappa(\mathop{\smile}\limits^{\frown}) \leq \mu \leq \lambda$ and $\mu^{<\ell(\mathop{\smile}\limits^{\frown})} = \mu$, then*

$$NT(\mu, \lambda, (2^\mu)^+) \leq \lambda^{\kappa(\mathop{\smile}\limits^{\frown})} + 2^\mu.$$

- $\mathbf{K}$ *does not have the tree property.*

4.7.1 Lascar rank

The Lascar rank was extended to the AEC context by Boney and the first author in [BoGr17].

Definition 4.7.7 ([BoGr17, 7.2])**.** We define U with domain a type and range an ordinal or ∞ by, for any $p \in \mathbf{gS}(M)$

1. $U(p) \geq 0$.
2. $U(p) \geq \alpha$ for α limit ordinal if and only if $U(p) \geq \beta$ for each $\beta < \alpha$.
3. $U(p) \geq \beta + 1$ if and only if there are $M' \geq_{\mathbf{K}} M$ and $p' \in \mathbf{gS}(M')$ with $\|M'\| = \|M\|$, p' is a forking extension of p and $U(p') \geq \beta$.
4. $U(p) = \alpha$ if and only if $U(p) \geq \alpha$ and it is not the case that $U(p) \geq \alpha + 1$.

5. $U(p) = \infty$ if and only if $U(p) \geq \alpha$ for each α ordinal.

The next couple of results show that U is a well-behaved rank. The proofs are similar to the ones presented in [BoGr17, §7], but we fix a minor mistake of [BoGr17, §7]. The arguments of [BoGr17, §7] only work when the models under consideration are all of the same size, we are able to extend the arguments for models of different sizes by using the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property, specifically Proposition 4.6.5.

Lemma 4.7.8. *Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$ be a simple independence relation with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons, then the U-rank satisfies:*

1. *([BoGr17, 7.4]) Invariance: If $p \in \mathbf{gS}(M)$ and $f : M \cong M'$, then $U(p) = U(f(p))$.*
2. *Monotonicity: If $M \leq_{\mathbf{K}} N$, $p \in \mathbf{gS}(M)$, $q \in \mathbf{gS}(N)$ and $p \leq q$, then $U(q) \leq U(p)$.*

Proof. We provide a proof for (2) based on [BoGr17, 7.3]. We prove by induction on α that: if $p \leq q$, then if $U(q) \geq \alpha$, then $U(p) \geq \alpha$. The base step and limit step are trivial so assume that $\alpha = \beta + 1$ and that $U(q) \geq \beta + 1$. By definition there is $N' \geq_{\mathbf{K}} N$ and $q' \in \mathbf{gS}(N')$ with $\|N'\| = \|N\|$, $q' \geq q$, q' forks over N and $U(q') \geq \beta$. Observe that by monotonicity q' forks over M and clearly $q' \geq p$. Then by Proposition 4.6.5 there is $M' \geq_{\mathbf{K}} M$ with $\|M'\| = \|M\|$, $q' \restriction_{M'} \geq p$ and $q' \restriction_{M'}$ forks over M. Since $q' \restriction_{M'} \leq q'$, by induction hypothesis $U(q' \restriction_{M'}) \geq \beta$. Therefore, by the definition of the U-rank $U(p) \geq \beta + 1$. □

Lemma 4.7.9. *Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$ be a simple independence relation with $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. Let $M \leq_{\mathbf{K}} N$, $p \in \mathbf{gS}(M)$ and $q \in \mathbf{gS}(N)$ with $p \leq q$ and $U(p), U(q) < \infty$. Then:*

$$U(p) = U(q) \text{ if and only if } q \text{ is a non-forking extension of } p.$$

Proof. $\Rightarrow$: Assume for a sake of contradiction that q forks over p. Then by Proposition 4.6.5 there is $M^* \in \mathbf{K}$ with $\|M^*\| = \|M\|$, $q \restriction_{M^*} \geq p$ and $q \restriction_{M^*}$ forks over M. Then from monotonicity of the rank and the definition of the U-rank, we can conclude that $U(p) \geq U(q) + 1$, which clearly contradicts our hypothesis.

$\Leftarrow$: The same argument given in [BoGr17, 7.7] can be carried out in our context due to Proposition 4.5.9. □

Fact 4.7.10. ([BoGr17, 7.8]) Let $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$ be a simple independence relation with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. For each $\mu \geq \mathrm{LS}(\mathbf{K})$, there is some $\alpha_{\mathbf{K},\mu} < (2^\mu)^+$ such that for any $M \in \mathbf{K}_\mu$, if $U(p) \geq \alpha_{\mathbf{K},\mu}$, then $U(p) = \infty$.

The proof of the following lemma is similar to that of [BoGr17, 7.9].

Lemma 4.7.11. *Let $\mathop{\smile\!\!\!\!\!\!\;|}$ be a simple independence relation with the $(< \mathrm{LS}(\mathbf{K})^+)$-witness property for singletons. Let $M \in \mathbf{K}_\mu$ and $p \in \mathbf{gS}(M)$. The following are equivalent.*

1. *$U(p) = \infty$*

2. *There is an increasing chain of types $\{p_n : n < \omega\}$ such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for each $n < \omega$.*

Proof. $\Rightarrow$: Let $\alpha_{\mathbf{K},\mu}$ be the ordinal given by Fact 4.7.10. We build $\{M_n : n < \omega\}$ and $\{p_n \in \mathbf{gS}(M_n) : n < \omega\}$ by induction such that:

1. $p_0 = p$.

2. $M_n \in \mathbf{K}_\mu$.

3. p_{n+1} is a forking extension of p_n for every $n < \omega$.

4. $U(p_n) \geq \alpha_{K,\mu} + 1$.

The base step is given by condition (1). As for the induction step, we have by induction that $U(p_n) \geq \alpha_{K,\mu}+1$. Then by definition of the U-rank there is $M_{n+1} \geq M_n$ and $p_{n+1} \in \mathbf{gS}(M_{n+1})$ a forking extension of p_n such that $\|M_{n+1}\| = \|M_n\| = \mu$ and $U(p_{n+1}) \geq \alpha_{\mathbf{K},\mu}$. Observe that since $U(p_{n+1}) \geq \alpha_{\mathbf{K},\mu}$ and $M_{n+1} \in \mathbf{K}_\mu$, we have that $U(p_{n+1}) = \infty$, so $U(p_{n+1}) \geq \alpha_{\mathbf{K},\mu} + 1$.

$\Leftarrow$: Let $\{p_n : n < \omega\}$ be an increasing chain of types such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for each $n < \omega$. We prove by induction on α that: $U(p_n) \geq \alpha$ for every $n < \omega$. The base step and limit case are trivial so assume that $\alpha = \beta + 1$ and take $n \in \omega$. By induction hypothesis $U(p_{n+1}) \geq \beta$ and by hypothesis p_{n+1} is a forking extension of p_n. Then by Proposition 4.6.5 there is $M^* \in \mathbf{K}$ with $\|M^*\| = \|dom(p_n)\|$, $p_{n+1} \restriction_{M^*} \geq p_n$ and $p_{n+1} \restriction_{M^*}$ forks over $dom(p_n)$. Then by monotonicity of the rank and the definition of the U-rank we can conclude that $U(p_n) \geq \beta + 1 = \alpha$. $\square$

With this we obtain our main result regarding the relationship between a supersimple independence relations and the U-rank . This generalizes a characterization of supersimplicity for first-order theories [Kim14, 2.5.16].

Theorem 4.7.12. *Let $\mathop{\smile\!\!\!\!\!\!\;|}$ be a simple independence relation with the $(< \aleph_0)$-witness property for singletons. The following are equivalent.*

1. *$\mathop{\smile\!\!\!\!\!\!\;|}$ is a supersimple independence relation.*

2. *If $M \in \mathbf{K}$ and $p \in \mathbf{gS}(M)$, then $U(p) < \infty$.*

Proof. $\Rightarrow$: Suppose there are $M \in \mathbf{K}$ and $p \in \mathbf{gS}(M)$ such that $U(p) = \infty$. Then, by Lemma 4.7.11, there is an increasing chain of types $\{p_n : n < \omega\}$ such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for every $n < \omega$.

Since we have that $\{p_n : n < \omega\}$ is an increasing chain of types, by [Bal09, 11.3], there is $p_\omega \in \mathbf{gS}(\bigcup_{n<\omega} dom(p_n))$ such that $p_\omega \geq p_n$ for each $n < \omega$. Then, by the definition of supersimplicty, there is $n < \omega$ such that p_ω does not fork over $dom(p_n)$. Hence by monotonicity $p_\omega \restriction_{dom(p_{n+1})} = p_{n+1}$ does not fork over $dom(p_n)$, which contradicts the fact that p_{n+1} is a forking extension of p_n.

$\Leftarrow$: Assume for the sake of contradiction that $\perp\!\!\!\!\perp$ is not a supersimple independence relation, then there are δ a limit ordinal, $\{N_i : i \leq \delta\}$ an increasing and continuous chain and $p \in \mathbf{gS}(N_\delta)$, such that p forks over N_i for every $i < \delta$.

We first show that for every $i < \delta$ there is $j_i \in (i, \delta)$ such that $p \restriction_{N_{j_i}}$ forks over N_i. Let $i < \delta$ and suppose for the sake of contradiction that $p \restriction_{N_j}$ does not fork over N_i for each $j \in (i, \delta)$. Then using the $(< \aleph_0)$-witness property for singletons, as in Proposition 4.6.5, one can show that p does not fork over N_i, contradicting the hypothesis that p forks over N_i.

Then one can build by induction $\{i_n : n < \omega\} \subseteq \delta$ increasing such that $\{p_{i_n} : n < \omega\}$ is an increasing chain of types with $p_{i_{n+1}}$ a forking extension of p_{i_n} for each $n < \omega$ where $p_{i_n} = p \restriction_{N_{i_n}}$. Therefore, by Lemma 4.7.11, we can conclude that $U(p_{i_0}) = \infty$. This contradicts the fact that $U(p_{i_0}) < \infty$ by hypothesis. $\square$

4.7.2 A family of classes with the $(< \aleph_0)$-witness property

In this subsection we show that in classes that admit intersections one obtains the $(< \aleph_0)$-witness property for singletons from supersimplicity. Similar results assuming the existence of a superstable-like independence relation are obtained in Appendix C of [Vas17c]. We begin by recalling the definition of classes that admit intersections, these were introduced by Baldwin and Shelah.

Definition 4.7.13 ([BaSh08, 1.2])**.** An AEC *admits intersections* if for every $N \in \mathbf{K}$ and $A \subseteq |N|$ there is $M_0 \leq_{\mathbf{K}} N$ such that $|M_0| = \bigcap\{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$. For $N \in \mathbf{K}$ and $A \subseteq |N|$, let $cl^N_{\mathbf{K}}(A) = \bigcap\{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$, if it is clear from the context we will drop the $\mathbf{K}$. We write $cl(A)$ instead of $cl^{\mathcal{C}}_{\mathbf{K}}(A)$ if $\mathcal{C}$ is a monster model of $\mathbf{K}$ and $\mathbf{K}$ is clear from the context.

Below we provide the properties of AECs that admit intersections that we will use, for a more detailed introduction to AECs that admit intersections the reader can consult [Vas17c, §2].

Fact 4.7.14. Let $\mathbf{K}$ be an AEC that admits intersections.

1. If $A \subseteq B \subseteq N$, then $cl^N(A) \leq_{\mathbf{K}} cl^N(B)$.

2. If $A \subseteq M$ and $M \in \mathbf{K}$, then $cl(A) \leq_{\mathbf{K}} M$.

3. (Finite character) Let $M \in \mathbf{K}$ and $a \in cl^M(B)$, then there is $B_0 \subseteq_{fin} B$ such that $a \in cl^M(B_0)$.

Proof. (1) and (2) are trivial and (3) is [Vas17c, 2.14]. $\square$

We show that finite local character is actually witnessed by a finite set in classes with intersections.

Lemma 4.7.15. *Let* $\mathbf{K}$ *be an AEC with a monster model that admits intersections and* $\perp$ *be a simple independence relation. The following are equivalent.*

1. *(Finite local character) For every* δ *limit ordinal,* $\{M_i : i \leq \delta\}$ *increasing and continuous chain and* $p \in \mathbf{gS}(M_\delta)$*, there is* $i < \delta$ *such that* p *does not fork over* M_i.

2. *For every* $M \in \mathbf{K}$ *and* $p \in \mathbf{gS}(M)$*, there is* $D \subseteq_{fin} M$ *such that* p *does not fork over* $cl(D)$.

Proof. The backward direction follows trivially using monotonicity, so we show the forward direction.

Let $M \in \mathbf{K}$ and $p \in \mathbf{gS}(M)$, we show by induction on $\lambda \leq \|M\|$ the following:

$$(*)_\lambda : \text{ For every } A \in \mathcal{P}_\lambda(M) \text{ and } p \in \mathbf{gS}(cl(A)) \text{ , there is } D \subseteq_{\text{fin}} M \text{ s.t. } p \text{ does not fork over } cl(D).$$

Observe that this is enough as $cl(M) = M$. So let us do the proof.

<u>Base:</u> If λ is finite $(*)_\lambda$ is clear because given $p \in \mathbf{gS}(cl(A))$, p does not fork over $cl(A)$. So let us do the case when $\lambda = \aleph_0$. Let $A = \{a_i : i < \omega\}$ be an enumeration without repetitions and $p \in \mathbf{gS}(cl(A))$. Let $M_i = cl(\{a_j : j < i\})$ for every $i < \omega$ and $M_\omega = \bigcup_{i<\omega} M_i$. Observe that $\{M_i : i \leq \omega\}$ is an increasing and continuous chain and $\bigcup_{i<\omega} M_i = cl(A)$ by the finite character of the closure operator. Then by (1) there is $i < \omega$ such that p does not fork over $M_i = cl(\{a_j : j < i\})$. So $D = \{a_j : j < i\}$ is as needed.

<u>Induction step:</u> Let λ be an uncountable cardinal and suppose that $(*)_\mu$ holds for every $\mu < \lambda$. In this case the proof is similar to that of the base step when $\lambda = \aleph_0$. The only difference is that on top of using (1), one uses the induction hypothesis, and transitivity of the independence relation. $\square$

Corollary 4.7.16. *Let* $\mathbf{K}$ *be a class that admits intersections. If* $\perp$ *is a supersimple independence relation, then* $\perp$ *has the* $(< \aleph_0)$*-witness property for singletons.*

Proof. Let $M \leq_{\mathbf{K}} N$ and $a \in \mathcal{C}$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_M B$ for every $B \subseteq_{\text{fin}} N$.

By the previous lemma there is $D \subseteq_{\text{fin}} N$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{cl(D)} N$, then by base monotonicity $a \mathop{\smile\!\!\!\!\!\!\;|}_{cl(DM)} N$. On the other hand, by hypothesis $a \mathop{\smile\!\!\!\!\!\!\;|}_M D$, then by normality, monotonicity and Proposition 4.5.9 it follows that $a \mathop{\smile\!\!\!\!\!\!\;|}_M cl(DM)$. Therefore, applying transitivity to $a \mathop{\smile\!\!\!\!\!\!\;|}_M cl(DM)$ and $a \mathop{\smile\!\!\!\!\!\!\;|}_{cl(DM)} N$ we obtain that $a \mathop{\smile\!\!\!\!\!\!\;|}_M N$. $\square$

4.8 Future work

In [KiPi97, 4.2] it is shown that if a complete first-order theory is simple, then there is a canonical independence relation satisfying the type-amalgamation property. In [BGKV16] it is shown that stable independence relations are canonical. So it is natural to ask if the same holds true for simple and supersimple independence relations.

Question 4.8.1. If $\mathbf{K}$ has $\mathop{\smile\!\!\!\!\!\!\;|}$ a simple or supersimple independence relation, is $\mathop{\smile\!\!\!\!\!\!\;|}$ canonical?

Remark 4.8.2. Theorem 1.1 of [Kam20] gives a positive answer to the above question under the assumptions that $\mathop{\smile\!\!\!\!\!\!\;|}$ has the $(< \aleph_0)$-witness property.

It is known that for a complete first-order theory T, T is simple if and only if T does not have the tree property (see for example [GIL02, 3.10]). In Sections 5 and 6 we showed some instances of the forward direction for simple independence relations (Corollary 4.5.14 and Corollary 4.6.3). So we ask the following:

Question 4.8.3. If $\mathbf{K}$ does not have the tree property, does $\mathbf{K}$ have $\mathop{\smile\!\!\!\!\!\!\;|}$ a simple independence relation?

Another notion that we studied in this paper is that of the witness property for independence relations. This seems to be a very strong hypothesis that can be taken for granted in first-order theories as forking has finite character. Regarding it we ask:

Question 4.8.4. Can Fact 4.5.7 be extended to simple independence relations? More precisely, if $\mathbf{K}$ is fully $(< \theta)$-tame and -type-short and $\mathop{\smile\!\!\!\!\!\!\;|}$ is a simple independence relation, does $\mathop{\smile\!\!\!\!\!\!\;|}$ have the $(< \theta)$-strong witness property?

A related question is the following:

Question 4.8.5. Is Corollary 4.7.16 true for all AECs with a monster model?

Moreover, we used the witness properties a few times in this paper, see for example Lemma 4.6.1 and Theorem 4.7.12. An interesting question would be if the use of the witness property is necessary in those arguments where we use it.

In [LRV19, 8.16] it is shown that the existence of a stable independence relation implies that the AEC is tame. We extended this result for almost-stable independence relations in Lemma 4.4.10, so a natural question to ask is:

Question 4.8.6. If $\mathbf{K}$ has $\mathop{\smile\!\!\!\!\!\!\!\!\;\overline{}}$ a simple or supersimple independence relation, is $\mathbf{K}$ tame?

Finally, as it was mentioned in the introduction, we think that it is premature to focus on applications. Nevertheless, we acknowledge the importance of *good examples*. Below is a list of the type of examples that we are interested in.

Question 4.8.7.

- Find an example of a simple independence relation that is not a stable independence relation in an AEC that is not fully $(< \aleph_0)$-tame and type-short.
- Find an example of a supersimple independence relation that is not a stable independence relation in an AEC that is not fully $(< \aleph_0)$-tame and type-short.
- Find an example of a strictly simple independence relation without the $(< \aleph_0)$-witness property.
- Find an example of a strictly simple independence relation without the witness property.
- Find an example of a strictly simple independence relation without the witness property in an AEC that is fully $(< \aleph_0)$-tame and type-short.

Part II

Applications to abelian group theory and ring theory

Chapter 5

Algebraic description of limit models in classes of abelian groups

This chapter is based on [Ch. 5].

Abstract

We study limit models in the class of abelian groups with the subgroup relation and in the class of torsion-free abelian groups with the pure subgroup relation. We show:

Theorem 5.0.1.

1. *If G is a limit model of cardinality λ in the class of abelian groups with the subgroup relation, then $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p\ prime}(\oplus_\lambda \mathbb{Z}(p^\infty))$.*

2. *If G is a limit model of cardinality λ in the class of torsion-free abelian groups with the pure subgroup relation, then:*

   - *If the length of the chain has uncountable cofinality, then*

   $$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_{p\ prime}\overline{(\oplus_\lambda \mathbb{Z}_{(p)})}.$$

   - *If the length of the chain has countable cofinality, then G is not algebraically compact.*

We also study the class of finitely Butler groups with the pure subgroup relation, we show that it is an AEC, stable and $(< \aleph_0)$-tame and short.

5.1 Introduction

Abstract elementary classes (AECs for short) were introduced in the late seventies by Shelah [Sh88] to capture the semantic structure of non-first-order theories, Shelah was interested in capturing logics like $\mathbb{L}_{\lambda^+,\omega}(\mathbf{Q})$. The setting is general enough to encompass many examples, but it still allows a development of a rich theory as witnessed by Shelah's two volume book on the subject [Sh:h] and many dozens of publications by several researchers. As a first approximation, an AEC is a class of structures with morphisms that is closed under colimits and such that every set is contained in a small model in the class.

Definition 5.1.1. An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

1. K is a class of τ-structures, for some fixed language $\tau = \tau(\mathbf{K})$.
2. $\leq_{\mathbf{K}}$ is a partial ordering on K.
3. $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.
4. If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
5. Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$.
6. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:
   (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.
   (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.
7. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

The main objective in the study of AECs is to develop a classification theory like the one of first-order model theory. The notions of non-forking, superstability and stability have been extended to this more general setting. The main test question is Shelah's eventual categoricity conjecture which asserts that if an AEC is categorical in *some* large cardinal then it is categorical in *all* large cardinals. Many partial results have been obtained in this direction as witnessed by for example [Sh87a], [Sh87b], [Sh394], [Sh:h], [GrVan06b], [GrVan06c], [Bon14b], [Vas17b], [Vas17d], [Vas17c], [Vas19] and [ShVas].[1]

The notion of limit model was introduced in [KolSh96] as a substitute for saturation in the non-elementary setting (see Definition 5.2.9). If $\lambda > \mathrm{LS}(\mathbf{K})$ is a regular cardinal and $\mathbf{K}$ is an AEC with joint embedding, amalgamation and no maximal models, then: M is λ-saturated if and only if M is a (λ, λ)-limit model ([GrVas17, 2.8]).

Limit models have proven to be an important concept in tackling Shelah's eventual categoricity conjecture as witnessed by for example [ShVi99], [GrVan06] and [Vas19]. The key question has been the uniqueness of limit models of the same cardinality but with chains of different lengths. This has been studied thoroughly [ShVi99], [Van06], [GVV16], [Bon14], [Van16], [BoVan], [ViZa16] and [Vas19]. In this same line, [GrVas17] and [Vas16c] showed that if a class has a monster model and is tame then uniqueness of limit models on a tail of cardinals is equivalent to being superstable[2].

Despite the importance of limit models in the understanding of AECs, explicit examples have never been studied. This paper ends this by studying examples of limit models in some classes of abelian groups. The need to analyze examples is also motivated by the regular inquiry of the model theory community when presenting results on AECs. In particular, the analysis of limit models in the class of torsion-free abelian groups provides a missing example needed for [BoVan].

In this article, we study limit models in the class of abelian groups with the subgroup relation and in the class of torsion-free abelian groups with the pure subgroup relation[3]. Observe that both classes are first-order axiomatizable, but since we are studying them with a strong substructure relation that is different from elementary substructure, their study is outside of the framework of first-order model theory. This freedom in choosing the strong substructure relation is a key feature of our examples

[1]For a more detailed introduction to the theory of AECs we suggest the reader to look at [Gro02], [Bal09] or [BoVas17b] (this only covers tame AECs, but the AECs that we will study in this paper are all tame).

[2]We say that $\mathbf{K}$ is *superstable* if there is $\mu < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ such that $\mathbf{K}$ is λ-stable for every $\lambda \geq \mu$. Under the assumption of joint embedding, amalgamation, no maximal models and $\mathrm{LS}(\mathbf{K})$-tameness (which hold for all the classes studied in this paper, except perhaps the one introduced in the last section) by [GrVas17] and [Vas18] the definition of the previous line is equivalent to any other definition of superstability given in the context of AECs.

[3]Recall that H is a pure subgroup of G if for every $n \in \mathbb{N}$ it holds that $nG \cap H = nH$.

and in the context of AECs has only been exploited in [BCG+] and [BET07].

The case of limit models in the class of abelian groups is simple.

Theorem 5.3.7. *Let $\alpha < \lambda^+$ be a limit ordinal. If G is a (λ, α)-limit model in the class of abelian groups with the subgroup relation, then we have that:*

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p\ prime}(\oplus_\lambda \mathbb{Z}(p^\infty)).$$

The case of torsion-free abelian groups (with the pure subgroup relation) is more interesting and the examination of limit models is divided into two cases. In the first one, we study limit models with chains of uncountable cofinality and by showing that they are algebraically compact we are able to give a full structure theorem. In the second one, we study limit models with chains of countable cofinality and we show that they are not algebraically compact. More precisely we obtain the following.

Theorem 5.4.26. *Let $\alpha < \lambda^+$ be a limit ordinal. If G is a (λ, α)-limit model in the class of torsion-free abelian groups with the pure subgroup relation, then we have that:*

1. *If the cofinality of α is uncountable, then*

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_{p\ prime}\overline{(\oplus_\lambda \mathbb{Z}_{(p)})}.$$

2. *If the cofinality of α is countable, then G is not algebraically compact.*

In particular, the class does not have uniqueness of limit models for any infinite cardinal.

The paper is organized as follows. Section 2 presents necessary background. Section 3 characterizes limit models in the class of abelian groups with the subgroup relation. Section 4 studies the class of torsion-free abelian groups with the pure subgroup relation. We show that limit models of uncountable cofinality are algebraically compact (and characterize them) while those of countable cofinality are not. Section 5 studies basic properties of the class of finitely Butler groups.

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5.2 Preliminaries

We present the basic concepts of abstract elementary classes that are used in this paper. These are further studied in [Bal09, §4 - 8] and [Gro2X, §2, §4.4]. Regarding the background on abelian groups, we assume that the reader has some familiarity with it and introduce the necessary concepts throughout the text.[4]

5.2.1 Basic notions

Before we introduce some concepts let us fix some notation.

Notation 5.2.1.

- If $M \in K$, $|M|$ is the underlying set of M.
- If λ is a cardinal, $\mathbf{K}_\lambda = \{M \in K : \|M\| = \lambda\}$.
- Let $M, N \in K$. If we write "$f : M \to N$" we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. Observe that in particular $\mathbf{K}$-embeddings are always monomorphisms.

All the examples that we consider in this paper have the additional property of admitting intersections. This class of AECs was introduced in [BaSh08] and further studied in [Vas17c, §2].

Definition 5.2.2. An AEC *admits intersections* if for every $N \in K$ and $A \subseteq |N|$ there is $M_0 \leq_{\mathbf{K}} N$ such that $|M_0| = \bigcap\{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$. For $N \in K$ and $A \subseteq |N|$, we denote by $cl^N_{\mathbf{K}}(A) = \bigcap\{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$, if it is clear from the context we will drop the $\mathbf{K}$.

Since an AEC is a semantic object, the notion of syntactic type (first-order type) does not interact well with the strong substructure relation of the AEC. Even when the AEC is axiomatizable in some extension of first-order logic, syntactic types do not behave well since equality of types does not imply the existence of $\mathbf{K}$-embeddings between the models mentioned in the types. For this reason Shelah introduced a notion of semantic type called Galois-type. We use the terminology of [Ch. 3, 2.5].

Definition 5.2.3. Let $\mathbf{K}$ be an AEC.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in K$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.

[4] An excellent encyclopedic resource is [Fuc15]. We recommend the reader to keep a copy of [Fuc15] nearby since we will cite frequently from it, specially in the last section.

2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E_{\mathrm{at}} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, $\ell(\mathbf{b}_1) = \ell(\mathbf{b}_2)$ and there exists $f_\ell : N_\ell \underset{A}{\to} N$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$.

3. Note that E_{at} is a symmetric and reflexive relation on $\mathbf{K}^3$. We let E be the transitive closure of E_{at}.

4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$. We call such an equivalence class a *Galois-type*. Usually, $\mathbf{K}$ will be clear from context and we will omit it.

5. For $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N)$ and $C \subseteq A$, $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) \upharpoonright_C := [(\mathbf{b}, C, N)]_E$.

In classes that admit intersections types are easier to describe as it was shown in [Vas17c, 2.18].

Fact 5.2.4. Let $\mathbf{K}$ be an AEC that admits intersections. $\mathbf{tp}(\mathbf{a}_1/A; N_1) = \mathbf{tp}(\mathbf{a}_2/A; N_2)$ if and only if there is $f : cl^{N_1}(\mathbf{a}_1 \cup A) \cong_A cl^{N_2}(\mathbf{a}_2 \cup A)$ such that $f(\mathbf{a}_1) = \mathbf{a}_2$.

The notion of stability generalizes that of a stable first-order theory. Since it will play an important role, as witness by Fact 10.2.10, we recall it.

Definition 5.2.5.

- An AEC is λ-*stable* if for any $M \in \mathbf{K}_\lambda$ it holds that $|\mathbf{gS}(M)| \leq \lambda$, where $\mathbf{gS}(M) = \{\mathbf{tp}(a/M; N) : M \leq_{\mathbf{K}} N \text{ and } a \in N\}$. Observe that $\mathbf{gS}(M)$ denotes the 1-ary Galois-types over M.
- An AEC is *stable* if there is a $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is λ-stable.

Tameness (for saturated models) appears implicitly in the work of Shelah [Sh394], but it was not until Grossberg and VanDieren isolated it in [GrVan06] that it became a central notion in the study of AECs. Tameness was first used to prove a stability spectrum theorem in [GrVan06] and to prove an upward categoricity transfer theorem in [GrVan06b]. For further details on tameness the reader can consult the survey by Boney and Vasey [BoVas17b].

Definition 5.2.6. $\mathbf{K}$ is $(<\kappa)$-*tame* if for any $M \in K$ and $p \neq q \in \mathbf{gS}(M)$, there is $A \subseteq M$ such that $|A| < \kappa$ and $p \upharpoonright_A \neq q \upharpoonright_A$.

Later, Boney isolated an analogous notion to tameness which he called type shortness in [Bon14b].

Definition 5.2.7. $\mathbf{K}$ is $(<\kappa)$-*short* if for any $M, N \in K$, $\bar{a} \in M^\alpha$, $\bar{b} \in N^\alpha$ and $\mathbf{tp}(\bar{a}/\emptyset, M) \neq \mathbf{tp}(\bar{b}/\emptyset, N)$, there is $I \subseteq \alpha$ such that $|I| < \kappa$ and $\mathbf{tp}(\bar{a} \upharpoonright_I /\emptyset; M) \neq \mathbf{tp}(\bar{b} \upharpoonright_I /\emptyset; N)$.

5.2.2 Limit models

Before introducing the concept of limit model we recall the concept of universal model.

Definition 5.2.8. M is *universal over* N if and only if $N \leq_{\mathbf{K}} M$, $\|M\| = \|N\| = \lambda$ and for any $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \xrightarrow[N]{} M$.

Recall that an increasing chain $\{M_i : i < \alpha\} \subseteq \mathbf{K}$ (for α an ordinal) is a *continuous chain* if $M_i = \bigcup_{j<i} M_j$ for every $i < \alpha$ limit ordinal. With this we are ready to introduce the main concept of this paper, it was originally introduced in [KolSh96].

Definition 5.2.9. Let $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)*-limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that $M_0 := N$, M_{i+1} is universal over M_i for each $i < \alpha$ and $M = \bigcup_{i<\alpha} M_i$. We say that $M \in \mathbf{K}_\lambda$ is a (λ, α)-limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. We say that $M \in \mathbf{K}_\lambda$ is a limit model if there is $\alpha < \lambda^+$ limit such that M is a (λ, α)-limit model.

Fact 5.2.10.

1. If $M \in \mathbf{K}_\lambda$ is universal over N and $M \leq_{\mathbf{K}} M^* \in \mathbf{K}_\lambda$, then M^* is universal over N.

2. Let $\mathbf{K}$ be an AEC with joint embedding. If M is a limit model of cardinality λ, then for any $N \in \mathbf{K}_\lambda$ there is $f : N \to M$.

Proof. The first assertion is trivial so we prove the second one.

Fix $\alpha < \lambda^+$, $\{M_i : i < \alpha\}$ a witness to the fact that M is a (λ, α)-limit model and let $N \in \mathbf{K}_\lambda$. By the joint embedding property applied to M_0 and N and using the Löwenheim-Skolem-Tarski axiom there is $N^* \in \mathbf{K}_\lambda$ and $g : N \to N^*$ such that $M_0 \leq_{\mathbf{K}} N^*$. Then since M_1 is universal over M_0, there is $h : N^* \xrightarrow[M_0]{} M_1$. Hence $f := h \circ g : N \to M$. □

The following fact gives conditions for the existence of limit models.

Fact 5.2.11. Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models. If $\mathbf{K}$ is λ-stable, then for every $N \in \mathbf{K}_\lambda$ and $\alpha < \lambda^+$ limit there is M a (λ, α)-limit model over N. Conversely, if $\mathbf{K}$ has a limit model of cardinality λ, then $\mathbf{K}$ is λ-stable

Proof. The forward direction is claimed in [Sh600] and proven in [GrVan06, 2.9]. The backward direction is straightforward. □

As mentioned in the introduction, the uniqueness of limit models of the same cardinality is a very interesting assertion. When the lengths of the cofinalities of the chains are equal, an easy back-and-forth argument gives the following.

Fact 5.2.12. Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models. If M is a (λ, α)-limit model and N is a (λ, β)-limit model such that $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$, then $M \cong N$.

The question of uniqueness is intriguing when the cofinalities of the lengths of the chains are different. This question has been studied in many papers, among them [ShVi99], [Van06], [GVV16], [Bon14], [Van16], [BoVan], [ViZa16] and [Vas19].

5.3 Abelian groups

In this third section, we study limit models in the class of abelian groups with the subgroup relation. Since this class was studied in great detail in [BCG+] and [BET07], the section will be short and we will cite several times.

Definition 5.3.1. Let $\mathbf{K}^{ab} = (K^{ab}, \leq)$ where K^{ab} is the class of abelian groups in the language $L_{ab} = \{0\} \cup \{+, -\}$ and $\leq$ is the subgroup relation, which is the same as the substructure relation in L_{ab}.

Fact 5.3.2.

1. $\mathbf{K}^{ab}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{ab}) = \aleph_0$.
2. $\mathbf{K}^{ab}$ admits intersections.
3. $\mathbf{K}^{ab}$ has joint embedding, amalgamation and no maximal models.
4. $\mathbf{K}^{ab}$ is a universal class.
5. $\mathbf{K}^{ab}$ is $(< \aleph_0)$-tame and short.

Proof. (1) and (3) are shown in [BCG+, 3.3] and (2) is clear, so we show the last two assertions:

4. It follows from the fact that K^{ab} is axiomatizable by a set of universal first-order sentences in the language $L_{ab} = \{0\} \cup \{+, -\}$. It is fundamental that we have "$-$" in the language.
5. It follows from (4) and [Vas17c, 3.7, 3.8].

□

The following fact is implied by [BCG+, 3.4, 3.5].

Fact 5.3.3. Let $G \leq H$ and $a, b \in H$, the following are equivalent:

1. There exists $f : cl^H_{\mathbf{K}^{ab}}(G \cup \{a\}) \cong_G cl^H_{\mathbf{K}^{ab}}(G \cup \{b\})$ such that $f(a) = b$.
2. - $\langle a \rangle \cap G = 0 = \langle b \rangle \cap G$, or
   - There are $n \in \mathbb{N}$ and $g^* \in G$ such that $na = g^* = nb$ and $ma, mb \notin G$ for all $m < n$.

In particular, $\mathbf{K}^{ab}$ is λ-stable for every λ infinite cardinal.

Remark 5.3.4. Since $\mathbf{K}^{ab}$ has joint embedding, amalgamation and no maximal models, $\mathbf{K}^{ab}$ has limit models in every infinite cardinal by Fact 5.3.3 and Fact 5.2.11.

Recall that a group G is *divisible* if for each $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ such that $nh = g$. In the next lemma we show that limit models in $\mathbf{K}^{ab}$ are divisible groups.

Lemma 5.3.5. *If G is a (λ, α)-limit model, then G is a divisible group.*

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α)-limit model. Let $g \in G$ and $n \in \mathbb{N}$, we want to show that $n|g$. Since $G = \bigcup_{i<\alpha} G_i$, there is $i < \alpha$ such that $g \in G_i$. Recall that every group can be embedded as a subgroup into a divisible group (see [Fuc15, §4.1.4]), so there is $D \in \mathbf{K}_\lambda$ divisible group such that $G_i \leq D$. In particular there is $d \in D$ with $nd = g$. Since G_{i+1} is universal over G_i, there is $f : D \underset{G_i}{\longrightarrow} G$. Hence $nf(d) = f(g) = g$ and $f(d) \in G$. □

Using the following structure theorem for divisible groups we can characterize the limit models of $\mathbf{K}^{ab}$. A proof of this fact appears in [Fuc15, §4.3.1].

Fact 5.3.6. If G is a divisible group, then we have that:

$$G \cong (\oplus_\kappa \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\kappa_p} \mathbb{Z}(p^\infty))$$

where the cardinal numbers κ, κ_p (for all p prime number) correspond to the ranks $rk_0(G)$, $rk_p(G)$ (for all p prime number)[5].

[5]The $rk_0(G)$ is the cardinality of a maximal linearly independent subset of elements of infinite order in G and $rk_p(G)$ is the cardinality of a maximal linearly independent subset of elements of order a power of p in G. The notion of linear independence in the context of abelian groups differs slightly from that of vector spaces, the reader can consult [Fuc15, p. 91] for the definition of linear independence in this setting.

From it we are able to show our first theorem.

Theorem 5.3.7. *If G is a (λ, α)-limit model in $\mathbf{K}^{ab}$, then we have that:*

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p\ prime}(\oplus_\lambda \mathbb{Z}(p^\infty)).$$

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α)-limit model. Observe that $G_0 \leq G_0 \oplus (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p \text{ prime}}(\oplus_\lambda \mathbb{Z}(p^\infty))$, therefore there is

$$f : G_0 \oplus (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p \text{ prime}}(\oplus_\lambda \mathbb{Z}(p^\infty)) \underset{G_0}{\longrightarrow} G.$$

In particular, $rk_0(G) = \lambda$ and $rk_p(G) = \lambda$ for all p prime, then by the structure theorem for divisible groups we have that $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p \text{ prime}}(\oplus_\lambda \mathbb{Z}(p^\infty))$. $\square$

As a simple corollary we obtain the following.

Corollary 5.3.8. $\mathbf{K}^{ab}$ *has uniqueness of limit models for every infinite cardinal.*

Remark 5.3.9. Fact 5.3.3 and Fact 5.3.2.(3) together with [Vas18, 3.7, 11.3, 11.7] imply that $\mathbf{K}^{ab}$ has uniqueness of limit models above $\beth_{(2^{\aleph_0})^+}$, so the result of the above corollary is only new for small cardinals.

5.4 Torsion-free abelian groups

In this fourth section, we study the class of torsion-free abelian groups with the pure subgroup relation. In the first half of the section we examine basic properties of the class while in the second one we look at limit models. As we will see in this case the theory becomes more interesting.

Definition 5.4.1. Let $\mathbf{K}^{tf} = (K^{tf}, \leq_p)$ where K^{tf} is the class of torsion-free abelian groups in the language $L_{ab} = \{0\} \cup \{+, -\}$ and $\leq_p$ is the pure subgroup relation. Recall that H is a *pure subgroup* of G if for every $n \in \mathbb{N}$ it holds that $nG \cap H = nH$.

5.4.1 Basic properties

Before analyzing the set of limit models, we obtain a few basic properties for the class of torsion-free abelian groups. As for abelian groups the basic properties of torsion-free abelian groups were studied in [BCG+] and [BET07].

Fact 5.4.2.

1. $\mathbf{K}^{tf}$ is an AEC with $\text{LS}(\mathbf{K}^{tf}) = \aleph_0$.
2. $\mathbf{K}^{tf}$ admits intersections.
3. $\mathbf{K}^{tf}$ has joint embedding, amalgamation and no maximal models.

Proof. (1) and (3) are shown in [BCG+, 3.3] and [BET07] and (2) is known to hold (an argument for this is given in [Fuc15, §5.1]). $\square$

The following proposition characterizes the closure operator in $\mathbf{K}^{tf}$, since the proof is a straightforward induction we omit it.

Proposition 5.4.3. *If $A \subseteq H$, then $cl^H_{\mathbf{K}^{tf}}(A) = \bigcup_{n<\omega} A_n$ where:*

- $A_0 = A$.
- $A_{2k+1} = \{-h : h \in A_{2k}\} \cup \{\Sigma_{i=0}^n h_i : h_0, ..., h_n \in A_{2k}, n \in \mathbb{N}\}$.
- $A_{2k+2} = \{h \in H :$ *there are* $h^* \in A_{2k+1}$ *and* $n \in \mathbb{N}$ *s.t.* $nh = h^*\}$.

Recall the following definition from [Vas17c, 3.1].

Definition 5.4.4. $\mathbf{K}$ is a *pseudo-universal class* if it admits intersections and for any $N_1, N_2 \in K$ and $\bar{a}_1 \in N_1$, $\bar{a}_2 \in N_2$, if $\mathbf{tp}(\bar{a}_1/\emptyset; N_1) = \mathbf{tp}(\bar{a}_2/\emptyset; N_2)$ and $f, g : cl^{N_1}(\bar{a}_1) \cong cl^{N_2}(\bar{a}_2)$ are such that $f(\bar{a}_1) = g(\bar{a}_1) = \bar{a}_2$, then $f = g$.

The reason pseudo-universal classes will be of interest to us is due to the following statement showed in [Vas17c, 3.7].

Fact 5.4.5. If $\mathbf{K}$ is a pseudo-universal class, then $\mathbf{K}$ is $(< \aleph_0)$-tame and short.

With this let us prove the following lemma.

Lemma 5.4.6. $\mathbf{K}^{tf}$ *is a pseudo-universal class. In particular,* $\mathbf{K}^{tf}$ *is $(< \aleph_0)$-tame and short.*

Proof. Let $H \in K^{tf}$, $\bar{a}, \bar{b} \in H$ with $\mathbf{tp}(\bar{a}/\emptyset; H) = \mathbf{tp}(\bar{b}/\emptyset; H)$ and $f, g : cl^H_{\mathbf{K}^{tf}}(\bar{a}) \cong cl^H_{\mathbf{K}^{tf}}(\bar{b})$ such that $f(\bar{a}) = g(\bar{a}) = \bar{b}$. We show by induction that $f \restriction_{A_n} = g \restriction_{A_n}$ for all $n < \omega$, where the A_n's are obtained by applying Proposition 5.4.3 to $cl^H_{\mathbf{K}^{tf}}(\bar{a})$. The base step is the hypothesis, so we do the induction step. The odd step is straightforward, so we do the even step. Let $h \in A_{2k+2}$, by definition there is $h^* \in A_{2k+1}$ and $n \in \mathbb{N}$ such that $nh = h^*$, then since f, g are isomorphisms we have that $nf(h) = f(h^*)$ and $ng(h) = g(h^*)$. By induction hypothesis $f(h^*) = g(h^*)$, so $nf(h) = ng(h)$; using that divisors in torsion-free groups are unique, we obtain that $f(h) = g(h)$. Hence $\mathbf{K}^{tf}$ is pseudo-universal. The fact that $\mathbf{K}^{tf}$ is $(< \aleph_0)$-tame and short follows from Fact 5.4.5. $\square$

In [BET07, 0.3] the following key result is obtained.

Fact 5.4.7. $\mathbf{K}^{tf}$ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$. In particular, $\mathbf{K}^{tf}$ is a stable AEC.

5.4.2 Limit models

In this subsection we classify the limit models in the class of torsion-free groups. It is clear that they are not divisible groups because if G is not divisible then G can not be a pure subgroup of a divisible group, but as we will show they are the next best thing, at least when the cofinality of the chain is uncountable. The examination of limit models will be done in two cases, we will first look at chains of uncountable cofinality and then at those of countable cofinality.

Remark 5.4.8. Since $\mathbf{K}^{tf}$ has joint embedding, amalgamation and no maximal models, $\mathbf{K}^{tf}$ has limit models when $\lambda^{\aleph_0} = \lambda$ (and only in those cardinals) by Fact 5.4.7 and Fact 5.2.11.

Recall the following characterization of algebraically compact groups [Fuc15, §6.1.3]. For more on algebraically compact groups the reader can consult [Fuc15, §6].

Definition 5.4.9. A group G is algebraically compact if given $\mathbb{E} = \{f_i(x_{i_0}, ..., x_{i_{n_i}}) = a_i : i < \omega\}$ a set of linear equations over G, $\mathbb{E}$ is finitely solvable in G if and only if $\mathbb{E}$ is solvable in G.

Lemma 5.4.10. *If G is a (λ, α)-limit model and* $\mathrm{cf}(\alpha) \geq \omega_1$*, then G is algebraically compact.*

Proof. Fix $\{G_\beta : \beta < \alpha\}$ a witness to the fact that G is a (λ, α)-limit model. Let $\mathbb{E} = \{f_i(x_{i_0}, ..., x_{i_{n_i}}) = a_i : i < \omega\}$ a set of linear equations finitely solvable in G. Since $\mathrm{cf}(\alpha) \geq \omega_1$ there is $\beta^* < \alpha$ such that $\{a_i : i < \omega\} \subseteq G_{\beta^*}$. Add new constants $\{c_i : i < \omega\}$ and let Σ be the following set of formulas:

$$\{f_i(c_{i_0}, ..., c_{i_{n_i}}) = a_i : i < \omega\} \cup ED(G_{\beta^*}) \cup T_{tf} \cup \{\neg\exists x(nx = g) : G_{\beta^*} \vDash \neg\exists x(nx = g), n \in \mathbb{N}, g \in G_{\beta^*}\},$$

where T_{tf} is the first-order theory of torsion-free abelian groups and $ED(G_{\beta^*})$ is the elementary diagram of G_{β^*}.

Since $\mathbb{E}$ is finitely solvable in G and $G_{\beta^*} \leq_p G$, it is easy to show that any finite subset of Σ is realized in G. Then by compactness and Löwenheim-Skolem-Tarski there is $H \in K^{tf}_\lambda$ such that $G_{\beta^*} \leq_p H$ (G_{β^*} is a pure subgroup by the last element in the definition of Σ) and $H \vDash \{f_i(c_{i_0}, ..., c_{i_{n_i}}) = a_i : i < \omega\}$. Using the fact that G_{β^*+1} is universal over G_{β^*}, there is $f : H \underset{G_{\beta^*}}{\longrightarrow} G_{\beta^*+1}$ and it is easy to show that $\{f(c_i^H) : i < \omega\}$ is a set of solutions to $\mathbb{E}$ which is contained in G. $\square$

As a simple corollary we obtain a new proof for the following well-known assertion, the assertion without the torsion-free hypothesis appears for example in [Fuc15, §6 1.10].

Corollary 5.4.11. *Every torsion-free group can be embedded as a pure subgroup in a torsion-free algebraically compact group.*

Proof. Follows from the joint embedding property, Fact 5.2.10 and the previous lemma. $\square$

Before proving a theorem parallel to Theorem 5.3.7, we prove the following proposition. In it the group $\mathbb{Z}_{(p)}$ will play a crucial role, recall that $\mathbb{Z}_{(p)} = \{n/m : (m,p) = 1\}$.

Proposition 5.4.12. *If G is a (λ, α)-limit model, then $dim_{\mathbb{F}_p}(G/pG) = \lambda$ for all p prime.*[6]

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α)-limit model. Notice that $G_0 \leq_p G_0 \oplus (\oplus_\lambda \mathbb{Z}_{(p)})$, then using that G_1 is universal over G_0, there is $f : G_0 \oplus (\oplus_\lambda \mathbb{Z}_{(p)}) \underset{G_0}{\longrightarrow} G$. In particular, we may assume that $(\oplus_\lambda \mathbb{Z}_{(p)}) \leq_p G$.

Claim: $\{e_i : i < \lambda\} \subseteq (\oplus_\lambda \mathbb{Z}_{(p)}) \subseteq G$ satisfy that for every $g \in G$, $A \subseteq_{fin} \lambda$ and $(n_i)_{i \in A} \in \{0, ..., p-1\}^{|A|} \backslash \{\bar{0}\}$ the following holds:

$$\Sigma_{i \in A} n_i e_i \neq pg.$$

Where each e_i is the i^{th}-element of the canonical basis.

Proof of Claim: Suppose for the sake a contradiction that it is not the case, then there is $g \in G$, $A \subseteq_{fin} \lambda$ and $(n_i)_{i \in A} \in \{0, ..., p-1\}^{|A|} \backslash \{\bar{0}\}$ such that

$$\Sigma_{i \in A} n_i e_i = pg.$$

Since $(\oplus_\lambda \mathbb{Z}_{(p)}) \leq_p G$ and $G \in K^{tf}$, we have that $g \in (\oplus_\lambda \mathbb{Z}_{(p)})$. Then $g = \Sigma_{i \in B} g_i$ for $B \subseteq_{fin} \lambda$ and unique $(g_i)_{i \in B} \in \mathbb{Z}_{(p)}^{|B|}$. Hence using the above equality it follows that $n_i = pg_i$ for each $i \in A$. Then p would divide the denominator of g_i for some $i \in A$, contradicting the fact that each $g_i \in \mathbb{Z}_{(p)}$, or $g = 0$, contradicting the linear independence of the e_i's.$\dagger_{\text{Claim}}$

From the above claim it follows that $\{e_i + pG : i < \lambda\}$ is a linearly independent set over $\mathbb{F}_p$. Hence $dim_{\mathbb{F}_p}(G/pG) = \lambda$. $\square$

The following fact puts together the information from [EkFi72, §1] that we will need in this paper.[7]

Fact 5.4.13. If G is a torsion-free algebraically compact group, then:

$$G \cong (\oplus_\delta \mathbb{Q}) \oplus \Pi_{p \text{ prime}} \overline{(\oplus_{\beta_p} \mathbb{Z}_{(p)})}.$$

[6]Notice that the proposition includes the case when the cofinality of α is countable.

[7] We recommend the reader to take a look at [EkFi72, §1] or [Fuc15, §6.3].

Where:

1. $\beta_p = dim_{\mathbb{F}_p}(G/pG)$ for all p prime ([EkFi72, 1.7.a]).
2. $\delta = rk_0(G_d)$, where G_d is the maximal divisible subgroup of G ([EkFi72, 1.10]).
3. $\mathbb{Z}_{(p)} = \{n/m : (m,p) = 1\}$ for p prime and the overline refers to the completion[8] (look at the discussion between [EkFi72, 1.4] and [EkFi72, 1.6]).

Lemma 5.4.14. *If G is a (λ,α)-limit model and G is algebraically compact, then*

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_{p\ prime}\overline{(\oplus_\lambda \mathbb{Z}_{(p)})}.$$

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ,α)-limit model. Since by hypothesis G is algebraically compact, by Fact 5.4.13 it is enough to show that $\beta_p = \lambda$ for all p prime and that $\delta = \lambda$.

By Fact 5.4.13.(1) and Proposition 5.4.12 it follows that $\beta_p = dim_{\mathbb{F}_p}(G/pG) = \lambda$ for all p prime, so we just need to show that $\delta = \lambda$. Observe that $G_0 \leq_p G_0 \oplus (\oplus_\lambda \mathbb{Q})$, then there is $f : G_0 \oplus (\oplus_\lambda \mathbb{Q}) \underset{G_0}{\longrightarrow} G$, from which it follows that $rk_0(G_d) = \lambda$ since $f[(\oplus_\lambda \mathbb{Q})] \subseteq G_d$. Hence by Fact 5.4.13.(2), we have that $\delta = \lambda$. □

With this we obtain our main result on limit models of uncountable cofinality.

Theorem 5.4.15. *If G is a (λ,α)-limit model and $\mathrm{cf}(\alpha) \geq \omega_1$, then*

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_{p\ prime}\overline{(\oplus_\lambda \mathbb{Z}_{(p)})}.$$

Proof. By Lemma 5.4.10 G is algebraically compact. Then the result follows from Lemma 5.4.14. □

The following corollary follows directly from Theorem 5.4.15.

Corollary 5.4.16. *If G is a (λ,α)-limit model and H is a (λ,β)-limit model such that $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) \geq \omega_1$, then $G \cong H$.*

Remark 5.4.17. Since $\mathbf{K}^{tf}$ has joint embedding, amalgamation, no maximal models and is $(< \aleph_0)$-tame, by [Vas18, 3.7] non-splitting has weak continuity and then by [Vas18, 11.3, 11.7] it follows that $\mathbf{K}^{tf}$ has uniqueness of limit models for large λ and $\mathrm{cf}(\alpha)$. Therefore, the result of the above corollary is only new for small cardinals.

The next corollary follows from the above corollary doing a similar construction to [GrVas17, 2.8.(3)].

[8] For the reader familiar with abelian group theory, this is precisely the pure-injective hull (see [Fuc15, §6.4]).

Corollary 5.4.18. *If G is a (λ, α)-limit model and $\mathrm{cf}(\alpha) \geq \omega_1$, then G is λ-saturated.*[9]

This finishes the characterization of G when G is a (λ, α)-limit model and the cofinality of α is uncountable, we know tackle the question when the cofinality of α is countable. Regarding it, we will only have negative results, i.e., we will show that if G is a (λ, α)-limit model then G is not algebraically compact. In order to do that, we will use some deep results on AECs which appear in [GrVas17] and [Vas16c]. Realize that since limit models with lengths of chains of the same cofinality are isomorphic, we only need to study (λ, ω)-limit models.

The proof will be divided into two parts. In the first we will use [GrVas17] and [Vas16c] to show that for λ big (λ, ω)-limit models are not algebraically compact and in the second we will reflect the big groups into smaller cardinalities.

The following fact contains the information we will need from [GrVas17] and [Vas16c]. For the readers not familiar with the theory of AECs this can be taken as a black box.

Fact 5.4.19. Assume that $\mathbf{K}$ has joint embedding, amalgamation, no maximal models, $\mathrm{LS}(\mathbf{K}) = \aleph_0$ and is $(< \aleph_0)$-tame. Let $\lambda \geq \beth_{(2^{\aleph_0})^+ + \omega}$ be such that $\mathbf{K}$ is λ-stable and there is a saturated model of cardinality λ. If every limit model of cardinality λ is saturated, then $\mathbf{K}$ is χ-stable for every $\chi \geq \lambda$.

Proof sketch. By [GrVas17, 3.2] $\mathbf{K}$ does not have the $\aleph_0$-order property of length $\beth_{(2^{\aleph_0})^+}$. Then by [GrVas17, 3.18] $\mathbf{K}$ has no long splitting chains in λ. Since $\mathbf{K}$ has no long splitting chains in λ, is λ-stable and is $(< \aleph_0)$-tame by [Vas16c, 5.6] we can conclude that $\mathbf{K}$ is χ-stable for every $\chi \geq \lambda$. □

Lemma 5.4.20. *Let $\lambda \geq \beth_{(2^{\aleph_0})^+ + \omega}$. If G is a (λ, ω)-limit model, then G is not algebraically compact.*

Proof. Since G is a (λ, ω)-limit model, it follows that $\mathbf{K}^{tf}$ is λ-stable by Fact 5.2.11.

Assume for the sake of contradiction that G is algebraically compact, then by Lemma 5.4.14 $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_{p \text{ prime}} \overline{(\oplus_\lambda \mathbb{Z}_{(p)})}$. Then by Theorem 5.4.15 $\mathbf{K}^{tf}$ has uniqueness of limit models of cardinality λ. Hence every limit model of cardinality λ is saturated by [GrVas17, 2.8.(3)].

By Fact 5.4.2 and Lemma 5.4.6 $\mathbf{K}^{tf}$ has joint embedding, amalgamation, no maximal models, $\mathrm{LS}(\mathbf{K}^{tf}) = \aleph_0$ and is $(< \aleph_0)$-tame. Then by Fact 5.4.19 $\mathbf{K}^{tf}$ is χ-stable for every $\chi \geq \lambda$. But this contradicts Fact 5.4.7, since there is $\chi \geq \lambda$ such that $\chi^{\aleph_0} \neq \chi$. □

Lemma 5.4.21. *Let $\lambda < \beth_{(2^{\aleph_0})^+ + \omega}$. If G is a (λ, ω)-limit model, then G is not algebraically compact.*

[9] Recall that G is λ-saturated if for every $H \leq_{\mathbf{K}} G$ and $p \in \mathbf{gS}(H)$ such that $\|H\| < \lambda$, p is realized in G. G is saturated if it is $\|G\|$-saturated.

Proof. Since G is a (λ, ω)-limit model, it follows that $\mathbf{K}^{tf}$ is λ-stable by Fact 5.2.11.

Let $\mu \geq \beth_{(2^{\aleph_0})^+ + \omega}$ such that $\mu^{\aleph_0} = \mu$, by Fact 5.4.7 $\mathbf{K}^{tf}$ is μ-stable. Let G^* a (μ, ω)-limit model witnessed by $\{G^*_i : i < \omega\}$. By Lemma 5.4.20 G^* is not algebraically compact, so there is $\mathbb{E} = \{f_k(x_{k_0}, ..., x_{k_{n_k}}) = a_k : k < \omega\}$ a set of linear equations finitely solvable in G^* but not solvable in G^*.

We build $\{r_i : i < \omega\} \subseteq \mathbb{N}$, $\{S_i : i < \omega\}$ and $\{H_i : i < \omega\}$ by induction such that:

1. $\{r_i : i < \omega\}$ is strictly increasing.
2. $a_i \in H_i$.
3. $S_i \subseteq H_i$ and S_i is a finite set.
4. S_i has a solution to $\{f_k(x_{k_0}, ..., x_{k_{n_k}}) = a_k : k \leq i\}$.
5. $H_i \leq_p G^*_{r_i}$.
6. $H_i \in \mathbf{K}^{tf}_\lambda$.
7. H_{i+1} is universal over H_i.

Before we do the construction, let us show that this is enough. Let $H_\omega := \bigcup_{i<\omega} H_i$, by (6) and (7) it follows that H_ω is a (λ, ω)-limit model. Since limit models of the same cofinality are isomorphic by Fact 5.2.12, it follows that $H_\omega \cong G$. So it is enough to show that H_ω is not algebraically compact. Assume for the sake of contradiction that H_ω is algebraically compact. Since $\mathbb{E} = \{f_k(x_{k_1}, ..., x_{k_{n_k}}) = a_k : k < \omega\}$ is finitely solvable in H_ω by (4), it follows that there is $\mathbf{a} \in H_\omega^\omega$ a solution for $\mathbb{E}$. But this contradicts the fact that $\mathbb{E}$ is not solvable in G^*, since $H_\omega \leq_p G^*$ by (5). Therefore, H_ω is not algebraically compact.

Now let us do the construction.

$\boxed{\text{Base}}$ Let $\{b_0, ..., b_l\} \subseteq G^*$ a solution to $f_0(x_{0_0}, ..., x_{0_{n_0}}) = a_0$, this exists by finite solvability of $\mathbb{E}$ in G^*, and $r < \omega$ such that $\{b_0, ..., b_l, a_0\} \subseteq G^*_r$. Let $r_0 := r$, $S_0 := \{b_0, ..., b_l\}$ and applying Löwenheim-Skolem-Tarski axiom to $\{b_0, ..., b_l, a_0\}$ in $G^*_{r_0}$ we get $H_0 \in \mathbf{K}^{tf}_\lambda$ such that $H_0 \leq_p G^*_{r_0}$ and $\{b_0, ..., b_l, a_0\} \subseteq H_0$. It is easy to see that this works.

$\boxed{\text{Induction step}}$ By construction there are $r_i \in \mathbb{N}$ and $H_i \leq_p G^*_{r_i}$. Since $\mathbf{K}^{tf}$ is λ-stable we can build $H \in \mathbf{K}^{tf}_\lambda$ such that H is universal over H_i by Fact 5.2.11. Using that $H_i \leq_p G^*_{r_i}$, the amalgamation property and that $G^*_{r_i+1}$ universal over $G^*_{r_i}$, there is $f : H \underset{H_i}{\longrightarrow} G^*_{r_i+1}$.

Let $\{b_0, ..., b_l\} \subseteq G^*$ a solution to $\{f_k(x_{k_0}, ..., x_{k_{n_k}}) = a_k : k \leq i+1\}$ and take $r \geq r_i + 1$ such that $\{b_0, ..., b_l, a_{i+1}\} \subseteq G^*_r$. Let $r_{i+1} := r$, $S_{i+1} := \{b_0, ..., b_l\}$ and applying Löwenheim-Skolem-Tarski axiom to $f[H] \cup \{b_0, ..., b_l, a_{i+1}\}$ in $G^*_{r_{i+1}}$ we get

$H_{i+1} \in \mathbf{K}^{tf}_\lambda$ such that $H_{i+1} \leq_p G^*_{r_{i+1}}$ and $f[H] \cup \{b_0, ..., b_l, a_{i+1}\} \subseteq H_{i+1}$. Using that $H_i \leq_p f[H] \leq_p H_{i+1}$ and that $f[H]$ is universal over H_i, it is easy to show that (1) through (7) hold. $\square$

Putting together the last two lemmas we obtain the following.

Theorem 5.4.22. *If G is a (λ, ω)-limit model, then G is not algebraically compact.*

Proof. If $\lambda \geq \beth_{(2^{\aleph_0})^+ + \omega}$ it follows from Lemma 5.4.20 and if $\lambda < \beth_{(2^{\aleph_0})^+ + \omega}$ it follows from Lemma 5.4.21. $\square$

Remark 5.4.23. After discussing Theorem 5.4.22 with Sebastien Vasey, he realized that by applying [Vas18, 4.12] instead of [GrVas17, 3.18] one could prove Theorem 5.4.22 without dividing the proof into cases. The proof using [Vas18, 4.12] is similar to that of Lemma 5.4.20. We decided to keep our original argument since the proof presented here shows how to transfer the failure of being algebraically compact and since we believe that showing that there are cofinally many (λ, ω)-limit models that are not algebraically compact is provable using only group theoretic methods.

Since (λ, ω)-limit models are not algebraically compact we ask:

Question 5.4.24. Is there a natural class of groups that contain the (λ, ω)-limit models?

Regarding the structure of (λ, ω)-limit models, using the fact that every group is a direct sum of a divisible group and a reduced group[10] (see [Fuc15, §4.2.5]), it is straightforward to show that if G is a (λ, ω)-limit model, then $G \cong (\oplus_\lambda \mathbb{Q}) \oplus G_r$ where $G_r \cong G/G_d$, G_d is the maximal divisible subgroup of G and G_r is reduced. So it is natural to ask the following.

Question 5.4.25. Is there a structure theorem for (λ, ω)-limit models similar to that of Theorem 5.4.15?

Let us conclude with the main theorem of this section.

Theorem 5.4.26. *If G is a (λ, α)-limit model in $\mathbf{K}^{tf}$, then we have that:*

1. *If the cofinality of α is uncountable, then $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_{p\ prime} \overline{(\oplus_\lambda \mathbb{Z}_{(p)})}$.*
2. *If the cofinality of α is countable, then G is not algebraically compact.*

In particular, $\mathbf{K}^{tf}$ does not have uniqueness of limit models for any infinite cardinal.

Proof. The first part is Theorem 5.4.15 and the second one is Theorem 5.4.22. The "in particular" follows from the fact that limit models with chains of uncountable cofinality are algebraically compact by (1), while those with chains of countable cofinality are not algebraically compact by (2). $\square$

[10]Recall that a group H is reduced if its only divisible subgroup is 0.

5.5 Finitely Butler Groups

In this last section, we look at some basic properties of the class of finitely Butler groups. The results in this section are weaker than those of the previous two sections and in some sense incomplete, but we decided to present them since we see this section as a stepping stone and moreover finitely Butler groups had never been isolated as an AEC.

Butler groups were introduced by Butler in [But65], while finitely Butler groups were first studied in [BiSa83] and given a name in [FuVi90]. We follow the exposition of [Fuc15, §14] and recommend the reader to consult it for further details.

Definition 5.5.1. A torsion-free group G of finite rank[11] is a *Butler group* if G is a pure subgroup of a finite rank completely decomposable group. Recall that a torsion-free group is *completely decomposable* if and only if it is the direct sum of groups of rank one.

Definition 5.5.2. A torsion-free group G is a *finitely Butler group* (B_0-group) if every pure subgroup of finite rank of G is a Butler group.

Let us introduce the class we will study.

Definition 5.5.3. Let $\mathbf{K}^{B_0} = (K^{B_0}, \leq_p)$ where K^{B_0} is the class of finitely Butler groups in the language $L_{ab} = \{0\} \cup \{+, -\}$ and $\leq_p$ is the pure subgroup relation.

Remark 5.5.4. Notice that if $G \in K^{B_0}$ and $H \leq_p G$, then $H \in K^{B_0}$.

Our first assertion is that indeed $\mathbf{K}^{B_0}$ is an AEC.

Lemma 5.5.5. *$\mathbf{K}^{B_0} = (K^{B_0}, \leq_p)$ is an AEC with $\mathrm{LS}(\mathbf{K}^{B_0}) = \aleph_0$ that admits intersections.*

Proof. From the closure under pure subgroups and the fact that $\mathbf{K}^{tf}$ is an AEC, it follows that $\mathbf{K}^{B_0}$ satisfies all the axioms of an AEC except the first Tarski-Vaught axiom. We show that it holds.[12]

Let $\{G_i : i < \delta\}$ such that $G_i \leq_p G_j$ for all $i < j$ and $G = \bigcup_{i<\delta} G_i$. It is clear that $G_i \leq_p G$ for all $i < j$, so we only need to show that $G \in K^{B_0}$, so let $H \leq_p G$ of finite rank.

Take X a finite maximal linearly independent subset of H, it exists because H has finite rank. Since X is finite, there is $i < \delta$ such that $X \subseteq G_i$. Since X is maximal linearly independent $H \subseteq span_{\mathbb{Q}}(X)$. Then using that $G_i \leq_p G$ and G_i is torsion-free, it follows that $H \leq_p G_i$. Therefore, since $G_i \in K^{B_0}$, we conclude that H is a Butler group.

[11] Given G a torsion-free group, the rank of G is $rk_0(G)$ (see footnote 6 for the definition).
[12] This is exercise [Fuc15, §14.4.1].

Moreover, the class admits intersections because $\mathbf{K}^{tf}$ admits intersections and the closure of $\mathbf{K}^{B_0}$ under pure subgroups. □

Fact 5.5.6. $\mathbf{K}^{B_0}$ has joint embedding and no maximal models.

Proof. By [Fuc15, §14.5.(B)] $\mathbf{K}^{B_0}$ is closed under direct sums so the result follows. □

Regarding the amalgamation property, we are only able to provide the following partial solution. We actually think that the amalgamation property might not hold for the class.

Lemma 5.5.7. *If $G \in K^{B_0}$ and G is divisible, then G is an amalgamation base, i.e., if $G \leq_p H_i \in K^{B_0}$ for $i \in \{1, 2\}$, then there are $H \in K^{B_0}$ and $f_i : H_i \to H$ for $i \in \{1, 2\}$ such that $f_1 \restriction_G = f_2 \restriction_G$.*

Proof. Let $G \leq_p H_i$ for $i \in \{1, 2\}$. Let $H := H_1 \oplus H_2/G^*$ where $G^* := \{(g, -g) : g \in G\}$, $f_1 : H_1 \to H$ be $f(h) = (h, 0) + G^*$ and $f_2 : H_2 \to H$ be $f(h) = (0, h) + G^*$. In [BCG+, 3.27] it is shown that $H \in K^{tf}$, f_1, f_2 are pure embeddings and $f_1 \restriction_G = f_2 \restriction_G$. So we only need to show that $H \in K^{B_0}$.

Let $E \subseteq H_1 \oplus H_2$ such that $E/G^* \leq_p H_1 \oplus H_2/G^*$ and E/G^* has rank n. Take $\{e_i + G^* : i < n\}$ a maximal linearly independent subset of E/G^*.

Observe that $E \leq_p H_1 \oplus H_2$, because $G^* \leq_p H_1 \oplus H_2$ and $E/G^* \leq_p H_1 \oplus H_2/G^*$. Moreover, $cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\}) \leq_p H_1 \oplus H_2$, $cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})$ has finite rank and $H_1 \oplus H_2 \in K^{B_0}$ (see [Fuc15, §14.5.(B)]), so it follows that $cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})$ is a Butler group (where the closure is the one described in Proposition 5.4.3 by Remark 11.4.21).

Claim: $E = G^* + cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})$.

Proof of Claim: Let $e \in E$, since $\{e_i + G^* : i < n\}$ is maximal linearly independent $e + G^* \in span_{\mathbb{Q}}(\{e_i + G^* : i < n\})$, then there are $\{m, k_0, ..., k_{n-1}\} \subseteq \mathbb{N}$ and $g_0^* \in G^*$ such that:

$$me = \Sigma_{i=0}^{n-1} k_i e_i + g_0^*.$$

Since G is divisible, G^* is divisible so there is $g_1^* \in G^*$ such that $mg_1^* = g_0^*$. Then $m(e - g_1^*) = \Sigma_{i=0}^{n-1} k_i e_i$, thus $e - g_1^* \in cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})$. Hence $e \in G^* + cl^{E}{}_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})$.$\dagger_{\text{Claim}}$

Then $E/G^* \cong G^* + cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})/G^* \cong cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})/cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\}) \cap G^*$. By the fact that torsion-free epimorphic images of Butler groups are Butler groups (see [Fuc15, §14.1.6]) and that $cl^E_{\mathbf{K}^{B_0}}(\{e_0, ..., e_{n-1}\})$ is a Butler group, we conclude that E/G^* is a Butler group. Hence $H \in K^{B_0}$. □

The next proposition is straightforward, but we include it because of its strong consequences.

Proposition 5.5.8. *If $G, H \in K^{B_0}$, $\mathbf{a} \in G$, $\mathbf{b} \in H$ and $A \subseteq G, H$, then $\mathbf{tp}_{\mathbf{K}^{B_0}}(\mathbf{a}/A; G) = \mathbf{tp}_{\mathbf{K}^{B_0}}(\mathbf{b}/A; H)$ if and only if $\mathbf{tp}_{\mathbf{K}^{tf}}(\mathbf{a}/A; G) = \mathbf{tp}_{\mathbf{K}^{tf}}(\mathbf{b}/A; H)$.*

Proof. Since $\mathbf{K}^{B_0}$ is closed under pure subgroups by Remark 11.4.21, using the minimality of the closures, it is easy to show that for all $H' \in K^{B_0}$ and $B \subseteq H'$ it holds that $cl^{H'}_{\mathbf{K}^{B_0}}(B) = cl^{H'}_{\mathbf{K}^{tf}}(B)$. Then using that $\mathbf{K}^{B_0}$ and $\mathbf{K}^{tf}$ admit intersections and Fact 5.2.4 the result follows. $\square$

Corollary 5.5.9.

- *$\mathbf{K}^{B_0}$ is $(< \aleph_0)$-tame and short.*
- *If $\lambda = \lambda^{\aleph_0}$, then $\mathbf{K}^{B_0}$ is λ-stable. In particular, $\mathbf{K}^{B_0}$ is a stable AEC.*

Proof. The proof follows directly from Proposition 5.5.8 and the fact that $\mathbf{K}^{tf}$ satisfies both of the properties we are trying to show. $\square$

Question 5.5.10. Do we have as in $\mathbf{K}^{tf}$ that: if $\mathbf{K}^{B_0}$ is λ-stable, then $\lambda = \lambda^{\aleph_0}$?

We were unable to answer the above question, but we have a partial solution (see Lemma 5.5.12). In order to present it, we will need some results from [Fuc15, §12.1] and the following definitions.

Definition 5.5.11. Let G be a torsion-free abelian group and $a \in G$:

- Given a prime p the *p-height* of a (denoted by $h_p(a)$) is the maximum $n \in \mathbb{N}$ such that $p^n | a$ or ∞ if the maximum does not exist.
- The *characteristic* of a is $\chi_G(a) = (h_{p_n}(a))_{n<\omega}$ where $\{p_n : n < \omega\}$ is an increasing enumeration of the prime numbers.
- Given $\eta, \nu \in (\mathbb{N} \cup \{\infty\})^\omega$ we define the equivalence relation $\sim$ as $\eta \sim \nu$ if and only if η and ν differ on finitely many natural numbers and when they differ they are both finite. A *type* $\boldsymbol{t}$ is an element of $(\mathbb{N} \cup \{\infty\})^\omega / \sim$ and the *type of* a is $\mathbf{t}_G(a) = \chi_G(a)/\sim$.
- We say that G has type $\mathbf{t}$, if for every $b \neq 0 \in G$ it holds that $\mathbf{t} = \mathbf{t}_G(b)$.

The proof of the following lemma uses similar ideas to those of [KojSh95, 3.7].

Lemma 5.5.12. *If $\lambda < 2^{\aleph_0}$, then $\mathbf{K}^{B_0}$ is not λ-stable.*

Proof. Let $G \in \mathbf{K}^{B_0}_\lambda$ and $\{\mathbf{t}_\eta : \eta \in 2^\omega\}$ an enumeration of all the types (in the sense of the previous definition). For each $\eta \in 2^\omega$, let G_η a group of rank one with type $\mathbf{t}_\eta$, it exists by [Fuc15, §12.1.1]. Let $H = G \oplus (\oplus_{\eta \in 2^\omega} G_\eta)$. Since $\mathbf{K}^{B_0}$ is closed under direct sums (see [Fuc15, §14.5.(B)]) and rank one groups are in $\mathbf{K}^{B_0}$, because they are completely decomposable, we have that $H \in K^{B_0}$.

For each $\eta \in 2^\omega$ take $a_\eta \in G_\eta$ with $a_\eta \neq 0$ and let $p_\eta := \mathbf{tp}(a_\eta/G; H)$. We show that all the Galois-types in the set $\{p_\eta : \eta \in 2^\omega\}$ are different.

Claim: If $\eta \neq \nu \in 2^\omega$, then $p_\eta \neq p_\nu$.

Proof of Claim: Suppose for the sake of contradiction that $\mathbf{tp}(a_\eta/G; H) = \mathbf{tp}(a_\nu/G; H)$, then by Fact 5.2.4 there is $f : cl^H_{\mathbf{K}^{B_0}}(\{a_\eta\} \cup G) \cong_G cl^H_{\mathbf{K}^{B_0}}(\{a_\nu\} \cup G)$ with $f(a_\eta) = a_\nu$. Then since the closures give rise to pure subgroups of H we have that $\chi_H(a_\eta) = \chi_H(a_\nu)$, so $\mathbf{t}_H(a_\eta) = \mathbf{t}_H(a_\nu)$. This contradicts the fact that $\mathbf{t}_H(a_\eta) = \mathbf{t}_{G_\eta}(a_\eta) = \mathbf{t}_\eta \neq \mathbf{t}_\nu = \mathbf{t}_{G_\nu}(a_\nu) = \mathbf{t}_H(a_\nu)$, the first and last equality follow from the fact that $G_\eta, G_\nu \leq_p H$. $\dagger_{\text{Claim}}$

Therefore, $|\mathbf{gS}(G)| \geq 2^{\aleph_0}$. Since $\lambda < 2^{\aleph_0}$, $\mathbf{K}$ is not λ-stable. $\square$

As we mentioned in the introduction we are interested in limit models, therefore we ask the following:

Question 5.5.13. Do limit models exist in $\mathbf{K}^{B_0}$? If they exist, what is their structure?

Regarding the first part of the question, realize that if $\mathbf{K}^{B_0}$ has the amalgamation property, then by Corollary 5.5.9 and Fact 5.2.11 limit models would exist. As for the second part, even if they existed the techniques to characterize them would have to be different from the ones presented in section four since finitely Butler groups do not seem to be first-order axiomatizable.

Besides the function of this article as a pool of examples of limit models in the context of AECs. We believe that the study of limit models (in different classes of groups) as a classes of infinite rank groups could be an interesting area of research on its own. We think this is possible since limit model are tame enough to be analyzable, but their theory is nontrivial as showcased in this article. A good place to look for new classes of limit models is [BET07].

Chapter 6

A model theoretic solution to a problem of László Fuchs

This chapter is based on [Ch. 6].

Abstract

Problem 5.1 in page 181 of [Fuc15] asks to find the cardinals λ such that there is a universal abelian p-group for purity of cardinality λ, i.e., an abelian p-group U_λ of cardinality λ such that every abelian p-group of cardinality $\leq \lambda$ purely embeds in U_λ. In this paper we use ideas from the theory of abstract elementary classes to show:

Theorem 6.0.1. *Let p be a prime number. If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then there is a universal abelian p-group for purity of cardinality λ. Moreover for $n \geq 2$, there is a universal abelian p-group for purity of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.*

As the theory of abstract elementary classes has barely been used to tackle algebraic questions, an effort was made to introduce this theory from an algebraic perspective.

6.1 Introduction

The aim of this paper is to address Problem 5.1 in page 181 of [Fuc15]. The problem stated by Fuchs is the following:

Main Problem. For which cardinals λ is there a universal abelian p-group for purity? We mean an abelian p-group U_λ of cardinality λ such that every abelian p-group of cardinality $\leq \lambda$ embeds in U_λ as a pure subgroup. The same question for torsion-free abelian groups.

The question for torsion-free abelian groups has been thoroughly studied as witnessed by [KojSh95], [Sh622], [Sh820] and [Ch. 7]. Due to this, we focus in this paper in the case of abelian p-groups[1]. Regarding abelian p-groups, there are some results for the subclass of separable abelian p-groups, these results appeared in [KojSh95], [Sh622] and [Sh820].

The solution we provide for the Main Problem extends the ideas presented in [Ch. 7] to the class of abelian p-groups.[2] We show that there are many cardinals with universal abelian p-groups for purity:

Theorem 6.3.6. *Let p be a prime number. If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then there is a universal abelian p-group for purity of cardinality λ.*

The proof has three main steps.[3] First, we identify the class of abelian p-groups with pure embeddings as an abstract elementary class and show that it has amalgamation, joint embedding and no maximal models (see Fact 6.3.1). Then, we show that the class of abelian p-groups with pure embeddings is λ-stable if $\lambda^{\aleph_0} = \lambda$ (see Theorem 6.3.5). Finally, the assertion follows from using some general results on abstract elementary classes (see Theorem 6.3.6).

Using some results of [KojSh95] it is possible to show that if some cardinal inequalities hold, then there are some cardinals where there can not be universal abelian p-groups for purity (see Lemma 6.3.8). The techniques used to obtain this result are explained in detail in [KojSh95], [Dža05] and [Bal20, §2.4].

As a simple corollary of Theorem 6.3.6 and what was mentioned in the paragraph above, we obtain a complete solution to the Main Problem below $\aleph_\omega$ with the exception of $\aleph_0$ and $\aleph_1$.

Corollary 6.3.9. *Let p be a prime number. For $n \geq 2$, there is a universal abelian p-group for purity of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.*

---

[1] Recall that G is an abelian p-group if it is an abelian group and every element of G different from zero has order p^n for some $n \in \mathbb{N}$.

[2] For the reader familiar with first-order logic and [Ch. 7], the reason we can not simply apply the results of [Ch. 7] to abelian p-groups is because the class of abelian p-groups is not first-order axiomatizable.

[3] All the definitions needed to understand this paragraph are given in Section 2 with many algebraic examples to showcase them.

We address the case of $\aleph_1$ in Lemma 6.3.10 and show that the answer depends on the value of the continuum and a combinatorial principle. We leave open the case of $\aleph_0$ (see Question 6.3.11 and the remark below it).

In Section 2, we make an effort to present all the necessary notions of abstract elementary classes that are needed to understand the proof of the main theorem (Theorem 6.3.6). We present them from an algebraic perspective and give many examples. In particular, we do not assume that the reader is familiar with logic.

The paper is divided into four sections. Section 2 presents an introduction to abstract elementary classes from an algebraic point of view. Section 3 has the main results. Section 4 presents how the main results can be generalized to other classes.

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6.2 An introduction to AECs from an algebraic point of view

This section will introduce the basic notions of abstract elementary classes that are used in this paper and will hopefully motivate the use of abstract elementary classes to tackle algebraic questions. This section assumes no familiarity with logic, with the exception of Example 6.2.7.

Abstract elementary classes (AECs for short) were introduced by Shelah [Sh88] in the mid-seventies to capture the semantic structure of non-first-order theories. The definition of AEC will mention the following logical notions: a language τ, τ-structures and the substructure relation. In this paper, the language τ will always be $\{0, +, -\} \cup \{r\cdot : r \in R\}$ where R is a fixed ring and $r\cdot$ is interpreted as multiplication by r for every $r \in R$. So, a class of τ-structures will be a class of R-modules for a fixed ring R. Moreover, being a substructure will mean being a submodule. The definitions of all of these notions can be found in [Mar02, §I]. We will give many algebraic examples of AECs right after its definition. For a structure M, we denote by $|M|$ the underlying set of M and by $\|M\|$ its cardinality.

Definition 6.2.1. An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

1. K is a class of τ-structures, for some fixed language $\tau = \tau(\mathbf{K})$.

2. $\leq_{\mathbf{K}}$ is a partial ordering on K extending the substructure relation.

3. $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.

4. Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$.

5. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:

   (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.

   (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.

6. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

Below we introduce many examples of AECs in the context of algebra. Recall that (for abelian groups) G is a *pure subgroup* of H, denoted by $G \leq_p H$, if and only if $nH \cap G = nG$ for every $n \in \mathbb{N}$. For R-modules M and N, we say that M is a pure submodule of N if for every L right R-module $L \otimes M \to L \otimes N$ is a monomorphism.

Example 6.2.2. We begin by giving some examples of abstract elementary classes contained in the class of abelian groups:

- $\mathbf{K}^{Ab}_{\leq} := (Ab, \leq)$ where Ab is the class of abelian groups.

- $\mathbf{K}^{Ab} := (Ab, \leq_p)$ where Ab is the class of abelian groups.

- $\mathbf{K}^{p\text{-grp}} = (K^{p\text{-grp}}, \leq_p)$ where $K^{p\text{-grp}}$ is the class of abelian p-groups for p a prime number. A group G is a p-group if every element $g \neq 0$ has order p^n for some $n \in \mathbb{N}$.

- $\mathbf{K}^{Tor} = (\mathrm{Tor}, \leq_p)$ where Tor is the class of abelian torsion groups. A group G is a torsion group if every element $g \neq 0$ has finite order.

- $\mathbf{K}^{tf} = (K^{tf}, \leq_p)$ where K^{tf} is the class of torsion-free abelian groups. A group G is a torsion-free group if every element has infinite order.

- $\mathbf{K}^{rtf} = (K^{rtf}, \leq_p)$ where K^{rtf} is the class of reduced torsion-free abelian groups. A group G is reduced if it does not have non-trivial divisible subgroups.

- $\mathbf{K}^{B_0} = (K^{B_0}, \leq_p)$ where K^{B_0} is the class of finitely Butler groups. A group G is a finitely Butler group if G is torsion-free and every pure subgroup of finite rank is a pure subgroup of a finite rank completely decomposable group (see [Fuc15, §14.4] for more details).
- $\mathbf{K}^{\aleph_1\text{-free}} = (\aleph_1\text{-free}, \leq_p)$ where $\aleph_1$-free is the class of $\aleph_1$-free groups. A group G is $\aleph_1$-free if every countable subgroup of G is free.

Many of these examples can be extended to arbitrary rings. Below are some examples of AECs in classes of modules:

- $(R\text{-Mod}, \subseteq_R)$ where R-Mod is the class of R-modules.
- $(R\text{-Mod}, \leq_p)$ where R-Mod is the class of R-modules.
- $(R\text{-Flat}, \leq_p)$ where R-Flat is the class of flat R-modules. An R-module F is flat if $(-) \otimes F$ is an exact functor.

Let us now introduce some notation.

Notation 6.2.3.

- For λ an infinite cardinal, $\mathbf{K}_\lambda = \{M \in K : \|M\| = \lambda\}$.
- Let $M, N \in K$. If we write "$f : M \to N$" we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. In the classes studied in this paper a $\mathbf{K}$-embedding is either a monomorphism or a pure monomorphism.

The next three properties are properties that an AEC may or may not have. The first one is a weakening of the notion of pushout.

Definition 6.2.4.

- $\mathbf{K}$ has the amalgamation property if for every $M, N_1, N_2 \in K$ such that $M \leq_{\mathbf{K}} N_1, N_2$ there are $N \in K$, and $\mathbf{K}$-embeddings $f_1 : N_1 \to N$ and $f_2 : N_2 \to N$ such that $f_1 \restriction_M = f_2 \restriction_M$.
- $\mathbf{K}$ has the joint embedding property if for every $M, N \in K$ there are $L \in K$ and $\mathbf{K}$-embeddings f, g such that $f : M \to L$ and $g : N \to L$.
- $\mathbf{K}$ has no maximal models if for every $M \in K$, there is $N \in K$ such that $M \leq_{\mathbf{K}} N$ and $M \neq N$.

Example 6.2.5. All the AECs introduced in Example 6.2.2 have joint embedding and no maximal models, this is the case as they are all closed under direct sums. As for the amalgamation property, all the AECs introduced in Example 6.2.5 have it with the exception of $\mathbf{K}^{\aleph_1\text{-free}}$, and the possible exception of $\mathbf{K}^{B_0}$. The problem remains open in the latter case.

The next notion was introduced by Shelah in [Sh300], it extends the notion of first-order type to this setting. It is one of the key notions that let us answer the Main Problem.

Definition 6.2.6. Let $\mathbf{K}$ be an AEC.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in K$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.

2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E_{\text{at}}^{\mathbf{K}} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exist $\mathbf{K}$-embeddings $f_\ell : N_\ell \underset{A}{\to} N$ for $\ell \in \{1, 2\}$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$ and $N \in \mathbf{K}$.

3. Note that $E_{\text{at}}^{\mathbf{K}}$ is a symmetric and reflexive relation on $\mathbf{K}^3$. We let $E^{\mathbf{K}}$ be the transitive closure of $E_{\text{at}}^{\mathbf{K}}$.

4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_{E^{\mathbf{K}}}$. We call such an equivalence class a *Galois-type*. Usually, $\mathbf{K}$ will be clear from the context and we will omit it.

5. For $M \in K$, $\mathbf{gS}_{\mathbf{K}}(M) = \{\mathbf{tp}_{\mathbf{K}}(b/M; N) : M \leq_{\mathbf{K}} N \in K \text{ and } b \in N\}$.

Example 6.2.7. This is the only place where we assume the reader is familiar with basic logic notions. The necessary logic background is presented in [Ch. 7, §2.2].

- ([Ch. 5, 3.12]) In $\mathbf{K} = \mathbf{K}^{Ab}_{\leq}$ we have that for any G, G_1, G_2 in $\mathbf{K}$ with $G \leq G_1, G_2$, $\bar{b}_1 \in G_1^{<\omega}$ and $\bar{b}_2 \in G_2^{<\omega}$,

  $\mathbf{tp}_{\mathbf{K}}(\bar{b}_1/G; G_1) = \mathbf{tp}_{\mathbf{K}}(\bar{b}_2/G; G_2)$ if and only if $qf\text{-}tp(\bar{b}_1/G, G_1) = qf\text{-}tp(\bar{b}_2/G, G_2)$.

  Where $qf\text{-}tp(\bar{b}_1/G, G_1)$ is the set of quantifier-free formulas with parameters in G that hold for $\bar{b}_1$ in G_1.

- ([Ch. 7, 3.14]) In $\mathbf{K} = \mathbf{K}^{Ab}$ or $\mathbf{K} = \mathbf{K}^{tf}$ we have that for G, G_1, G_2 in $\mathbf{K}$ with $G \leq_p G_1, G_2$, $\bar{b}_1 \in G_1^{<\omega}$ and $\bar{b}_2 \in G_2^{<\omega}$,

  $$\mathbf{tp}_{\mathbf{K}}(\bar{b}_1/G; G_1) = \mathbf{tp}_{\mathbf{K}}(\bar{b}_2/G; G_2) \text{ if and only if } \operatorname{pp}(\bar{b}_1/G, G_1) = \operatorname{pp}(\bar{b}_2/G, G_2).$$

  Where $\operatorname{pp}(\bar{b}_1/G, G_1)$ is the set of positive primitive formulas with parameters in G that hold for $\bar{b}_1$ in G_1.

A direct consequence of Lemma 6.4.5 (see below) is that the second bullet is also true for $\mathbf{K}^{p\text{-grp}}$ for any prime number p.

On a different direction, it is not known if the analogue is true for the class $(R\text{-Flat}, \leq_p)$. It is shown that this is the case under extra hypothesis in [Ch. 9, 4.4].

One of the main objectives of the theory of AECs is to find dividing lines analogous to those of first-order theories. A dividing line is a property such that the classes satisfying such a property have some nice behaviour while those not satisfying it have a bad one. An introduction to dividing lines for mathematicians not working in mathematical logic can be found in [Sh1151, Part I] and [Bal20]. One of the first dividing lines that was studied is that of stability.

Definition 6.2.8. An AEC $\mathbf{K}$ is λ-*stable* if for any $M \in \mathbf{K}_\lambda$, $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$. An AEC is stable if there is a $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is λ-stable.

As the following example will be used in the main section of the paper we recall it as a fact.

Fact 6.2.9.

1. ([BCG+], [Ch. 8, 3.12]) $\mathbf{K}^{Ab}_{\leq}$ is λ-stable for every $\lambda \geq \aleph_0$.
2. ([Ch. 7, 3.16]) $\mathbf{K}^{Ab}$ is λ-stable for every λ such that $\lambda^{\aleph_0} = \lambda$.
3. ([BET07, 0.3], [Sh820, 1.2], [Ch. 5, 5.9]) $\mathbf{K}^{tf}$, $\mathbf{K}^{rtf}$ and $\mathbf{K}^{B_0}$ are λ-stable for every λ such that $\lambda^{\aleph_0} = \lambda$.
4. $\mathbf{K}^{\aleph_1\text{-free}}$ is λ-stable for every λ such that $\lambda^{\aleph_0} = \lambda$.
5. ([Ch. 8, 3.6], [Ch. 7, 3.16]) $(R\text{-Mod}, \subseteq_R)$ and $(R\text{-Mod}, \leq_p)$ are λ-stable for every λ such that $\lambda^{|R|+\aleph_0} = \lambda$.
6. ([LRV1a, 6.20, 6.21]) $(R\text{-Flat}, \leq_p)$ is stable.

Proof sketch. The only bullet point that is missing references is (4). The proof is similar to the one of $\mathbf{K}^{B_0}$ (see [Ch. 5, 5.8]). It follows from the fact that $\aleph_1$-free groups are closed under pure subgroups. $\square$

Remark 6.2.10. We will obtain in Lemma 6.3.5 (see below) that $\mathbf{K}^{p\text{-grp}}$ is also λ-stable if $\lambda^{\aleph_0} = \lambda$.

As witnessed by Fact 6.2.9, Lemma 6.3.5 and Corollary 6.4.8, all the AECs of abelian groups and modules identified in this paper are stable. Moreover, a classical result of first-order model theory assures us that:

Fact 6.2.11 (Fisher, Baur, see e.g. [Pre88, 3.1])**.** If T is a complete first-order theory extending the theory of modules, then $(Mod(T), \leq_p)$ is stable.

The same result is obtained for incomplete first-order theories of modules closed under direct sums in [Ch. 7, 3.16]. So we ask if the above is still true for all AECs of modules.

Question 6.2.12. Let R be an associative ring with an identity element.

If $(K, \leq_p)$ is an AEC such that $K \subseteq R$-Mod, is $(K, \leq_p)$ stable? Is this true if $R = \mathbb{Z}$?

Let us recall the classical notion of a universal model.

Definition 6.2.13. Let $\mathbf{K}$ be an AEC and λ be an infinite cardinal. We say that $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is a $\mathbf{K}$-embedding $f : N \to M$. We say that $\mathbf{K}$ has a universal model of cardinality λ if there is a universal model in $\mathbf{K}_\lambda$.

Remark 6.2.14. Let p be a prime number. Observe that the phrase *there is a universal abelian p-group for purity of cardinality* λ means precisely that $\mathbf{K}^{p\text{-}grp}$ *has a universal model of cardinality* λ. We will use the latter in the rest of the paper.

One of the nice consequences of being stable is that it is easy to build universal models in many cardinals.

Fact 6.2.15 ([Ch. 7, 3.20])**.** Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models. Assume there is $\theta_0 \geq \mathrm{LS}(\mathbf{K})$ and κ such that for all $\theta \geq \theta_0$, if $\theta^\kappa = \theta$, then $\mathbf{K}$ is θ-stable.

Suppose $\lambda > \theta_0$. If $\lambda^\kappa = \lambda$ or $\forall \mu < \lambda(\mu^\kappa < \lambda)$, then $\mathbf{K}$ has a universal model of cardinality λ.

This is all the theory of AECs that is required to follow the proof of the main theorem (Theorem 6.3.6). Connections between algebra and AECs were studied in: [GrSh83], [GrSh86], [BET07], [ŠaTr12], [Ch. 5], [Ch. 7], [BCG+], [Bon20, §5], [LRV1a, §6], [Ch. 8] and [Ch. 9]. Finally, the reader interested in a more in depth introduction to AECs can consult: [Gro02], [Bal09] and [Sh:h].

6.3 Main result

In this section we will study the class of abelian p-groups with pure embeddings for any prime number p. We introduced these classes as the third bullet point of Example 6.2.2. Following the notation of Example 6.2.2, we will denote them by $\mathbf{K}^{p\text{-grp}}$ for any prime number p.

The following assertion contains many known facts about abelian p-groups.

Fact 6.3.1. Assume p is a prime number.

1. $\mathbf{K}^{p\text{-grp}} = (K^{p\text{-grp}}, \leq_p)$ is an AEC with $\mathrm{LS}(\mathbf{K}^{p\text{-grp}}) = \aleph_0$.

2. $\mathbf{K}^{p\text{-grp}}$ has the amalgamation property, the joint embedding property and no maximal models.

3. The class of abelian p-groups is not closed under pure-injective envelopes.

4. The class of abelian p-groups is not first-order axiomatizable.

Proof. (1) is trivial and (2) follows from the closure of abelian p-groups under direct sums and pushouts. As for closure under pure-injective envelopes, recall that the pure-injective envelope of $\oplus_n \mathbb{Z}(p^n)$ is $\Pi_n \mathbb{Z}(p^n)$, which is not an abelian p-group. That the class of abelian p-groups is not first-order axiomatizable follows from the last line together with the fact that the pure-injective envelope of a module is an elementary extension. $\square$

Remark 6.3.2. Items (3) and (4) of the above assertion will not be used, but hint to the difficulty of dealing with this class from a model theoretic perspective.

Remark 6.3.3. It is worth mentioning that if the Generalized Continuum Hypothesis (GCH) holds[4], then just from the fact that $\mathbf{K}^{p\text{-grp}}$ is an AEC with amalgamation, joint embedding and no maximal models, one can easily show that $\mathbf{K}^{p\text{-grp}}$ has a universal model of cardinality λ for every λ uncountable cardinal.

We begin by characterizing Galois-types in the class of abelian p-groups. Recall, from Example 6.2.2, that $\mathbf{K}^{Ab}$ is the class of abelian groups with pure embeddings.

Theorem 6.3.4. *Assume p is a prime number. Let $G_1, G_2 \in K^{p\text{-}grp}$, $A \subseteq G_1, G_2$, $\bar{b}_1 \in G_1^{<\omega}$ and $\bar{b}_2 \in G_2^{<\omega}$, then:*

$\mathbf{tp}_{\mathbf{K}^{Ab}}(\bar{b}_1/A; G_1) = \mathbf{tp}_{\mathbf{K}^{Ab}}(\bar{b}_2/A; G_2)$ *if and only if* $\mathbf{tp}_{\mathbf{K}^{p\text{-}grp}}(\bar{b}_1/A; G_1) = \mathbf{tp}_{\mathbf{K}^{p\text{-}grp}}(\bar{b}_2/A; G_2)$.

Proof. The backward direction is trivial as $\mathbf{K}^{Ab}$ and $\mathbf{K}^{p\text{-grp}}$ are both AECs with respect to pure embeddings. We prove the forward direction. Assume that $\mathbf{tp}_{\mathbf{K}^{Ab}}(\bar{b}_1/A; G_1) = \mathbf{tp}_{\mathbf{K}^{Ab}}(\bar{b}_2/A; G_2)$, then by definition of Galois-types there are $G \in K^{Ab}$, $f_1 : G_1 \to G$ and $f_2 : G_2 \to G$ such that $f_1 \restriction_A = f_2 \restriction_A$ and $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$. Let $L = f[G_1] + f[G_2] \leq G$. Observe that $f[G_1] + f[G_2]$ is an abelian p-group and that$f[G_1], f[G_2] \leq_p f[G_1] + f[G_2]$ since f_1, f_2 are pure embeddings. Therefore, the following square is a commutative square in $\mathbf{K}^{p\text{-grp}}$:

$$\begin{tikzcd} G_2 \arrow[r, "f_1"] & f_1[G_1] + f_2[G_2] \\ A \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & G_2 \arrow[u, "f_2"'] \end{tikzcd}$$

[4]GCH states that $2^\lambda = \lambda^+$ for every infinite cardinal λ.

and $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$. Hence $\mathbf{tp}_{\mathbf{K}^{p\text{-grp}}}(\bar{b}_1/A; G_1) = \mathbf{tp}_{\mathbf{K}^{p\text{-grp}}}(\bar{b}_2/A; G_2)$. $\square$

We show that $\mathbf{K}^{p\text{-grp}}$ is stable for every prime number p.

Lemma 6.3.5. *Assume p is a prime number. If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{p\text{-}grp}$ is λ-stable.*

Proof. Let $G \in K^{p\text{-grp}}_\lambda$ and $\{\mathbf{tp}_{\mathbf{K}^{p\text{-grp}}}(b_i/G; G_i) : i < \alpha\}$ be an enumeration without repetitions of $\mathbf{gS}_{\mathbf{K}^{p\text{-grp}}}(G)$. Let $\Phi : \mathbf{gS}_{\mathbf{K}^{p\text{-grp}}}(G) \to \mathbf{gS}_{\mathbf{K}^{Ab}}(G)$ be given by $\Phi(\mathbf{tp}_{\mathbf{K}^{p\text{-grp}}}(b_i/G; G_i)) = \mathbf{tp}_{\mathbf{K}^{Ab}}(b_i/G; G_i)$. Then by Theorem 6.3.4, Φ is a well-defined injective function, so $|\mathbf{gS}_{\mathbf{K}^{p\text{-grp}}}(G)| \leq |\mathbf{gS}_{\mathbf{K}^{Ab}}(G)|$. Then as $\mathbf{K}^{Ab}$ is λ-stable, by Fact 6.2.9.(2), we conclude that $|\mathbf{gS}_{\mathbf{K}^{p\text{-grp}}}(G)| \leq \lambda$. $\square$

With this we are able to obtain many cardinals such that there are universal models for abelian p-groups for purity.

Theorem 6.3.6. *Let p be a prime number. If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then $\mathbf{K}^{p\text{-}grp}$ has a universal model of cardinality λ.*

Proof. By Lemma 6.3.5 $\mathbf{K}^{p\text{-grp}}$ is λ-stable if $\lambda^{\aleph_0} = \lambda$. As $\mathbf{K}^{p\text{-grp}}$ has amalgamation, joint embedding and no maximal models by Fact 6.3.1, it follows by Fact 6.2.15 that there is a universal model of cardinality λ if $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$. $\square$

Remark 6.3.7. Recall that for an abelian group G, $t_p(G) = \{g \in G :$ there exists an $n \in \mathbb{N}$ s.t. $p^n g = 0\}$. One can show that if $G \in K^{Ab}$ is a universal group in $\mathbf{K}^{Ab}_\lambda$, then $t_p(G) \in K^{p\text{-grp}}$ is a universal group in $\mathbf{K}^{p\text{-grp}}_\lambda$. In light of this observation, the above result also follows from [Ch. 7, 3.19]. The reason we decided to present the above argument for Theorem 6.3.6 is to showcase how the technology of AECs can be used to obtain universal models. Moreover, the method presented in this section gives us more information about $\mathbf{K}^{p\text{-grp}}$ (it shows that $\mathbf{K}^{p\text{-grp}}$ is stable) and can be generalized to other classes (see Section 4).

The above theorem is the main result in the positive direction. The next result shows that if certain inequalities hold, then there are cardinals where there are no universal models.

Lemma 6.3.8. *Let λ be a regular cardinal and μ be a regular cardinal. If $\mu^+ < \lambda < \mu^{\aleph_0}$, then $\mathbf{K}^{p\text{-}grp}$ does not have a universal model of cardinality λ.*

Proof sketch. In [KojSh95, 3.3] it is shown that there is no universal model of cardinality λ in the class of separable abelian p-groups for purity. What is presented there can be used to conclude the stronger result that there is no universal model of cardinality λ in the class of abelian p-groups for purity. $\square$

As a simple corollary we obtain:

Corollary 6.3.9. *For $n \geq 2$, $\mathbf{K}^{p\text{-}grp}$ has a universal model of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.*

Proof. $\boxed{\rightarrow}$ Assume for the sake of contradiction that $2^{\aleph_0} > \aleph_n$, then $\aleph_0^+ < \aleph_n < \aleph_0^{\aleph_0}$. So we get a contradiction by Lemma 6.3.8.

$\boxed{\leftarrow}$ If $2^{\aleph_0} \leq \aleph_n$, then $\aleph_n^{\aleph_0} = 2^{\aleph_0}\aleph_n = \aleph_n$ where the first equality follows from Hausdorff formula. Therefore, by Theorem 6.3.6, there is a universal model of cardinality $\aleph_n$. □

The above corollary gives a complete solution to the Main Problem below $\aleph_\omega$ except for the cases of $\aleph_0$ and $\aleph_1$. The next lemma addresses the case of $\aleph_1$.

Lemma 6.3.10.

1. *If $2^{\aleph_0} = \aleph_1$, then $\mathbf{K}^{p\text{-}grp}$ has a universal model of cardinality $\aleph_1$.*
2. *If $2^{\aleph_0} > \aleph_1$ and the combinatorial principle ♣ holds*[5], *then $\mathbf{K}^{p\text{-}grp}$ does not have a universal model of cardinality $\aleph_1$.*

Proof. (1) follows directly from Theorem 6.3.6, so we focus on (2). It is easy to show that ♣ implies the existence of a club guessing sequence in $\aleph_1$ in the sense of [KojSh95, 1.5]. Then the result follows from [KojSh95, 3.3] by a similar argument to the one given in Lemma 6.3.8. □

So for cardinals below $\aleph_\omega$ we are only left with $\aleph_0$. So we ask:

Question 6.3.11. Is there a universal model in $\mathbf{K}^{p\text{-grp}}$ of cardinality $\aleph_0$?

Remark 6.3.12. In the case of $\aleph_0$, we think that the question can be answered using group theoretic methods. For instance, using Exercise 6 of [Fuc15, §11.1] it is possible to build an abelian p-group G of cardinality $\aleph_1$ such that every countable abelian p-group is purely embeddable into G.

Remark 6.3.13. For cardinals greater than or equal to $\aleph_\omega$, Theorem 6.3.6 gives many instances for existence of universal models in $\mathbf{K}^{p\text{-grp}}$. For example, Theorem 6.3.6 implies the existence of a universal model in $2^{\aleph_1}$ or $\beth_\omega$. On the other hand, Lemma 6.3.8 gives instances where there are no universal models if GCH fails. There are still some cardinals that are not covered by any of these cases, but based on what is known about reduced torsion-free groups and separable torsion groups (see [Sh1151, §10.(B)], in particular Table 1 at the end of that paper), we think that the answer in those cases depends even more on set theoretic hypotheses and not on the theory of AECs.

[5] ♣ is a combinatorial principle similar to $\diamondsuit$, but weaker. For the definition and what is known about ♣, the reader can consult [Sh:f, §I.7].

6.4 Some generalizations

The key ideas we used to understand $\mathbf{K}^{p\text{-grp}}$ were that $\mathbf{K}^{p\text{-grp}}$ fits nicely inside $\mathbf{K}^{Ab}$ and that the class $\mathbf{K}^{Ab}$ is well-understood and well-behaved. As we think that these ideas might have further applications, we abstract this set up in the next definition.

Definition 6.4.1. Let $\mathbf{K} = (K, \leq_{\mathbf{K}})$ and $\mathbf{K}^\star = (K^\star, \leq_{\mathbf{K}})$ be a pair of AECs. We say that $\mathbf{K}^\star$ is nicely generated inside $\mathbf{K}$ if:

1. $K^\star \subseteq K$.
2. For any $N_1, N_2 \in K^\star$ and $N \in K$, if $N_1, N_2 \leq_{\mathbf{K}} N$, then there is $L \in K^\star$ such that $N_1, N_2 \leq_{\mathbf{K}} L \subseteq N$.

Example 6.4.2.

1. $\mathbf{K}^{p\text{-grp}}$ is nicely generated inside $\mathbf{K}^{Ab}$. Given $N_1, N_2 \in K^{p\text{-grp}}$ and $N \in Ab$ such that $N_1, N_2 \leq_p N$, then $L = N_1 + N_2 \in K^{p\text{-grp}}$ and it satisfies that $N_1, N_2 \leq_p L \subseteq N$.
2. $\mathbf{K}^{\text{Tor}}$ is nicely generated inside $\mathbf{K}^{Ab}$.
3. $\mathbf{K}^{p\text{-grp}}_{\leq} = (K^{p\text{-grp}}, \leq)$ and $\mathbf{K}^{\text{Tor}}_{\leq} = (K^{\text{Tor}}, \leq)$ are nicely generated inside $\mathbf{K}^{Ab}_{\leq}$.
4. $\mathbf{K}^{R\text{-Tor}} = (K^{R\text{-Tor}}, \leq_p)$ is nicely generated inside $(R\text{-Mod}, \leq_p)$ where R is an integral domain and $K^{R\text{-Tor}}$ is the class of R-torsion modules. An R-module M is an R-torsion module if for every $m \in M$ there is $r \neq 0 \in R$ such that $rm = 0$.
5. $\mathbf{K}^{R\text{-Div}} = (K^{R\text{-Div}}, \leq_p)$ is nicely generated inside $(R\text{-Mod}, \leq_p)$ where R is an integral domain and $K^{R\text{-Div}}$ is the class of R-divisible modules. An R-module M is an R-divisible module if for every $m \in M$ and $r \neq 0 \in R$, there is an $n \in M$ such that $rn = m$.
6. $\mathbf{K}^{\text{Div}}_{\leq} = (K^{\text{Div}}, \leq)$ is nicely generated inside $\mathbf{K}^{Ab}_{\leq}$, where K^{Div} is the class of divisible abelian groups.

The next lemma is the key observation.

Lemma 6.4.3. *Assume $\mathbf{K}^\star = (K^\star, \leq_{\mathbf{K}})$ is nicely generated inside $\mathbf{K} = (K, \leq_{\mathbf{K}})$. If $M \leq_{\mathbf{K}} N_1, N_2 \in K^\star$ and there are $N' \in K$ and $\mathbf{K}$-embeddings $f_1 : N_1 \to N'$ and $f_2 : N_2 \to N'$ such that $f_1 \restriction_M = f_2 \restriction_M$, then there are $L \in K^\star$ and $\mathbf{K}^\star$-embeddings $g_1 : N_1 \to L$ and $g_2 : N_2 \to L$ such that $L \subseteq N'$ and $i \circ g_\ell = f_\ell$ for $\ell \in \{1, 2\}$ where $i : L \to N'$ is the inclusion.*

Corollary 6.4.4. *Assume $\mathbf{K}^\star = (K^\star, \leq_{\mathbf{K}})$ is nicely generated inside $\mathbf{K} = (K, \leq_{\mathbf{K}})$. If $\mathbf{K}$ has the amalgamation property, then $\mathbf{K}^\star$ has the amalgamation property.*

The next two result are proven by generalizing the proofs of Theorem 6.3.4 and Theorem 6.3.5 respectively.

Corollary 6.4.5. *Assume* $\mathbf{K}^\star = (K^\star, \leq_{\mathbf{K}})$ *is nicely generated inside* $\mathbf{K} = (K, \leq_{\mathbf{K}})$. *Let* $N_1, N_2 \in K^\star$, $A \subseteq N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ *and* $\bar{b}_2 \in N_2^{<\omega}$, *then:*

$$\mathbf{tp}_{\mathbf{K}}(\bar{b}_1/A; N_1) = \mathbf{tp}_{\mathbf{K}}(\bar{b}_2/A; N_2) \text{ if and only if } \mathbf{tp}_{\mathbf{K}^\star}(\bar{b}_1/A; N_1) = \mathbf{tp}_{\mathbf{K}^\star}(\bar{b}_2/A; N_2)$$

Theorem 6.4.6. *Assume* $\mathbf{K}^\star = (K^\star, \leq_{\mathbf{K}})$ *is nicely generated inside* $\mathbf{K} = (K, \leq_{\mathbf{K}})$. *If* $\mathbf{K}$ *is* λ*-stable and* $\lambda \geq \mathrm{LS}(\mathbf{K}^\star)$, *then* $\mathbf{K}^\star$ *is* λ*-stable.*

Remark 6.4.7. Another consequence of Corollary 6.4.5 is that if $\mathbf{K}^\star = (K^\star, \leq_{\mathbf{K}})$ is nicely generated inside $\mathbf{K} = (K, \leq_{\mathbf{K}})$ then: if $\mathbf{K}$ is $(< \aleph_0)$-tame then $\mathbf{K}^\star$ is $(< \aleph_0)$-tame. Therefore, we get that $\mathbf{K}^{p\text{-grp}}$ is $(< \aleph_0)$-tame as $\mathbf{K}^{Ab}$ is $(< \aleph_0)$-tame by [Ch. 7, 3.15]. Recall that an AEC is $(< \aleph_0)$-tame if its Galois-types are determined by their restrictions to finite sets (the reader can consult the definition in [Ch. 5, 1.6]).

Finally, these results can be used to obtain stability cardinals and universal models.

Corollary 6.4.8. *Let* p *be a prime number and* R *be an integral domain.*

1. *If* $\lambda^{\aleph_0} = \lambda$, *then* $\mathbf{K}^{p\text{-}grp}$ *and* $\mathbf{K}^{Tor}$ *are* λ*-stable. If* $\lambda^{\aleph_0} = \lambda$ *or* $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, *then* $\mathbf{K}^{p\text{-}grp}$ *and* $\mathbf{K}^{Tor}$ *have a universal model of cardinality* λ.
2. *If* $\lambda^{|R|+\aleph_0} = \lambda$, *then* $\mathbf{K}^{R\text{-}Tor}$ *and* $\mathbf{K}^{R\text{-}Div}$ *are* λ*-stable. If* $\lambda^{|R|+\aleph_0} = \lambda$ *or* $\forall \mu < \lambda(\mu^{|R|+\aleph_0} < \lambda)$, *then* $\mathbf{K}^{R\text{-}Tor}$ *and* $\mathbf{K}^{R\text{-}Div}$ *have a universal model of cardinality* λ.
3. $\mathbf{K}^{p\text{-}grp}_{\leq}$, $\mathbf{K}^{Tor}_{\leq}$ *and* $\mathbf{K}^{Div}_{\leq}$ *are* λ*-stable and have a universal model of cardinality* λ *for every* $\lambda \geq \aleph_0$.

Remark 6.4.9. Universal models for torsion groups with embeddings and divisible groups with embeddings can be obtained by algebraic methods. More precisely, $\oplus_{p \text{ prime}}(\oplus_\lambda \mathbb{Z}(p^\infty))$ is the universal model of size λ for torsion groups and $\oplus_{p \text{ prime}}(\oplus_\lambda \mathbb{Z}(p^\infty)) \oplus \oplus_\lambda \mathbb{Q}$ is the universal model of size λ for divisible groups.

Chapter 7

On universal modules with pure embeddings

This chapter is based on [Ch. 7] and is joint work with Thomas G. Kucera. In this chapter the *first author* is Thomas G. Kucera and the *second author* is Marcos Mazari-Armida.

Abstract

We show that certain classes of modules have universal models with respect to pure embeddings.

Theorem 7.0.1. *Let R be a ring, T be a first-order theory with an infinite model extending the theory of R-modules and $\mathbf{K}^T = (Mod(T), \leq_p)$ (where $\leq_p$ stands for pure submodule). Assume $\mathbf{K}^T$ has joint embedding and amalgamation.*

If $\lambda^{|T|} = \lambda$ or $\forall \mu < \lambda(\mu^{|T|} < \lambda)$, then $\mathbf{K}^T$ has a universal model of cardinality λ.

As a special case we get a recent result of Shelah [Sh820, 1.2] concerning the existence of universal reduced torsion-free abelian groups with respect to pure embeddings.

We begin the study of limit models for classes of R-modules with joint embedding and amalgamation. We show that limit models with chains of long cofinality are pure-injective and we characterize limit models with chains of countable cofinality. This can be used to answer Question 4.25 of [Ch. 5].

As this paper is aimed at model theorists and algebraists an effort was made to provide the background for both.

7.1 Introduction

The first result concerning the existence of universal uncountable objects in classes of modules was [Ekl71]. In it, Eklof showed that there exists a homogeneous universal R-module of cardinality λ in the class of R-modules if and only if $\lambda^{<\gamma} = \lambda$ (where γ is the least cardinal such that every ideal of R is generated by less than γ elements).

Grossberg and Shelah [GrSh83] used the weak continuum hypothesis to answer a question of Macintyre and Shelah [MaSh76] regarding the existence of universal locally finite groups in uncountable cardinalities. Kojman and Shelah [KojSh95] and Shelah [Sh456], [Sh552], [Sh622] and [Sh820] continued the study of universal groups for certain classes of abelian groups with respect to embeddings and pure embeddings. For further historical comments the reader can consult [Dža05, §6].

In this paper, we will give a positive answer to the question of whether certain classes of modules with pure embeddings have universal models in specific cardinals. More precisely, we obtain:

Theorem 7.3.19. *Let R be a ring, T a first-order theory with an infinite model extending the theory of R-modules and $\mathbf{K}^T = (Mod(T), \leq_p)$ (where $\leq_p$ stands for pure submodule). Assume $\mathbf{K}^T$ has joint embedding and amalgamation.*

If $\lambda^{|T|} = \lambda$ or $\forall \mu < \lambda(\mu^{|T|} < \lambda)$, then $\mathbf{K}^T$ has a universal model of cardinality λ.

There are many examples of theories satisfying the hypothesis of Theorem 7.3.19 (see Example 7.3.10). One of them is the theory of torsion-free abelian groups. So as straightforward corollary we get:

Corollary 7.3.22. *If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then the class of torsion-free abelian groups with pure embeddings has a universal group of cardinality λ.*

In [Sh820, 1.2] Shelah shows a result analogous to the above theorem, but instead of working with the class of torsion-free abelian groups he works with the class of reduced torsion-free abelian groups. The reason Corollary 7.3.22 transfers to Shelah's setting is because every abelian group can be written as a direct sum of a unique divisible subgroup and a unique up to isomorphism reduced subgroup (see [Fuc15, §4.2.5]). Shelah's statement is Corollary 7.3.26 in this paper.

The proof presented here is not a generalization of Shelah's original idea. We prove first that the class is λ-stable (for $\lambda^{|T|} = \lambda$) and then using that the class is an abstract elementary class we construct universal extensions of size λ (for $\lambda^{|T|} = \lambda$). By contrast, Shelah first constructs universal extensions of cardinality λ (for $\lambda^{\aleph_0} = \lambda$) and from it he concludes that the class is λ-stable.

The methods used in both proofs are also quite different. We exploit the fact that any theory of R-modules has *pp*-quantifier elimination and that our class is an abstract elementary class with joint embedding and amalgamation. By contrast, Shelah's argument seems to only work in the restricted setting of torsion-free abelian groups. This is the case since the main device of his argument is the existence of a metric in reduced torsion-free abelian groups and the completions obtained from this metric.

In [Ch. 5], the second author began the study of limit models in classes of abelian groups. In this paper we go one step further and begin the study of limit models in classes of R-modules with joint embedding and amalgamation. *Limit models* were introduced in [KolSh96] as a substitute for saturation in the context of AECs. Intuitively the reader can think of them as universal models with some level of homogeneity (see Definition 7.2.10). They have proven to be an important concept in tackling Shelah's eventual categoricity conjecture. The key question has been the uniqueness of limit models of the same cardinality but of different length.[1]

We show that limit models in $\mathbf{K}^T$ are elementary equivalent (see Lemma 7.4.3). We generalize [Ch. 5, 4.10] by showing that limit models with chains of cofinality greater than $|T|$ are pure-injective (see Theorem 7.4.5). We characterize limit models with chains of countable cofinality for classes that are closed under direct sums (see Theorem 7.4.9). The main feature is that there is a natural way to construct universal models over pure-injective modules. More precisely, given M pure-injective and U a universal model of size $\|M\|$, $M \oplus U$ is universal over M. As a by-product of our study of limit models and [Ch. 5, 4.15] we answer Question 4.25 of [Ch. 5].

Theorem 7.4.14. *If G is a (λ, ω)-limit model in the class of torsion-free abelian groups with pure embeddings, then* $G \cong \mathbb{Q}^{(\lambda)} \oplus \left(\prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}} \right)^{(\aleph_0)}$.

Finally, combining Corollary 7.3.22 and Theorem 7.4.14, we are able to construct universal extensions of cardinality λ for some cardinals such that the class of torsion-free groups with pure embeddings is not λ-stable (an example for such a λ is $\beth_\omega$). This is the first example of an AEC with joint embedding, amalgamation and no maximal models in which one can construct universal extensions of cardinality λ without the hypothesis of λ-stability.

The paper is organized as follows. Section 2 presents necessary background. Section 3 studies classes of the form $\mathbf{K}^T$, studies universal models in these classes and shows how [Sh820, 1.2] is a special case of the theory developed in the section. Section 4 begins the study of limit models for classes of R-modules with joint embedding and amalgamation. It also answers Question 4.25 of [Ch. 5].

This paper was written while the second author was working on a Ph.D. under the direction of Rami Grossberg at Carnegie Mellon University and he would like to

[1] A more detailed account of the importance of limit models is given in [Ch. 5, §1].

thank Professor Grossberg for his guidance and assistance in his research in general and in this work in particular. We would also like to thank Sebastien Vasey for several comments that helped improve the paper. We would also like to thank John T. Baldwin for introducing us to one another and for useful comments that improved the paper. We are grateful to the referees for their comments that significantly improved the paper.

7.2 Preliminaries

We introduce the key concepts of abstract elementary classes and the model theory of modules that are used in this paper. Our primary references for the former are [Bal09, §4 - 8] and [Gro2X, §2, §4.4]. Our primary references for the latter is [Pre88].

7.2.1 Abstract elementary classes

Abstract elementary classes (AECs) were introduced by Shelah in [Sh88]. Among the requirements we have that an AEC is closed under directed colimits and that every set is contained in a small model in the class. Given a model M, we will write $|M|$ for its underlying set and $\|M\|$ for its cardinality.

Definition 7.2.1. An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

1. K is a class of τ-structures, for some fixed language $\tau = \tau(\mathbf{K})$.
2. $\leq_{\mathbf{K}}$ is a partial order on K.
3. $(K, \leq_{\mathbf{K}})$ respects isomorphisms:

   If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$.

   In particular (taking $M = N$), K is closed under isomorphisms.
4. If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
5. Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $|M_0| \subseteq |M_1|$, then $M_0 \leq_{\mathbf{K}} M_1$.
6. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:

   (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.

(b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.

7. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

Notation 7.2.2.

- If λ is cardinal and $\mathbf{K}$ is an AEC, then $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$.
- Let $M, N \in \mathbf{K}$. If we write "$f : M \to N$" we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. In particular $\mathbf{K}$-embeddings are always monomorphisms.
- Let $M, N \in \mathbf{K}$ and $A \subseteq M$. If we write "$f : M \underset{A}{\to} N$" we assume that f is a $\mathbf{K}$-embedding and that $f \restriction_A = \mathrm{id}_A$.

Let us recall the following three properties. They are satisfied by all the classes considered in this paper, although not every AEC satisfies them.

Definition 7.2.3.

1. $\mathbf{K}$ has the *amalgamation property* if for every $M, N, R \in \mathbf{K}$ such that $M \leq_{\mathbf{K}} N, R$, there is $R^* \in \mathbf{K}$ with $R \leq_{\mathbf{K}} R^*$ and a $\mathbf{K}$-embedding $f : N \underset{M}{\longrightarrow} R^*$.

2. $\mathbf{K}$ has the *joint embedding property* if for every $M, N \in \mathbf{K}$, there is $R^* \in \mathbf{K}$ with $N \leq_{\mathbf{K}} R^*$ and a $\mathbf{K}$-embedding $f : M \to R^*$.

3. $\mathbf{K}$ has *no maximal models* if for every $M \in \mathbf{K}$, there is $M^* \in \mathbf{K}$ such that $M <_{\mathbf{K}} M^*$.

In [Sh300] Shelah introduced a notion of semantic type. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We follow the notation of [Ch. 3, 2.5].

Definition 7.2.4. Let $\mathbf{K}$ be an AEC.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.

2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E^{\mathbf{K}}_{\mathrm{at}} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \underset{A}{\rightarrow} N$ $\mathbf{K}$-embeddings such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$ and $N \in \mathbf{K}$.

3. Note that $E^{\mathbf{K}}_{\mathrm{at}}$ is a symmetric and reflexive relation on $\mathbf{K}^3$. We let $E^{\mathbf{K}}$ be the transitive closure of $E^{\mathbf{K}}_{\mathrm{at}}$.

4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_{E^{\mathbf{K}}}$. We call such an equivalence class a *Galois-type*. Usually, $\mathbf{K}$ will be clear from the context and we will omit it.

5. For $M \in \mathbf{K}$, $\mathbf{gS}_{\mathbf{K}}(M) = \{\mathbf{tp}_{\mathbf{K}}(b/M; N) : M \leq_{\mathbf{K}} N \in \mathbf{K} \text{ and } b \in N\}$

6. For $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N)$ and $C \subseteq A$, $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) \restriction_C := [(\mathbf{b}, C, N)]_E$.

Remark 7.2.5. If $\mathbf{K}$ has amalgamation, it is straightforward to show that $E^{\mathbf{K}}_{\mathrm{at}}$ is transitive.

Definition 7.2.6. An AEC is λ-*stable* if for any $M \in \mathbf{K}_\lambda$, $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$.

The following notion was isolated by Grossberg and VanDieren in [GrVan06].

Definition 7.2.7. $\mathbf{K}$ is $(< \kappa)$-*tame* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}(M)$, there is $A \subseteq M$ such that $|A| < \kappa$ and $p \restriction_A \neq q \restriction_A$. $\mathbf{K}$ is κ-*tame* if it is $(< \kappa^+)$-tame.

Let us recall the following concept that was introduced in [KolSh96].

Definition 7.2.8. Let $\mathbf{K}$ be an AEC. M is λ-*universal over* N if and only if $N \leq_{\mathbf{K}} M$ and for any $N^* \in \mathbf{K}_{\leq \lambda}$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\rightarrow} M$. M is *universal over* N if and only if $\|N\| = \|M\|$ and M is $\|M\|$-*universal over* N.

The next fact gives conditions for the existence of universal extensions.

Fact 7.2.9 ([Sh:h, §II], [GrVan06, 2.9])**.** Let $\mathbf{K}$ an AEC with joint embedding, amalgamation and no maximal models. If $\mathbf{K}$ is λ-stable, then for every $P \in \mathbf{K}_\lambda$, there is $M \in \mathbf{K}_\lambda$ such that M is universal over P.

The following notion was introduced in [KolSh96] and plays an important role in this paper.

Definition 7.2.10. Let $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that $M_0 := N$, M_{i+1} is universal over M_i for each $i < \alpha$ and $M = \bigcup_{i<\alpha} M_i$. We say that $M \in \mathbf{K}_\lambda$ is a (λ, α)-limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. We say that $M \in \mathbf{K}_\lambda$ is a limit model if there is $\alpha < \lambda^+$ limit such that M is a (λ, α)-limit model.

Observe that by iterating Fact 7.2.9 there exist limit models in stability cardinals for AECs with joint embedding, amalgamation and no maximal models.

In this paper, we deal with the classical global notion of universal model.

Definition 7.2.11. Let $\mathbf{K}$ an AEC and λ a cardinal. $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is $f : N \to M$.

Remark 7.2.12. When an abstract elementary class has joint embedding, then M is universal over N or M is a limit model implies that M is a universal model in $\mathbf{K}_{\|M\|}$. A proof is given in [Ch. 5, 2.10].

7.2.2 Model theory of modules

For most of the basic results of the model theory of modules, we use the comprehensive text [Pre88] of M. Prest as our primary source. The detailed history of these results can be found there.

The following definitions are fundamental and will be used throughout the text.

Definition 7.2.13. Let R be a ring and $L_R = \{0, +, -\} \cup \{r : r \in R\}$ be the language of R-modules.

- $\phi(\bar{v})$ is a *pp-formula* if and only if

$$\phi(\bar{v}) = \exists w_1 ... \exists w_l (\bigwedge_{j=1}^{m} \Sigma_{i=1}^{n} r_{i,j} v_i + \Sigma_{k=1}^{l} s_{k,j} w_k = 0),$$

  where $r_{i,j}, s_{k,j} \in R$ for every $i \in \{1, ..., n\}, j \in \{1, ..., m\}, k \in \{1, ..., l\}$.

- Given N an R-module, $A \subseteq N$ and $\bar{b} \in N^{<\omega}$ we define the *pp-type* of $\bar{b}$ over A in N as

$$pp(\bar{b}/A, N) = \{\phi(\bar{v}, \bar{a}) : \phi(\bar{v}, \bar{w}) \text{ is a pp-formula, } \bar{a} \in A \text{ and } N \models \phi[\bar{b}, \bar{a}]\}.$$

- Given M, N R-modules we say that M is a *pure submodule* of N, written as $M \leq_p N$, if and only if $M \subseteq N$ and $pp(\bar{a}/\emptyset, M) = pp(\bar{a}/\emptyset, N)$ for every $\bar{a} \in M^{<\omega}$. Observe that in particular if $M \leq_p N$ then M is a submodule of N.

A key property of R-modules is that they have *pp*-quantifier elimination, i.e., every formula in the language of R-modules is equivalent to a boolean combination of *pp*-formulas.

Fact 7.2.14 (Baur-Monk-Garavaglia, see e.g. [Pre88, §2.4])**.** Let R be a ring and M be a (left) R-module. Every formula in the language of R-modules is equivalent modulo $Th(M)$ to a boolean combination of pp-formulas.

The above theorem makes the model theory of modules algebraic in character, and we will use many of its consequences throughout the text. See for example Facts 7.3.2, 7.3.3, 7.3.13 and 7.4.2.

Recall that given T a complete first-order theory and $A \subseteq M$ with M a model of T, $S^T(A)$ is the set of complete first-order types with parameters in A. A complete first-order theory T is λ-*stable* if $|S^T(A)| \leq \lambda$ for every $A \subseteq M$ with $|A| = \lambda$ and M a model of T. For a complete first-order theory T this is equivalent to $(Mod(T), \preceq)$ being λ-stable, where $\preceq$ is the elementary substructure relation.

Fact 7.2.15 (Fisher, Baur, see e.g. [Pre88, 3.1])**.** If T is a complete first-order theory extending the theory of R-modules and $\lambda^{|T|} = \lambda$, then T is λ-stable.

Pure-injective modules generalize the notion of injective module.

Definition 7.2.16. A module M is *pure-injective* if and only if for every module N, if $M \leq_p N$ then M is a direct summand of N.

There are many statements equivalent to the definition of pure-injectivity. The following equivalence will be used in the last section:

Fact 7.2.17 ([Pre88, 2.8])**.** Let M be an R-module. The following are equivalent:

1. M is pure-injective.
2. Every M-consistent pp-type $p(x)$ over $A \subseteq M$ with $|A| \leq |R| + \aleph_0$, is realized in M.[2]

That is, pure-injective modules are saturated with respect to *pp*-types. They often suffice as a substitute for saturated models in the model theory of modules.

We will also use the pure hull of a module. The next fact has all the information the reader will need about them. They are studied extensively in [Pre88, §4] and [Zie84, §3].

Fact 7.2.18.

1. For M a module the *pure hull of* M, denoted by $\overline{M}$, is a pure-injective module such that $M \leq_p \overline{M}$ and it is minimum with respect to this. Its existence follows from [Zie84, 3.6] and the fact that every module can be embedded in a pure-injective module.
2. [Sab70] For M a module, $M \preceq \overline{M}$.

[2] For an incomplete theory T we say that a pp-type $p(x)$ over $A \subseteq M$ is M-consistent if it is realized in an elementary extension of M.

7.2.3 Torsion-free groups

The following class will be studied in detail.

Definition 7.2.19. Let $\mathbf{K}^{tf} = (K^{tf}, \leq_p)$ where K^{tf} is the class of torsion-free abelian groups in the language $L_{\mathbb{Z}} = \{0, +, -\} \cup \{z : z \in \mathbb{Z}\}$ (the usual language of $\mathbb{Z}$-modules)and $\leq_p$ is the pure subgroup relation. Recall that H is a pure subgroup of G if for every $n \in \mathbb{N}$, $nG \cap H = nH$.

It is known that $\mathbf{K}^{tf}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{tf}) = \aleph_0$ that has joint embedding, amalgamation and no maximal models (see [BCG+], [BET07] or [Ch. 5, §4]). Furthermore limit models of uncountable cofinality were described in [Ch. 5].

Fact 7.2.20 ([Ch. 5, 4.15])**.** If $G \in \mathbf{K}^{tf}$ is a (λ, α)-limit model and $\mathrm{cf}(\alpha) \geq \omega_1$, then

$$G \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}}.$$

7.3 Universal models in classes of R-modules

In this section we will construct universal models for certain classes of R-modules.

Notation 7.3.1. Given R a ring, we denote by $\mathbf{Th}_R$ the theory of left R-modules. Given T a first-order theory (not necessarily complete) extending the theory of (left) R-modules, let $\mathbf{K}^T = (Mod(T), \leq_p)$ and $|T| = |R| + \aleph_0$.

Our first assertion will be that $\mathbf{K}^T$ is always an abstract elementary class. In order to prove this, we will use the following two corollaries of *pp*-quantifier elimination (Fact 7.2.14). Given $n \in \mathbb{N}$ and ϕ, ψ *pp*-formulas such that $\mathbf{Th}_R \vdash \psi \to \phi$ we denote by $\mathrm{Inv}(-, \phi, \psi) \geq n$ the first-order sentence satisfying: $M \models \mathrm{Inv}(-, \phi, \psi) \geq n$ if and only if $[\phi(M) : \psi(M)] \geq n$. Such a formula is called an *invariant condition.*

Fact 7.3.2 ([Pre88, 2.15])**.** Every sentence in the language of R-modules is equivalent, modulo the theory of R-modules, to a boolean combination of invariant conditions.

Fact 7.3.3 ([Pre88, 2.23(a)(b)])**.** Let M, N be R-modules and ϕ, ψ *pp*-formulas such that $\mathbf{Th}_R \vdash \psi \to \phi$.

1. If $M \leq_p N$, then $\mathrm{Inv}(M, \phi, \psi) \leq \mathrm{Inv}(N, \phi, \psi)$.
2. $\mathrm{Inv}(M \oplus N, \phi, \psi) = \mathrm{Inv}(M, \phi, \psi)\,\mathrm{Inv}(N, \phi, \psi)$.

Lemma 7.3.4. *If T is a first-order theory extending the theory of R-modules, then $\mathbf{K}^T$ is an abstract elementary class with* $\mathrm{LS}(\mathbf{K}^T) = |T|$.

Proof. It is easy to check that $\mathbf{K}^T$ satisfies all the axioms of an AEC except possibly the Tarski-Vaught axiom. Moreover if δ is a limit ordinal, $\{M_i \in \mathbf{K}^T : i < \delta\}$ is an increasing chain (with respect to $\leq_p$) and $N \in \mathbf{K}^T$ such that $\forall i < \delta(M_i \leq_p N)$, then $\forall i < \delta(M_i \leq_p M_\delta = \bigcup_{i<\delta} M_i \leq_p N)$. Therefore, we only need to show that if δ is a limit ordinal and $\{M_i \in \mathbf{K}^T : i < \delta\}$ is an increasing chain, then M_δ is a model of T.

First, by Fact 7.3.2, every $\sigma \in T$ is equivalent modulo $\mathbf{Th}_R$ to a boolean combination of invariant conditions. By putting that boolean combination in conjunctive normal form and separating the conjuncts we conclude that:

$$Mod(T) = Mod(\mathbf{Th}_R \cup \{\theta_\beta : \beta < \alpha\}),$$

where $\alpha \leq |T|$ and each θ_β is a finite disjunction of invariants statements of the form $\mathrm{Inv}(-, \phi, \psi) \geq k$ or of the form $\mathrm{Inv}(-, \phi, \psi) < k$ (for some *pp*-formulas ϕ, ψ such that $\mathbf{Th}_R \vdash \psi \to \phi$ and some positive integer k).

Let δ be a limit ordinal and $\{M_i \in \mathbf{K}^T : i < \delta\}$ an increasing chain. It is clear that $M_\delta \models \mathbf{Th}_R$ and that $M_i \leq_p M_\delta$ for all $i < \delta$. Take $\beta < \alpha$ and consider θ_β. There are two cases:

Case 1: Some disjunct of θ_β is of the form $\mathrm{Inv}(-, \phi, \psi) \geq k$ and for some $i < \delta$, $M_i \models \mathrm{Inv}(-, \phi, \psi) \geq k$. Since $M_i \leq_p M_\delta$, by Fact 7.3.3.(1) it follows that $\mathrm{Inv}(M_i, \phi, \psi) \leq \mathrm{Inv}(M_\delta, \phi, \psi)$, and so $M_\delta \models \theta_\beta$.

Case 2: Every disjunct of θ_β satisfied by a M_i, for $i < \delta$, is of the form $\mathrm{Inv}(-, \phi, \psi) < k$ (for some ϕ, ψ, and k). Since δ is a limit ordinal and θ_β is a finite disjunction, there is some cofinal subchain $\{M_{i'}\}$ of $\{M_i : i < \delta\}$, such that each $M_{i'}$ satisfies the same disjunct of θ_β. So without loss of generality we can assume that this is true of the entire chain, i.e, there are ϕ, ψ, and k such that $M_i \models \mathrm{Inv}(-, \phi, \psi) < k$ for all $i < \delta$ and $\mathrm{Inv}(-, \phi, \psi) < k$ is a disjunct of θ_β. A counterexample to $\mathrm{Inv}(M_\delta, \phi, \psi) < k$ would be witnessed by finitely many tuples from M_δ, hence by finitely many tuples from M_i for some $i < \delta$, a contradiction. Therefore, $M_\delta \models \theta_\beta$.

$\square$

Remark 7.3.5. If T has an infinite model, then $\mathbf{K}^T$ has no maximal models. An infinite model M of T has arbitrarily large elementary extensions, which are, *ipso facto*, models of T and pure extensions of M.

The reader might wonder if $\mathbf{K}^T$ satisfies any other of the structural properties of an AEC such as joint embedding or amalgamation. We show that if $\mathbf{K}^T$ is closed under direct sums, then $\mathbf{K}^T$ has both of these properties. This will be done in three steps.

Fact 7.3.6 ([Pre88, Exercise 1, §2.6])**.** Let $M, N_1, N_2 \in \mathbf{K}^T$. If $M \leq_p N_1$ and $M \preceq N_2$, then there are $N \in \mathbf{K}^T$ and $f : N_1 \underset{M}{\longrightarrow} N$ with f elementary embedding and $N_2 \leq_p N$.

Proof sketch. Introduce new distinct constant symbols for the elements of N_1 and N_2, agreeing on their common part M. Let $\Delta(N_1)$ be the (complete) elementary diagram of N_1, let $p^+(N_2) = \{\phi(\overline{a}) : \phi$ is a *pp*-formula, $\bar{a} \in N_2^{<\omega}$ and $N_2 \models \phi[\overline{a}]\}$, and let $p^-(N_2) = \{\neg\phi(\overline{a}) : \phi$ is a *pp*-formula, $\bar{a} \in N_2^{<\omega}$ and $N_2 \models \neg\phi[\overline{a}]\}$. Then it is straightforward to verify that

$$\Sigma = \Delta(N_1) \cup p^+(N_2) \cup p^-(N_2)$$

is finitely satisfiable in N_1 and any model N of Σ has the desired properties. $\square$

Proposition 7.3.7. *If* $\mathbf{K}^T$ *is closed under direct sums, then pure-injective modules are amalgamation bases*[3] *.*

Proof. Let $N \leq_p N_1, N_2$ all in $\mathbf{K}^T$ with N pure-injective. Since N is pure-injective there are submodules M_1, M_2 of N_1, N_2 respectively, such that for $l \in \{1, 2\}$ we have that $N_l = N \oplus M_l$. Let $L = N_1 \oplus N_2 = (N \oplus M_1) \oplus (N \oplus M_2)$. Since $\mathbf{K}^T$ is closed under direct sums $L \in \mathbf{K}^T$. Define $f_1 : N_1 \to L$ by $f_1(n, m_1) = (n, m_1, n, 0)$ and $f_2 : N_2 \to L$ by $f(n, m_2) = (n, 0, n, m_2)$. Clearly f_1, f_2 are pure embeddings with $f_1 \restriction_N = f_2 \restriction_N$. $\square$

Lemma 7.3.8. *If* $\mathbf{K}^T$ *is closed under direct sums, then:*

1. $\mathbf{K}^T$ *has joint embedding.*

2. $\mathbf{K}^T$ *has amalgamation.*

Proof. For the joint embedding property observe that given $M, N \in \mathbf{K}^T$, they embed purely in $M \oplus N$ which is in $\mathbf{K}^T$ by hypothesis.

Regarding the amalgamation property, let $M \leq_p N_1, N_2$ all in $\mathbf{K}^T$. For $\ell \in \{1, 2\}$, $M, N_\ell, \overline{M}$ satisfy the hypothesis of Fact 7.3.6, since $M \preceq \overline{M}$ by Fact 7.2.18.(2). Then for $\ell \in \{1, 2\}$, there are $N_\ell^* \in \mathbf{K}^T$ and $f_\ell : N_\ell \underset{M}{\longrightarrow} N_\ell^*$, with f_ℓ an elementary embedding and $\overline{M} \leq_p N_\ell^*$.

Since $\overline{M} \leq_p N_1^*, N_2^*$ and $\overline{M}$ is pure-injective by Fact 7.2.18.(1), it follows from Proposition 7.3.7 that there are $N \in \mathbf{K}^T$, $g_1 : N_1^* \to N$ and $g_2 : N_2^* \to N$ with $g_1 \restriction_{\overline{M}} = g_2 \restriction_{\overline{M}}$ and g_1, g_2 both $\mathbf{K}^T$-embeddings. Finally, observe that $g_1 \circ f_1 : N_1 \to N$ and $g_2 \circ f_2 : N_2 \to N$ are $\mathbf{K}^T$-embeddings such that $g_1 \circ f_1 \restriction_M = g_2 \circ f_2 \restriction_M$. $\square$

From the algebraic perspective the natural hypothesis is to assume that $\mathbf{K}^T$ is closed under direct sums. On the other hand, from the model theoretic perspective it is more natural to assume that $\mathbf{K}^T$ has joint embedding and amalgamation. This

[3]Recall that $N \in \mathbf{K}$ is an *amalgamation base*, if given $N \leq_{\mathbf{K}} N_1, N_2 \in \mathbf{K}$, there are $L \in \mathbf{K}$ and $f : N_2 \underset{M}{\longrightarrow} L$ such that $N_1 \leq_{\mathbf{K}} L$.

is always the case when T is a complete theory, which is precisely Example 7.3.10.(2) below.

Since we just showed that in $\mathbf{K}^T$ closure under direct sums implies joint embedding and amalgamation, we will assume these throughout the paper.

Hypothesis 7.3.9. Let R be a ring and T a first-order theory (not necessarily complete) with an infinite model extending the theory of R-modules such that:

1. $\mathbf{K}^T$ has joint embedding.
2. $\mathbf{K}^T$ has amalgamation.

Even after this discussion the reader might wonder if there are any natural classes that satisfy the above hypothesis. We give some examples:

Example 7.3.10.

1. $\mathbf{K}^{tf} = (K^{tf}, \leq_p)$ where K^{tf} is the class of torsion-free abelian groups. In this case T is a first-order axiomatization of torsion-free abelian groups. Since torsion-free abelian groups are closed under direct sums, by Lemma 7.3.8 $\mathbf{K}^{tf}$ has joint embedding and amalgamation.

2. $\mathbf{K}^T = (Mod(T), \leq_p)$ where T is a complete theory extending $\mathbf{Th}_R$. This follows from the fact that if $M, N \models T$, then $M \leq_p N$ if and only if $M \preceq N$ by pp-quantifier elimination.

3. $\mathbf{K}^{\mathbf{Th}_R} = (Mod(\mathbf{Th}_R), \leq_p)$. It is clear that $\mathbf{K}^{\mathbf{Th}_R}$ is closed under direct sums, so by Lemma 7.3.8 $\mathbf{K}^{\mathbf{Th}_R}$ has joint embedding and amalgamation.

4. $\mathbf{K} = (\chi, \leq_p)$ where χ is a definable category of modules in the sense of [Pre09, §3.4]. In this case $T = \{\forall x(\phi(x) \to \psi(x)) : \mathbf{Th}_R \vdash \psi \to \phi \text{ and } \phi(M) = \psi(M) \text{ for every } M \in \chi\}$ and $\mathbf{K}$ has joint embedding and amalgamation because $\mathbf{K}$ is closed under direct sums (by [Pre09, 3.4.7]) and by Lemma 7.3.8.

5. $\mathbf{K} = (\mathbf{C}, \leq_p)$ where $\mathbf{C}$ is a universal Horn class. In this case $T = T_{\mathbf{C}}$ (where $T_{\mathbf{C}}$ is an axiomatization of $\mathbf{C}$) and $\mathbf{K}$ has joint embedding and amalgamation because $\mathbf{K}$ is closed under direct sums (by [Pre88, 15.8]) and by Lemma 7.3.8.

6. $\mathbf{K} = (\mathcal{F}_r, \leq_p)$ where r is a radical of finite type and $\mathcal{F}_r$ is the class of r-torsion-free modules. In this case T exists by [Pre88, 15.9] and $\mathbf{K}$ has joint embedding and amalgamation because $\mathbf{K}$ is closed under direct sums (by [Pre88, 15.8]) and by Lemma 7.3.8.

7. $\mathbf{K} = (\mathcal{T}_r, \leq_p)$ where r is a left exact radical, $\mathcal{T}_r$ is the class of r-torsion modules and $\mathcal{T}_r$ is closed under products. In this case T exists by [Pre88, 15.14] and $\mathbf{K}$ has joint embedding and amalgamation by a similar reason to (5).

8. $\mathbf{K} = (K_{\text{flat}}, \leq_p)$ where K_{flat} is the class of (left) flat R-modules over a right coherent ring. In this case T exists by [Pre88, 14.18] and $\mathbf{K}$ has joint embedding and amalgamation because the class of flat modules is closed under direct sums and by Lemma 7.3.8.

The following example shows that Hypothesis 7.3.9 is not trivial, i.e., given T a first-order theory with an infinite model extending the theory of R-modules Hypothesis 7.3.9 does not necessarily hold.

Example 7.3.11. Let $T = \mathbf{Th}_{\mathbb{Z}} \cup \{\text{Inv}(-, x = x, 3x = 0) < 6\}$.

Let A be an abelian group satisfying T and B the subgroup of A defined by $3x = 0$. Then $|A/B| \in \{1, 2, 3, 4, 5\}$ and so $A/B \cong A_0$, where A_0 is one of the finite groups $\{0\}$, $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/4$, $\mathbb{Z}/5$, or $\mathbb{Z}/3$.

In particular, if $B = 0$, observe that the first five A_0's just listed are models of T. On the other hand, if $B \neq 0$, then $B \cong (\mathbb{Z}/3)^{(\kappa)}$ for some finite or infinite cardinal κ, and since 3 is a prime, it has no non-trivial extensions by any of the groups A_0. There is one exceptional case, as $\mathbb{Z}/9$ is an extension of $\mathbb{Z}/3$ by itself.

Since the invariants multiply across direct sums (Fact 7.3.3), then all the models of T are $\mathbb{Z}/9$ or of the form A_0 or $(\mathbb{Z}/3)^{(\kappa)} \oplus A_0$, for some choice of A_0 and κ a finite or infinite cardinal.

Therefore, there are many examples of failures of the joint embedding property: amongst them we have that $\mathbb{Z}/2$ and $\mathbb{Z}/5$ do not have a common extension to a model of T, and since the zero module is pure-injective, this is an example of the failure of amalgamation over pure-injectives. Since $(\mathbb{Z}/3)^{(\aleph_0)}$ is pure-injective, $(\mathbb{Z}/3)^{(\aleph_0)} \oplus \mathbb{Z}/2$ and $(\mathbb{Z}/3)^{(\aleph_0)} \oplus \mathbb{Z}/5$ provide an infinite example.

It is worth pointing out that there is an easy first-order argument to find universal models if one assumes the hypothesis that $\mathbf{K}^T$ is closed under direct sums.[4]

Lemma 7.3.12. *If $\mathbf{K}^T$ is closed under direct sums and $\lambda^{|T|} = \lambda$, then $\mathbf{K}^T_\lambda$ has a universal model.*

Proof. Observe that T has no more than $2^{|T|}$ complete extensions. Each such extension is λ-stable, see Fact 7.2.15, and so has a saturated model of cardinality λ. Take the direct sum U of all of these; it has cardinality $2^{|T|}\lambda = \lambda$. We claim that $U \in \mathbf{K}^T_\lambda$ and is universal in $\mathbf{K}^T_\lambda$. But $\mathbf{K}^T$ is closed under direct sums, so $U \in \mathbf{K}^T$; and we have already observed that $\|U\| = \lambda$.

If $N \in \mathbf{K}^T_\lambda$, then N is elementarily embedded in the λ-saturated model of $\text{Th}(N)$ which is a summand of U, and hence N is purely embedded in U. $\square$

[4]This was discovered after we had a proof using the theory of abstract elementary classes (see Lemma 7.3.17).

7.3.1 Stability

The following consequence of *pp*-quantifier elimination will be the key to the arguments in this subsection:

Fact 7.3.13 ([Pre88, 2.17])**.** Let $N \in \mathbf{K}^T$, $A \subseteq N$ and $\bar{b}_1, \bar{b}_2 \in N^{<\omega}$. Then:

$$\mathrm{pp}(\bar{b}_1/A, N) = pp(\bar{b}_2/A, N) \text{ if and only if } \mathrm{tp}(\bar{b}_1/A, N) = \mathrm{tp}(\bar{b}_2/A, N).$$

With this, we are able to show that *pp*-types and Galois-types are the same over models.

Lemma 7.3.14. *Let $M, N_1, N_2 \in \mathbf{K}^T$, $M \leq_p N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ and $\bar{b}_2 \in N_2^{<\omega}$. Then:*

$$\mathbf{tp}(\bar{b}_1/M; N_1) = \mathbf{tp}(\bar{b}_2/M; N_2) \textit{ if and only if } \mathrm{pp}(\bar{b}_1/M, N_1) = \mathrm{pp}(\bar{b}_2/M, N_2).$$

Proof. $\underline{\rightarrow:}$ Suppose $\mathbf{tp}(\bar{b}_1/M; N_1) = \mathbf{tp}(\bar{b}_2/M; N_2)$. Since $\mathbf{K}^T$ has amalgamation, there are $N \in \mathbf{K}^T$ and $f : N_1 \to N$ a $\mathbf{K}^T$-embedding such that $f \restriction_M = \mathrm{id}_M$, $f(\bar{b}_1) = \bar{b}_2$ and $N_2 \leq_p N$. Then the result follows from the fact that $\mathbf{K}^T$-embeddings preserve and reflect *pp*-formulas by definition.

$\underline{\leftarrow:}$ Suppose $pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2)$. Since $M \in \mathbf{K}^T$ and $\mathbf{K}^T$ has amalgamation, there is $N \in \mathbf{K}^T$ and $f : N_1 \to N$ a $\mathbf{K}^T$-embedding such that $f \restriction_M = \mathrm{id}_M$ and $N_2 \leq_p N$. Using that $\mathbf{K}^T$-embeddings preserve *pp*-formulas we have that $\mathrm{pp}(f(\bar{b}_1)/M, N) = \mathrm{pp}(\bar{b}_2/M, N)$.

Then by Fact 7.3.13 it follows that $\mathrm{tp}(f(\bar{b}_1)/M, N) = \mathrm{tp}(\bar{b}_2/M, N)$. Let N^* an elementary extension of N such that there is $g \in Aut_M(N^*)$ with $g(f(\bar{b}_1)) = \bar{b}_2$. Observe that since $\mathbf{K}^T$ is first-order axiomatizable $N^* \in \mathbf{K}^T$. Consider $h := g \circ f : N_1 \to N^*$.

It is clear that $h(\bar{b}_1) = \bar{b}_2$, $h \restriction_M = \mathrm{id}_M$ and since being an elementary substructure is stronger than being a pure substructure it follows that $h : N_1 \to N^*$ is a $\mathbf{K}^T$-embedding and $N_2 \leq_p N^*$. Therefore, $\mathbf{tp}(\bar{b}_1/M; N_1) = \mathbf{tp}(\bar{b}_2/M; N_2)$. $\square$

The next corollary follows from the preceding lemma since we can witness that two Galois-types are different by a *pp*-formula.

Corollary 7.3.15. $\mathbf{K}^T$ *is* $(< \aleph_0)$*-tame.*

The next theorem is the main result of this subsection.

Theorem 7.3.16. *If $\lambda^{|T|} = \lambda$, then $\mathbf{K}^T$ is λ-stable.*

Proof. Let $M \in \mathbf{K}^T_\lambda$ and $\{p_i : i < \alpha\}$ an enumeration without repetitions of $\mathbf{gS}(M)$ where $\alpha \leq 2^\lambda$. Since $\mathbf{K}^T$ has amalgamation, there is $N \in \mathbf{K}^T$ and $\{a_i : i < \alpha\} \subseteq N$ such that $p_i = \mathbf{tp}(a_i/M; N)$ for every $i < \alpha$.

Let $\Phi : \mathbf{gS}(M) \to S_{pp}^{Th(N)}(M)$ be defined by $p_i \mapsto \mathrm{pp}(a_i/M, N)$. By Lemma 7.3.14 Φ is a well-defined injective function. By Fact 7.3.13 $|S_{pp}^{Th(N)}(M)| = |S^{Th(N)}(M)|$. Then it follows from Fact 7.2.15 that $|S^{Th(N)}(M)| \leq \lambda$, hence $|\mathbf{gS}(M)| \leq \lambda$. $\square$

7.3.2 Universal models

It is straightforward to construct universal models in $\mathbf{K}^T$ for λ's satisfying that $\lambda^{|T|} = \lambda$. This follows from Fact 7.2.9 and Remark 7.2.12.

Lemma 7.3.17. *If $\lambda^{|T|} = \lambda$, then $\mathbf{K}^T_\lambda$ has a universal model.*

The following lemma shows how to build universal models in cardinals where $\mathbf{K}^T$ might not be λ-stable.

Lemma 7.3.18. *If $\forall \mu < \lambda(\mu^{|T|} < \lambda)$, then $\mathbf{K}^T_\lambda$ has a universal model.*

Proof. We may assume that λ is a limit cardinal, because if it is not the case then we have that $\lambda^{|T|} = \lambda$ and we can apply Lemma 7.3.17. Let $\mathrm{cf}(\lambda) = \kappa \leq \lambda$. By using the hypothesis that $\forall \mu < \lambda(\mu^{|T|} < \lambda)$, it is easy to build $\{\lambda_i : i < \kappa\}$ an increasing continuous sequence of cardinals such that $\forall i(\lambda_{i+1}^{|T|} = \lambda_{i+1})$ and $sup_{i<\kappa}\lambda_i = \lambda$.

We build $\{M_i : i < \kappa\}$ an increasing continuous chain such that:

1. M_{i+1} is $\|M_{i+1}\|$-universal over M_i.
2. $M_i \in \mathbf{K}_{\lambda_i}$.

In the base step pick any $M \in \mathbf{K}^T_{\lambda_0}$ and if i is limit, let $M_i = \bigcup_{j<i} M_j$.

If $i = j+1$, by construction we are given $M_j \in \mathbf{K}^T_{\lambda_j}$. Using that $\mathbf{K}^T$ has no maximal models, we find $N \in \mathbf{K}^T_{\lambda_{j+1}}$ such that $M_j \leq_p N$. Since $\lambda_{j+1}^{|T|} = \lambda_{j+1}$, by Theorem 7.3.16 $\mathbf{K}^T$ is λ_{j+1}-stable. Then by Fact 7.2.9 applied to N, there is $M_{j+1} \in \mathbf{K}^T_{\lambda_{j+1}}$ universal over N. Using that $\mathbf{K}^T$ has amalgamation, it is straightforward to check that (1) holds.

This finishes the construction of the chain.

Let $M = \bigcup_{i<\kappa} M_i$. By (2) $\|M\| = \lambda$. We show that M is universal in $\mathbf{K}^T_\lambda$.

Let $N \in \mathbf{K}^T_\lambda$ and $\{N_i : i < \kappa\}$ an increasing continuous chain such that $\forall i(N_i \in \mathbf{K}^T_{\lambda_i})$ and $\bigcup_{i<\kappa} N_i = N$. We build $\{f_i : i < \kappa\}$ such that:

1. $f_i : N_i \to M_{i+1}$.
2. $\{f_i : i < \kappa\}$ is an increasing chain.

Observe that this is enough by taking $f = \bigcup_{i<\kappa} f_i : N = \bigcup_{i<\kappa} N_i \to \bigcup_{i<\kappa} M_{i+1} = M$.

Now, let us do the construction. In this case the base step is non-trivial. By joint embedding there is $g : N_0 \to M^*$ with $M_0 \leq_p M^* \in \mathbf{K}^T_{\lambda_0}$. Now, since M_1 is $\|M_1\|$-universal over M_0 there is $h : M^* \xrightarrow[M_0]{} M_1$. Let $f_0 := h \circ g$ and observe that this satisfies the requirements.

We do the induction steps.

If i is limit, let $f_i = \bigcup_{j<i} f_j : N_i = \bigcup_{j<i} N_j \to M_{i+1}$.

If $i = j+1$, by construction we have $f_j : N_j \to M_{j+1}$ and $N_j \leq_p N_{j+1}$. Since $\mathbf{K}^T$ has amalgamation there is $M' \in \mathbf{K}^T_{\lambda_{j+1}}$ and $g : N_{j+1} \to M'$ such that $M_{j+1} \leq_p M'$ and $f_j \restriction_{N_j} = g \restriction_{N_j}$. Since M_{j+2} is $\|M_{j+2}\|$-universal over M_{j+1}, there is $h : M' \xrightarrow[M_{j+1}]{} M_{j+2}$. Let $f_{j+1} := h \circ g$ and observe that this satisfies the requirements. $\square$

Putting together Lemma 7.3.17 and Lemma 7.3.18 we get one of our main results.

Theorem 7.3.19. *If $\lambda^{|T|} = \lambda$ or $\forall \mu < \lambda(\mu^{|T|} < \lambda)$, then $\mathbf{K}^T_\lambda$ has a universal model.*

The proof of Lemma 7.3.17 and Lemma 7.3.18 can be extended in a straightforward way to the following general setting.

Corollary 7.3.20. *Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models. Assume there is $\theta_0 \geq \mathrm{LS}(\mathbf{K})$ and κ such that for all $\theta \geq \theta_0$, if $\theta^\kappa = \theta$, then $\mathbf{K}$ is θ-stable.*

Suppose $\lambda > \theta_0$. If $\lambda^\kappa = \lambda$ or $\forall \mu < \lambda(\mu^\kappa < \lambda)$, then $\mathbf{K}_\lambda$ has a universal model.[5]

Remark 7.3.21. In [Vas16d, 4.13] it is shown that if $\mathbf{K}$ is an AEC with joint embedding, amalgamation and no maximal models, $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame and $\mathbf{K}$ is λ-stable for some $\lambda \geq \mathrm{LS}(\mathbf{K})$, then there are θ_0 and κ satisfying the hypothesis of Corollary 7.3.20.

7.3.3 Reduced torsion-free abelian groups

Recall that $\mathbf{K}^{tf}$ has joint embedding and amalgamation, so it satisfies Hypothesis 7.3.9. Moreover, $|T^{tf}| = \aleph_0$, therefore the next assertion follows directly from Theorem 7.3.16 and Theorem 7.3.19.

Corollary 7.3.22.

1. *If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{tf}$ is λ-stable.*

[5] In Lemma 7.3.17 and Theorem 7.3.18 $\theta_0 = \mathrm{LS}(\mathbf{K}^T) = |T|$ and $\kappa = |T|$.

2. *If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then $\mathbf{K}^{tf}_\lambda$ has a universal model.*

Remark 7.3.23. In [BET07, 0.3] it is shown that: $\mathbf{K}^{tf}$ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$. The argument given here differs substantially with that of [BET07, 0.3], their argument does not consider *pp*-formulas and instead exploits the property that $\mathbf{K}^{tf}$ admits intersections.

As mentioned in the introduction, Shelah's result [Sh820, 1.2] is concerned with reduced torsion-free groups instead of with torsion-free groups. The next two assertion show how we can recover his assertion from the above results. First let us introduce a new class of groups.

Definition 7.3.24. Let $\mathbf{K}^{rtf} = (K^{rtf}, \leq_p)$ where K^{rtf} is the class of reduced torsion-free abelian groups defined in the usual language $L_{\mathbb{Z}}$ of $\mathbb{Z}$-modules, and $\leq_p$ is the pure subgroup relation. Recall that a group G is reduced if its only divisible subgroup is 0.

Fact 7.3.25. Let λ an infinite cardinal. $\mathbf{K}^{tf}_\lambda$ has a universal model if and only if $\mathbf{K}^{rtf}_\lambda$ has a universal model.

Proof. The proof follows from the fact that divisible torsion-free abelian groups of cardinality $\leq \lambda$ are purely embeddable into $\mathbb{Q}^{(\lambda)}$ and that every group can be written as a direct sum of a unique divisible subgroup and a unique up to isomorphisms reduced subgroup (see [Fuc15, §4.2.4, §4.2.5]). □

The following is precisely [Sh820, 1.2].

Corollary 7.3.26.

1. *If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{rtf}_\lambda$ has a universal model.*
2. *If $\lambda = \Sigma_{n<\omega}\lambda_n$ and $\aleph_0 \leq \lambda_n = (\lambda_n)^{\aleph_0} < \lambda_{n+1}$, then $\mathbf{K}^{rtf}_\lambda$ has a universal model.*
3. *$\mathbf{K}^{rtf}$ has amalgamation, joint embedding, is an AEC and is λ-stable if $\lambda^{\aleph_0} = \lambda$.*

Proof. For (1) and (2), realize that λ either satisfies the first or second hypothesis of Corollary 7.3.22.(2), hence $\mathbf{K}^{tf}_\lambda$ has a universal model. Then by Fact 7.3.25 we conclude that $\mathbf{K}^{rtf}_\lambda$ has a universal model in either case.

For (3), the first three assertions are easy to show. As for the last one, this follows from Corollary 7.3.22.(1) and the fact that if $G, H \in \mathbf{K}^{rtf}$ and $a, b \in H$ then: $\mathbf{tp}_{\mathbf{K}^{rtf}}(a/G; H) = \mathbf{tp}_{\mathbf{K}^{rtf}}(b/G; H)$ if and only if $\mathbf{tp}_{\mathbf{K}^{tf}}(a/G; H) = \mathbf{tp}_{\mathbf{K}^{tf}}(b/G; H)$. □

Remark 7.3.27. It is worth noticing that Corollary 7.3.22.(2) not only implies [Sh820, 1.2.1, 1.2.2] (Corollary 7.3.26.(1) and Corollary 7.3.26.(2)), but the two assertions are equivalent. The backward direction follows from the fact that if λ satisfies $\mathrm{cf}(\lambda) \geq \omega_1$ and $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then $\lambda^{\aleph_0} = \lambda$.

Remark 7.3.28. It follows from Corollary 7.3.22.(2) that if $2^{\aleph_0} < \aleph_\omega$, then $\mathbf{K}^{tf}_{\aleph_\omega}$ has a universal model. On the other hand, it follows from [KojSh95, 3.7] that if $\aleph_\omega < 2^{\aleph_0}$, then $\mathbf{K}^{tf}_{\aleph_\omega}$ does not have a universal model. Hence the existence of a universal model in $\mathbf{K}^{tf}$ of cardinality $\aleph_\omega$ is independent of ZFC. Similarly one can show that the existence of a universal model in $\mathbf{K}^{tf}$ of cardinality $\aleph_n$ is independent of ZFC for every $n \geq 1$.

7.4 Limit models in classes of R-modules

In this section we will begin the study of limit models in classes of R-modules under Hypothesis 7.3.9. The existence of limit models in $\mathbf{K}^T$ for λ's satisfying $\lambda^{|T|} = \lambda$ follows directly from Theorem 7.3.16 and Fact 7.2.9.

Corollary 7.4.1. *If $\lambda^{|T|} = \lambda$, then there is a (λ, α)-limit model in $\mathbf{K}^T$ for every $\alpha < \lambda^+$ limit ordinal.*

We first show that any two limit models are elementarily equivalent. In order to do that, we will use one more consequence of *pp*-quantifier elimination (Fact 7.2.14).

Fact 7.4.2 ([Pre88, 2.18])**.** Let M and N R-modules. M is elementary equivalent to N if and only if $\mathrm{Inv}(M, \phi, \psi) = \mathrm{Inv}(N, \phi, \psi)$ for every ϕ, ψ *pp*-formulas in one free variable such that $\mathbf{Th}_R \vdash \psi \to \phi$.

Lemma 7.4.3. *If M, N are limit models, then M and N are elementary equivalent.*

Proof. Assume M is a (λ, α)-limit model for $\alpha < \lambda^+$ and let $\{M_i : i < \alpha\} \subseteq \mathbf{K}^T_\lambda$ be a witness for it. Similarly assume N is a (μ, β)-limit model for $\beta < \mu^+$ and let $\{N_i : i < \beta\} \subseteq \mathbf{K}^T_\mu$ be a witness for it.

By Fact 7.4.2, it is enough to show that for every ϕ, ψ , *pp*-formulas in one free variable such that $\mathbf{Th}_R \vdash \psi \to \phi$, and $n \in \mathbb{N}$: $\mathrm{Inv}(M, \phi, \psi) \geq n$ if and only if $\mathrm{Inv}(N, \phi, \psi) \geq n$. By the symmetry of this situation, we only need to prove one implication. So consider such *pp*-formulas ϕ, ψ and $n \in \mathbb{N}$ such that $\mathrm{Inv}(M, \phi, \psi) \geq n$. We show that $\mathrm{Inv}(N, \phi, \psi) \geq n$.

If $n = 0$, the result is clear. So assume that $n \geq 1$. Then since $\mathrm{Inv}(M, \phi, \psi) \geq n$, there are $m_0, ..., m_{n-1} \in M$ such that:

$$M \models \bigwedge_i \phi(m_i) \wedge \bigwedge_{i \neq j} \neg\psi(m_i - m_j).$$

Applying the downward Löwenheim-Skolem-Tarski axiom inside M to $\{m_i : i < n\}$, we get $M^* \leq_p M$ such that $M^* \in \mathbf{K}^T_{\mathrm{LS}(\mathbf{K})}$ and $\{m_i : i < n\} \subseteq M^*$. Then it is still

the case that
$$M^* \models \bigwedge_i \phi(m_i) \wedge \bigwedge_{i \neq j} \neg\psi(m_i - m_j).$$

By joint embedding there is g and $M^{**} \in \mathbf{K}^T_\mu$ such that $g : M^* \to M^{**}$ and $N_0 \leq_p M^{**}$. Then since N_1 is universal over N_0, there is $h : M^{**} \underset{N_0}{\longrightarrow} N_1$. Finally, observe that:
$$N \models \bigwedge_i \phi(h \circ g(m_i)) \wedge \bigwedge_{i \neq j} \neg\psi(h \circ g(m_i) - h \circ g(m_j)).$$

Hence $\mathrm{Inv}(N, \phi, \psi) \geq n$. □

Remark 7.4.4. Observe that in the proof of the above lemma we only used that $\mathbf{K}^T$ is an AEC of modules with the joint embedding property.

As in [Ch. 5, §4], limit models with chains of big cofinality are easier to understand than those of small cofinalities. Due to this we begin by studying the former.

Theorem 7.4.5. *Assume $\lambda \geq |T|^+ = \mathrm{LS}(\mathbf{K}^T)^+$. If M is a (λ, α)-limit model and $\mathrm{cf}(\alpha) \geq |T|^+$, then M is pure-injective.*

Proof. Fix $\{M_i : i < \alpha\}$ a witness to the fact that M is a (λ, α)-limit model. We show that M is pure-injective using the equivalence of Fact 7.2.17.

Let $p(x)$ be an M-consistent *pp*-type over $A \subseteq M$ and $|A| \leq |R| + \aleph_0 = |T|$. Then there is a module N and $b \in N$ with $M \preceq N \in \mathbf{K}^T_{\|M\|}$ and b realizing p. Since $|A| \leq |T|$ and $\mathrm{cf}(\alpha) \geq |T|^+$, there is $i < \alpha$ such that $A \subseteq M_i$.

Note that $M_i \leq_p N$. Then there is $f : N \underset{M_i}{\longrightarrow} M_{i+1}$, because M_{i+1} is universal over M_i. Since A is fixed by the choice of M_i, it is easy to see that $f(b) \in M_{i+1} \leq_p M$ realizes $p(x)$. Therefore, M is pure-injective. □

The following fact about pure-injective modules is a generalization of Bumby's result [Bum65]. A proof of it (and a discussion of the general setting) appears in [GKS18, 3.2]. We will use it to show uniqueness of limit models of big cofinalities.

Fact 7.4.6. Let M, N be pure-injective modules. If there is $f : M \to N$ a $\mathbf{K}^{\mathbf{Th}_R}$-embedding and $g : N \to M$ a $\mathbf{K}^{\mathbf{Th}_R}$-embedding, then $M \cong N$.

Corollary 7.4.7. *Assume $\lambda \geq |T|^+ = \mathrm{LS}(\mathbf{K}^T)^+$. If M is a (λ, α)-limit model and N is a (λ, β)-limit model such that $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) \geq |T|^+$, then M is isomorphic to N.*

Proof. It is straightforward to check that M and N are universal models in $\mathbf{K}^T_\lambda$ (see Remark 7.2.12). Since M and N are pure-injective by Theorem 7.4.5, then the result follows from Fact 7.4.6 because $\mathbf{K}^T$-embeddings and $\mathbf{K}^{\mathbf{Th}_R}$-embeddings are the same. □

Dealing with limit models of small cofinality is complicated. We will only be able to describe limit models of countable cofinality under the additional assumption that $\mathbf{K}^T$ is closed under direct sums. All the examples of Example 7.3.10, except Example 7.3.10.(2), satisfy this additional hypothesis.

Lemma 7.4.8. *Assume $\mathbf{K}^T$ is closed under direct sums. If $M \in \mathbf{K}^T_\lambda$ is pure-injective and $U \in \mathbf{K}^T_\lambda$ is a universal model in $\mathbf{K}^T_\lambda$, then $M \oplus U$ is universal over M.*

Proof. It is clear that $M \leq_p M \oplus U$ and that both modules have the same cardinality, so take $N \in \mathbf{K}^T_\lambda$ such that $M \leq_p N$. Since M is pure-injective we have that $N = M \oplus M'$ for some M'. Using that U is universal in $\mathbf{K}^T_\lambda$, there is $f' : M' \to U$ a pure embedding. Let $f : M \oplus M' \to M \oplus U$ be given by $f(a + b) = a + f'(b)$. It is easy to check that f is a $\mathbf{K}^T$-embedding that fixes M. $\square$

Theorem 7.4.9. *Assume $\lambda \geq |T|^+ = \mathrm{LS}(\mathbf{K}^T)^+$ and $\mathbf{K}^T$ is closed under direct sums. If M is a (λ, ω)-limit model and N is a $(\lambda, |T|^+)$-limit model, then $M \cong N^{(\aleph_0)}$.*

Proof. For every $i < \omega$, let N_i be given by i-many direct copies of N. Consider the increasing chain $\{N_i : i < \omega\} \subseteq \mathbf{K}^T_\lambda$.

By Theorem 7.4.5 $N \in \mathbf{K}^T$ is pure-injective. Since pure-injective modules are closed under finite direct sums, N_i is pure-injective for every $i < \omega$. Moreover, for each $i < \omega$, $N_{i+1} = N_i \oplus N$ is universal over N_i because N is universal in $\mathbf{K}^T_\lambda$, N_i is pure-injective and by Lemma 7.4.8. Therefore, $N_\omega := \bigcup_{i<\omega} N_i$ is a (λ, ω)-limit model.

Since N_ω and M are limit models with chains of the same cofinality, a back-and-forth argument shows that $N_\omega \cong M$. Hence $M \cong N^{(\aleph_0)}$. $\square$

Lemma 7.4.8 can also be used to characterize stability in classes closed under direct sums.

Corollary 7.4.10. *Assume $\mathbf{K}^T$ is closed under direct sums and $\lambda \geq |T|^+$ is an infinite cardinal. $\mathbf{K}^T$ is λ-stable if and only if $\mathbf{K}^T_\lambda$ has a pure-injective universal model.*

Proof. The forward direction follows from the fact that $(\lambda, |T|^+)$-limit models are pure-injective by Theorem 7.4.5. So we sketch the backward direction. Let $M \in \mathbf{K}^T_\lambda$ and $U \in \mathbf{K}^T_\lambda$ a pure-injective universal model. By universality of U we may assume that $M \leq_p U$. Then by minimality of the pure hull we have that $\overline{M} \leq_p U$, thus $\overline{M} \in \mathbf{K}^T_\lambda$. So by Lemma 7.4.8 $\overline{M} \oplus U$ is universal over $\overline{M}$. Therefore, every type over M is realized in $\overline{M} \oplus U$. Hence $|\mathbf{gS}(M)| \leq \|\overline{M} \oplus U\| = \lambda$. $\square$

Remark 7.4.11. Observe that by Corollary 7.4.7 we know that for every cardinal λ the number of non-isomorphic limit models is bounded by $|\{\alpha : \alpha \leq |T|, \alpha \text{ is limit and } \mathrm{cf}(\alpha) = \alpha\}| + 1$. So for example, when R is countable, we know that there are at most two non-isomorphic limit models.

We believe the following question is very interesting (see also Conjecture 2 of [BoVan]):

Question 7.4.12. Let $\mathbf{K}^T$ as in Hypothesis 7.3.9. How does the spectrum of limit models look like?

More precisely, given λ, how many non-isomorphic limit models are there of cardinality λ for a given $\mathbf{K}^T$? Is it always possible to find T such that $\mathbf{K}^T$ has the maximum number of non-isomorphic limit models?

We will be able to answer Question 7.4.12 when the ring is countable.

Theorem 7.4.13. *Let R be a countable ring. Assume $\mathbf{K}^T$ satisfies Hypothesis 7.3.9.*

1. *If $\mathbf{K}^T$ is superstable*[6]*, then there is $\mu < \beth_{(2^{\aleph_0})^+}$ such for every $\lambda \geq \mu$ there is a unique limit model of cardinality λ.*
2. *If $\mathbf{K}^T$ is not superstable, then $\mathbf{K}^T$ does not have uniqueness of limit models in any infinite cardinal $\lambda \geq \mathrm{LS}(\mathbf{K}^T)^+ = \aleph_1$. More precisely, if $\mathbf{K}^T$ is λ-stable there are exactly two non-isomorphic limit models of cardinality λ.*

Proof sketch. $\mathbf{K}^T$ has joint embedding, amalgamation and no maximal models and by Corollary 7.3.15 $\mathbf{K}^T$ is $(< \aleph_0)$-tame. Due to this we can use the results of [GrVas17] and [Vas18].

1. This follows on general grounds from [Vas18, 4.24] and [GrVas17, 5.5].
2. Let $\lambda \geq \aleph_1$ such that $\mathbf{K}^T$ is λ-stable. As in [Ch. 5, 4.19, 4.20, 4.21, 4.23] one can show that the limit models of countable cofinality are not pure-injective. Since we know that limit models of uncountable cofinality are pure-injective by Theorem 7.4.5, we can conclude that the (λ, ω)-limit model and the (λ, ω_1)-limit model are not isomorphic. Moreover, given N a (λ, α)-limit model, N is isomorphic to the (λ, ω)-limit model if $cf(\alpha) = \omega$ (by a back-and-forth argument) or N is isomorphic to the (λ, ω_1)-limit model if $cf(\alpha) > \omega$ (by Corollary 7.4.7).

□

[6]We say that $\mathbf{K}$ is *superstable* if there is $\mu < \beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$ such that $\mathbf{K}$ is λ-stable for every $\lambda \geq \mu$. Under the assumption of joint embedding, amalgamation, no maximal models and $\mathrm{LS}(\mathbf{K})$-tameness by [GrVas17] and [Vas18] the definition of the previous line is equivalent to any other definition of superstability given in the context of AECs.

7.4.1 Torsion-free abelian groups

In this section we will show how to apply the results we just obtained to answer Question 4.25 of [Ch. 5].

Recall that a group G is *algebraically compact* if given $\mathbb{E} = \{f_i(x_{i_0}, ..., x_{i_{n_i}}) = a_i : i < \omega\}$ a set of linear equations over G, $\mathbb{E}$ is finitely solvable in G if and only if $\mathbb{E}$ is solvable in G. It is well-known that an abelian group G is algebraically compact if and only if G is pure-injective (see e.g. [Fuc15, 1.2, 1.3]). The following theorem answers Question 4.25 of [Ch. 5].

Theorem 7.4.14. *If $G \in \mathbf{K}^{tf}$ is a (λ, ω)-limit model, then $G \cong \mathbb{Q}^{(\lambda)} \oplus \left(\prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}} \right)^{(\aleph_0)}$.*

Proof. The amalgamation property together with the existence of a limit model imply that $\mathbf{K}^{tf}$ is λ-stable. Then by Remark 7.3.23 $\lambda^{\aleph_0} = \lambda$, so by Corollary 7.4.1 there is H a (λ, ω_1)-limit model. Since $\mathbf{K}^{tf}$ is closed under direct sums, we have that $G \cong H^{(\aleph_0)}$ by Theorem 7.4.9.

In view of the fact that H is a (λ, ω_1)-limit model, by Fact 7.2.20 $H \cong \mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}}$. Therefore we have:

$$G \cong \left(\mathbb{Q}^{(\lambda)} \oplus \prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}} \right)^{(\aleph_0)} \cong \mathbb{Q}^{(\lambda)} \oplus \left(\prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}} \right)^{(\aleph_0)}.$$

□

In [Ch. 5, 4.22] it was shown that limit models of countable cofinality are not pure-injective. The argument given there uses some deep facts about the theory of AECs. Here we give a new argument that relies on some well-known properties of abelian groups.

Corollary 7.4.15. *If $G \in \mathbf{K}^{tf}$ is a (λ, ω)-limit model, then G is not pure-injective.*

Proof. By Theorem 7.4.14 we have that $G \cong \mathbb{Q}^{(\lambda)} \oplus \left(\prod_p \overline{\mathbb{Z}_{(p)}^{(\lambda)}} \right)^{(\aleph_0)}$. For every p, one can show that $\overline{\mathbb{Z}_{(p)}^{(\lambda)}}^{(\aleph_0)}$ is not pure-injective by a similar argument to the proof that $\overline{\mathbb{Z}_{(p)}}^{(\aleph_0)}$ is not pure-injective (an argument for this is given in [Pre88, §2]). Then using that a direct product is pure-injective if every component is pure-injective (see [Fuc15, §6.1.9]), it follows that G is not pure-injective. □

Combining the results of this section with the ones of the previous section we obtain:

Corollary 7.4.16. *If $\forall \mu < \lambda (\mu^{\aleph_0} < \lambda)$, then for any $G \in \mathbf{K}^{tf}_\lambda$ pure-injective there is a universal model over it.*

Proof. Let $G \in \mathbf{K}^{tf}_{\lambda}$ be pure-injective. Since λ satisfies the hypothesis of Corollary 7.3.22, there is $U \in \mathbf{K}^{tf}_{\lambda}$ universal model in $\mathbf{K}^{tf}_{\lambda}$. Then by Lemma 7.4.8 $G \oplus U$ is a universal model over G. $\square$

By the above corollary, given $G \in \mathbf{K}^{tf}_{\beth_\omega}$ pure-injective, for example $G = \mathbb{Q}^{(\beth_\omega)}$, there is $H \in \mathbf{K}^{tf}_{\beth_\omega}$ such that H is universal over G. Since $\beth_\omega^{\aleph_0} > \beth_\omega$, by Remark 7.3.23 we have that $\mathbf{K}^{tf}$ is not $\beth_\omega$-stable. This is the first example of an AEC with joint embedding, amalgamation and no maximal models in which one can construct universal extensions of cardinality λ without the hypothesis of λ-stability.

Chapter 8

Superstability, noetherian rings and pure-semisimple rings

This chapter is based on [Ch. 8].

Abstract

We uncover a connection between the model-theoretic notion of superstability and that of noetherian rings and pure-semisimple rings.

We characterize noetherian rings via superstability of the class of left modules with embeddings.

Theorem 8.0.1. *For a ring R the following are equivalent.*

1. *R is left noetherian.*
2. *The class of left R-modules with embeddings is superstable.*
3. *For every $\lambda \geq |R| + \aleph_0$, there is $\chi \geq \lambda$ such that the class of left R-modules with embeddings has uniqueness of limit models of cardinality χ.*
4. *Every limit model in the class of left R-modules with embeddings is Σ-injective.*

We characterize left pure-semisimple rings via superstability of the class of left modules with pure embeddings.

Theorem 8.0.2. *For a ring R the following are equivalent.*

1. *R is left pure-semisimple.*
2. *The class of left R-modules with pure embeddings is superstable.*
3. *There exists $\lambda \geq (|R| + \aleph_0)^+$ such that the class of left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*
4. *Every limit model in the class of left R-modules with pure embeddings is Σ-pure-injective.*

Both equivalences provide evidence that the notion of superstability could shed light in the understanding of algebraic concepts.

As this paper is aimed at model theorists and algebraists an effort was made to provide the background for both.

8.1 Introduction

An abstract elementary class (AEC) is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is class of structures and $\leq_{\mathbf{K}}$ is a strong substructure relation extending the substructure relation. Among the requirements we have that an AEC is closed under directed colimits and that every set is contained in a small model in the class (see Definition 8.2.1). These were introduced by Shelah in [Sh88]. Natural examples in the context of algebra are abelian groups with embeddings, torsion-free abelian groups with pure embeddings, first-order axiomatizable classes of modules with pure embeddings and flat modules with pure embeddings.[1]

Dividing lines in complete first-order theories were introduced by Shelah in the late sixties and early seventies. One of the best behaved classes is that of *superstable* theories. Extensions of superstability in a non-elementary setting were first considered in [GrSh86]. In the context of AECs, superstability was introduced in [Sh394] and until recently it was believed to suffer from "schizophrenia" [Sh:h, p. 19]. In [GrVas17, 1.3] and [Vas18], it was shown (under additional hypothesis that are satisfied by the classes studied in this paper[2]) that superstability is a well-behaved concept and many conditions that were believed to characterize superstability were found to be equivalent. Based on this and the key role *limit models* play in this paper, we will say

[1] The first two classes were studied in [BCG+], [BET07] and [Ch. 5], the next ones were studied in [Ch. 7] and the last one was studied in [LRV1a] and [Ch. 9].

[2] The hypothesis are amalgamation, joint embedding, no maximal models and LS($\mathbf{K}$)-tameness.

that an AEC is *superstable* if it has uniqueness of limit models in a tail of cardinals.[3] This particular definition of superstability appears for the first time in [GrVas17]. Intuitively the reader can think of limit models as universal models with some level of homogeneity (see Definition 8.2.8). Further details on the development of the notion of superstability can be consulted in the introduction of [GrVas17].

In this paper, we show that the notion of superstability has algebraic substance if one chooses the right context. More specifically, we characterize noetherian rings and pure-semisimple rings via superstability in certain classes of modules with embeddings and with pure embeddings respectively.

Noetherian rings are rings with the ascending chain condition for ideals. The precise equivalence we obtain is the following[4].

Theorem 8.3.12. *For a ring R the following are equivalent.*

1. *R is left noetherian.*
2. *The class of left R-modules with embeddings is superstable.*
3. *For every $\lambda \geq |R| + \aleph_0$, there is $\chi \geq \lambda$ such that the class of left R-modules with embeddings has uniqueness of limit models of cardinality χ.*
4. *For every $\lambda \geq |R| + \aleph_0$, the class of left R-modules with embeddings has uniqueness of limit models of cardinality λ.*
5. *For every $\lambda \geq (|R| + \aleph_0)^+$, the class of left R-modules with embeddings has a superlimit of cardinality λ.*
6. *For every $\lambda \geq |R| + \aleph_0$, the class of left R-modules with embeddings is stable.*
7. *Every limit model in the class of left R-modules with embeddings is Σ-injective.*

If R is left coherent, they are further equivalent to:

8. *(Lemma 8.4.30) The class of left absolutely pure modules with embeddings is superstable.*

A ring R is *left pure-semisimple* if every left R-module is pure-injective. It was pointed out to us by Daniel Simson that pure-semisimple rings were introduced by him in [Sim77]. There are many papers where several characterizations of pure-semisimple rings are obtained, for example [Cha60], [Aus74], [Aus76], [Z-H79], [Sim81] and [Pre84]. For additional information on what is known about pure-semisimple rings the reader can consult [Sim00], [Hui00] and [Pre09, §4.5.1].

[3] For a complete first-order theory T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory, i.e., T is λ-stable as for every $\lambda \geq 2^{|T|}$.

[4] Conditions (4) through (6) of the theorem below were motivated by [GrVas17, 1.3].

In this paper, we give several new characterizations of left pure-semimple rings via superstability. More precisely, we show[5].

Theorem 8.4.28. *For a ring R the following are equivalent.*

1. *R is left pure-semisimple.*
2. *The class of left R-modules with pure embeddings is superstable.*
3. *There exists $\lambda \geq (|R| + \aleph_0)^+$ such that the class of left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*
4. *For every $\lambda \geq |R| + \aleph_0$, the class of left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*
5. *For every $\lambda \geq (|R| + \aleph_0)^+$, the class of left R-modules with pure embeddings has a superlimit of cardinality λ.*
6. *For every $\lambda \geq |R| + \aleph_0$, the class of left R-modules with pure embeddings is λ-stable.*
7. *For every $\lambda \geq (|R| + \aleph_0)^+$, an increasing chain of λ-saturated models is λ-saturated in the class of left R-modules with pure embeddings.*
8. *There exists $\lambda \geq (|R| + \aleph_0)^+$ such that the class of left R-modules with pure embeddings has a Σ-pure-injective universal model of cardinality λ.*
9. *Every limit model in the class of left R-modules with pure embeddings is Σ-pure-injective.*

A key difference between our results and those of [GrVas17, 1.3] is that in [GrVas17] the cardinal where the *nice property* starts to show up is eventual (bounded by $\beth_{(2^{|R|+\aleph_0})^+}$), while in our case the cardinal is exactly $|R| + \aleph_0$ or $(|R| + \aleph_0)^+$. In the introduction of [GrVas17] is asked if it was possible to lower these bounds (see Theorem 8.4.23 and the remark below it).

Although the results of Theorem 8.3.12 and Theorem 8.4.28 are similar, the techniques used to prove the results differ significantly. The proof of Theorem 8.3.12 is more rudimentary and depends heavily on the fact that we are working with the class of all modules. On the other hand, Theorem 8.4.28 is a corollary of the theory of superstable classes of modules with pure embeddings closed under direct sums which is developed in the fourth section of the paper. One could give a proof of Theorem 8.4.28 similar to that of Theorem 8.3.12, but we think that the theory of superstable classes of modules with pure embeddings is an interesting theory that should be developed.

[5] Conditions (4) through (7) of the theorem below were motivated by [GrVas17, 1.3].

Another result of the paper is a positive solution above $\mathrm{LS}(\mathbf{K})^+$ to Conjecture 2 of [BoVan] in the case of classes of modules axiomatizable in first-order with joint embedding and amalgamation (see Theorem 8.4.34). This also provides a partial solution to Question 4.12 of [Ch. 7].

Algebraically the key idea is to identify limit models with well-understood classes of modules. First, we show that *long* limit models in the class of modules with embeddings are injective modules (see Lemma 8.3.7) and that *long* limit models in the class of modules with pure embeddings are pure-injective modules (see Fact 8.4.14) for arbitrary rings. Then by assuming that the ring is noetherian or pure-semisimple we show that all limit models are Σ-injective (see Theorem 8.3.12.(7)) or Σ-pure-injective (see Theorem 8.4.28.(9)) respectively. From these characterizations one can obtain the equivalence with superstability.

The paper is divided into four sections. Section 2 presents necessary background. Section 3 provides a new characterization of noetherian rings. Section 4 characterizes superstability with pure embeddings in classes of modules closed under direct sums and provides a new characterization of pure-semisimple rings. Moreover, a positive solution above $\mathrm{LS}(\mathbf{K})^+$ to Conjecture 2 of [BoVan] is given for certain classes of modules.

It was pointed out to us by Vasey that already in [Sh54] Shelah noticed some connections between superstability of the theory of modules, noetherian rings and pure-semisimple rings. Regarding noetherian rings, Shelah has a remark on page 299 immediately after an unproven theorem (Theorem 8.6) that indicates that he knew that superstability of the theory of modules implies that the ring is left noetherian. The precise equivalence he noticed is similar to that of (6) implies (1) of Theorem 8.3.12, but the equivalence of (1) and (6) in Theorem 3.12 is new. As for pure-semisimple rings, Shelah claims in Theorem 8.7 that superstability of the theory of modules is equivalent to the ring being pure-semisimple (without mentioning pure-semisimple rings). Shelah fails to provide a proof that if a ring R is pure-semisimple then the theory of R-modules is superstable ((2) to (1) of his Theorem 8.7). The precise equivalence he noticed is similar to that of (1) and (6) of Theorem 8.4.28. As the theory of modules is not a complete first-order theory, it is unclear the precise notion of superstability that Shelah refers to in his paper, but he seems to be working with syntactic superstability with respect to positive primitive formulas.

This paper was written while the author was working on a Ph.D. under the direction of Rami Grossberg at Carnegie Mellon University and I would like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work in particular. After reading a preliminary preprint, Sebastien Vasey informed us that he independently discovered the equivalence between superstability and noetherian rings (the equivalence between (1) and (2) of Theorem 8.3.12), but has not circulated it yet. His proof follows from [Vas17c, 3.7], [Ekl71, Proposition 3] and

[Vas18, 5.9]. I thank an anonymous referee for comments on another paper of mine that prompted the development of Subsection 4.4. I would also like to thank John T. Baldwin, Daniel Simson, Sebastien Vasey and a couple of referees for comments that helped improve the paper. Finally, I would like to dedicate this work to Marquititos, you will always be loved and remembered.

8.2 Preliminaries

We present the basic concepts of abstract elementary classes that are used in this paper. These are further studied in [Bal09, §4 - 8] and [Gro2X, §2, §4.4]. An introduction from an algebraic perspective is given in [Ch. 6, §2]. Regarding the background on module theory, we give a brief survey of the concepts we will use in this paper. An excellent resource for the module theory we will use in this paper are [Pre88] and [Pre09].

8.2.1 Basic concepts

Abstract elementary classes (AECs) were introduced by Shelah in [Sh88, 1.2]. Among the requirements we have that an AEC is closed under directed colimits and that every set is contained in a small model in the class. Given a model M, we will write $|M|$ for its underlying set and $\|M\|$ for its cardinality.

Definition 8.2.1. An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

1. K is a class of τ-structures, for some fixed language $\tau = \tau(\mathbf{K})$.
2. $\leq_{\mathbf{K}}$ is a partial order on K.
3. $(K, \leq_{\mathbf{K}})$ respects isomorphisms:

   If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$.

   In particular (taking $M = N$), K is closed under isomorphisms.
4. If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
5. Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$.
6. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:

(a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.

(b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.

7. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

Notation 8.2.2.

- If λ is a cardinal and $\mathbf{K}$ is an AEC, then $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$.
- Let $M, N \in \mathbf{K}$. If we write "$f : M \to N$" we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. In particular $\mathbf{K}$-embeddings are always monomorphisms.
- Let $M, N \in \mathbf{K}$ and $A \subseteq M$. If we write "$f : M \underset{A}{\to} N$" we assume that f is a $\mathbf{K}$-embedding and that $f \restriction_A = \mathrm{id}_A$.

In [Sh300] Shelah introduced a notion of semantic type. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We follow the notation of [Ch. 3, 2.5].

Definition 8.2.3. Let $\mathbf{K}$ be an AEC.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.
2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E_{\mathrm{at}} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exists $f_\ell : N_\ell \underset{A}{\to} N$ for $\ell \in \{1, 2\}$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$.
3. Note that E_{at} is a symmetric and reflexive relation on $\mathbf{K}^3$. We let E be the transitive closure of E_{at}.
4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$. We call such an equivalence class a *Galois-type*. Usually, $\mathbf{K}$ will be clear from context and we will omit it.
5. For $M \in \mathbf{K}$, $\mathbf{gS}_{\mathbf{K}}(M) = \{\mathbf{tp}_{\mathbf{K}}(b/M; N) : M \leq_{\mathbf{K}} N \in \mathbf{K} \text{ and } b \in N\}$.
6. For $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N)$ and $C \subseteq A$, $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) \restriction_C := [(\mathbf{b}, C, N)]_E$.

Definition 8.2.4. An AEC is λ-*stable* if for any $M \in \mathbf{K}_\lambda$, $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$.

Remark 8.2.5. Recall that given T a complete first-order theory and $A \subseteq M$ with M a model of T, $S^T(A)$ is the set of complete first-order types with parameters in A. For a complete first-order theory T and $\lambda \geq |T|$, $(Mod(T), \preceq)$ is λ-stable (where $\preceq$ is the elementary substructure relation) if and only if T is λ-stable as a first-order theory, i.e., $|S^T(A)| \leq \lambda$ for every $A \subseteq M$ where $|A| = \lambda$ and M is a model of T.

The following notion was isolated by Grossberg and VanDieren in [GrVan06].

Definition 8.2.6. $\mathbf{K}$ is $(< \kappa)$-*tame* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}(M)$, there is $A \subseteq |M|$ such that $|A| < \kappa$ and $p \restriction_A \neq q \restriction_A$.

8.2.2 Limit models, saturated models and superlimits

Before introducing the concept of limit model we recall the concept of universal extension.

Definition 8.2.7. M is *universal over* N if and only if $N \leq_{\mathbf{K}} M$, $\|M\| = \|N\| = \lambda$ and for any $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\rightarrow} M$.

With this we are ready to introduce limit models, they were originally introduced in [KolSh96].

Definition 8.2.8. Let λ be an infinite cardinal and $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that $M_0 := N$, M_{i+1} is universal over M_i for each $i < \alpha$ and $M = \bigcup_{i<\alpha} M_i$. We say that M is a (λ, α)-limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. We say that M is a limit model of cardinality λ if there exists a limit ordinal $\alpha < \lambda^+$ such that M is a (λ, α)-limit model.

Observe that if M is a (λ, α)-limit model, then M has cardinality λ.

Definition 8.2.9. Let $\mathbf{K}$ be an AEC and λ be a cardinal. $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is $f : N \to M$ a $\mathbf{K}$-embedding.

The following is a simple exercise, a proof is given in [Ch. 5, 2.10].

Fact 8.2.10. Let $\mathbf{K}$ be an AEC with joint embedding and amalgamation. If M is a limit model of cardinality λ, then M is a universal model in $\mathbf{K}_\lambda$.

The next fact gives conditions for the existence of limit models.

Fact 8.2.11 ([Sh:h, §II], [GrVan06, 2.9])**.** Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models. If $\mathbf{K}$ is λ-stable, then for every $N \in \mathbf{K}_\lambda$ and $\alpha < \lambda^+$ limit ordinal there is M a (λ, α)-limit model over N. Conversely, if $\mathbf{K}$ has a limit model of cardinality λ, then $\mathbf{K}$ is λ-stable

The key question regarding limit models is the uniqueness of limit models of a given cardinality but with chains of different lengths. This has been studied thoroughly in the context of abstract elementary classes [ShVi99], [Van06], [GVV16], [Bon14], [Van16], [BoVan] and [Vas19].

Definition 8.2.12. $\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has a limit model of cardinality λ and if any two limit models of cardinality λ are isomorphic.

Since we will only deal with AECs with amalgamation, joint embedding, no maximal models and LS($\mathbf{K}$)-tame and it is known (by [GrVas17, 1.3] and [Vas18]) that in this context the definition below is equivalent to every other definition of superstability considered in the context of AECs, we introduce the following as the definition of superstability.

Definition 8.2.13. $\mathbf{K}$ is *superstable* if and only if $\mathbf{K}$ has uniqueness of limit models in a tail of cardinals.

Remark 8.2.14. For a complete first-order T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory, i.e., T is λ-stable for every $\lambda \geq 2^{|T|}$.

It is important to point out that to establish that $\mathbf{K}$ has uniqueness of limit models of cardinality λ, one needs to show first the existence of limit models. Due to Fact 8.2.11, this is equivalent to λ-stability.

Another important class of models is that of saturated models.

Definition 8.2.15. $M \in \mathbf{K}$ is λ-*saturated* if for every $N \leq_{\mathbf{K}} M$ and $p \in \mathbf{gS}(N)$ with $\|N\| < \lambda$, there is $a \in M$ such that $p = \mathbf{tp}(a/N; M)$. M is saturated if M is $\|M\|$-saturated.

A model M is λ-*model-homogeneous* if for every $N, N' \in \mathbf{K}$ with $N \leq_{\mathbf{K}} M$, $N \leq_{\mathbf{K}} N'$ and $\|N'\| < \lambda$, there is $f : N' \xrightarrow[N]{} M$. Recall that for $\lambda > \mathrm{LS}(\mathbf{K})$, a model is λ-saturated if and only if it is λ-model-homogeneous. A proof of it appears in [Sh:h, §II.1.4].

Superlimit models were introduced in [Sh88, 3.1.(1)] as another possible notion of saturation on AECs.

Definition 8.2.16. Let $\mathbf{K}$ be an AEC. Let $M \in \mathbf{K}$ and $\lambda \geq \mathrm{LS}(\mathbf{K})$. M is a superlimit in λ if:

1. $M \in \mathbf{K}_\lambda$.

2. For every $N \in \mathbf{K}_\lambda$, there is $f : N \to M$ such that $f[N] \neq M$.

3. If $\{M_i : i < \delta\} \subseteq \mathbf{K}_\lambda$ is an increasing chain, $\delta < \lambda^+$ is an ordinal and $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i<\delta} M_i \cong M$.

The following fact has some known connections between limit models, saturated models and superlimits.

Fact 8.2.17 ([GrVas17, 2.8], [Dru13, 2.3.10])**.** Let $\mathbf{K}$ be an AEC with amalgamation, joint embedding and no maximal models.

1. If $\lambda > \mathrm{LS}(\mathbf{K})$ and M is a (λ, α)-limit model for $\alpha \in [\mathrm{LS}(\mathbf{K})^+, \lambda]$ a regular cardinal, then M is an α-saturated model.

2. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and $\mathbf{K}$ be λ-stable. $\mathbf{K}$ has uniqueness of limit models in λ if and only if every limit model of cardinality λ is saturated.

3. Let $\mathbf{K}$ be λ-stable. If M is a superlimit of cardinality λ, then M is a (λ, α)-limit model for every $\alpha < \lambda^+$ limit ordinal.

4. Let $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ be λ-stable and assume there exists a saturated model of size λ. $\mathbf{K}$ has a superlimit of cardinality λ if and only if the union of an increasing chain (of length less than λ^+) of saturated models in $\mathbf{K}_\lambda$ is saturated.

8.2.3 Module theory

All rings considered in this paper are associative with an identity element. A module M is *injective* if and only if for every module N, if $M \leq N$ then M is a direct summand of N. We say that M is Σ-*injective* if and only if $M^{(I)}$ is injective for every index set I. To consider only countable index sets one needs M to be injective.

Fact 8.2.18 ([Fai66, Proposition 3])**.** For M an injective module the following are equivalent.

1. M is Σ-injective.

2. $M^{(\aleph_0)}$ is injective.

Recall that a formula ϕ is a positive primitive formula (*pp*-formula for short), if ϕ is an existentially quantified system of linear equations. Given M and N R-modules, M is a *pure submodule* of N, denoted by $M \leq_p N$, if and only if M is a submodule of N and for every *pp*-formula ϕ it holds that $\phi[N] \cap M = \phi[M]$. Equivalently if for every L right R-module $L \otimes M \to L \otimes N$ is a monomorphism.

(Σ-)Pure-injective modules generalize the notion of (Σ-)injective modules. A module M is *pure-injective* if in the definition of injective module one substitutes "$\leq$" by "$\leq_p$". A module M is Σ-*pure-injective* if in the definition of Σ-injective module one substitutes "injective" for "pure-injective". In the case of Σ-pure-injectivity it is enough to consider countable index sets.

Fact 8.2.19 ([Zim77, 3.4])**.** M is Σ-*pure-injective* if and only if $M^{(\aleph_0)}$ is pure-injective.

A module M is *absolutely pure* if every extension of M is pure. The next fact relates Σ-injectivity and Σ-pure-injectivity.

Fact 8.2.20 ([Pre09, 4.4.16])**.** For M an R-module the following are equivalent.

1. M is Σ-injective.

2. M is absolutely pure and Σ-pure-injective.

Using the equivalence between Σ-pure-injectivity and the descending chain condition on *pp*-definable subgroups one can show the following (see for example [Pre88, 2.11]).

Fact 8.2.21.

- If N is Σ-pure-injective and $M \leq_p N$, then M is Σ-pure-injective.

- If N is Σ-pure-injective and M is elementary equivalent to N, then M is Σ-pure-injective.

We will also use that Σ-pure-injective modules are totally transcendental.

Fact 8.2.22 ([Pre88, 3.2])**.** If M is Σ-pure-injective, then $(Mod(Th(M)), \preceq)$ is λ-stable for every $\lambda \geq |Th(M)|$.

A ring R is *left noetherian* if every increasing chain of left ideals is stationary. These were introduced by Noether in [Noe21]. Following [Ekl71], denote by γ_R the smallest cardinal such that every left ideal of R is generated by less than γ_R elements. Observe that $\gamma_R \leq |R|^+$. We will use the following equivalence later in the paper. The equivalence between one and four is due to Cartan-Eilenberg-Bass-Papp and the equivalence between one and two is trivial

Fact 8.2.23 ([Pre09, 4.4.17])**.** For a ring R the following are equivalent.

1. R is left noetherian.

2. $\gamma_R \leq \aleph_0$.
3. Every injective left R-module is Σ-injective.
4. Every direct sum of injective left R-modules is injective.
5. Every absolutely pure left R-module is injective.

Recall the notion of a left pure-semisimple ring.

Definition 8.2.24. A ring R is *left pure-semisimple* if and only if every left R-module M is pure-injective.

Many equivalent conditions have been found for the notion of a pure-semisimple ring, see for example [Cha60], [Aus74], [Aus76], [Z-H79], [Sim81] and [Pre84]. A more updated set of equivalences is given in [Sim00] and [Pre09, §4.5.1]. Below we give some of the equivalent conditions for a ring to be left pure-semisimple.

Fact 8.2.25 ([Pre88, 11.3])**.** For a ring R the following are equivalent.

1. R is left pure-semisimple.
2. Every left R-module M is Σ-pure-injective.
3. Every left R-module is the direct sum of indecomposable submodules.

Recall Bumby's result [Bum65] and its generalization to pure-injective modules. A proof of both results (and a discussion of the general setting) appears in [GKS18, 2.5].

Fact 8.2.26.

- Let M, N be injective modules. If there are $f : M \to N$ an embedding and $g : N \to M$ an embedding, then M is isomorphic to N.
- Let M, N be pure-injective modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then M is isomorphic to N.

8.2.4 Notation

We will use the following notation which was introduced in [Ch. 7, 3.1].

Notation 8.2.27. Given R a ring, we denote by $\mathbf{Th}_R$ the theory of left R-modules. A (not necessarily complete) first-order theory T is *a theory of modules* if it extends $\mathbf{Th}_R$. For T a theory of modules, let $\mathbf{K}^T = (Mod(T), \leq_p)$ and $|T| = |R| + \aleph_0$.

Since we will also work with embeddings we introduce the following notation.

Notation 8.2.28. Given R a ring, we will use the standard notation $(R\text{-Mod}, \subseteq_R)$ instead of the model-theoretic notation $(Mod(\mathbf{Th}_R), \leq)$ to denote the AEC of left R-modules with embeddings.

8.3 A new characterization of noetherian rings

In this section we will work in the class of modules with embeddings. Since complete theories of modules only have *pp*-quantifier elimination, we do not think that in the case of classes of modules with embeddings there is a deep theory as the one we will develop in the next section for pure embeddings. Instead, using some more rudimentary methods, we will study the class of modules with embeddings.

Remark 8.3.1. It is well-known that $(R\text{-Mod}, \subseteq_R)$ is an AEC that has amalgamation, joint embedding and no maximal models.

The next assertion describes Galois-types in this context.

Lemma 8.3.2. *Let $M, N_1, N_2 \in R$-Mod, $M \subseteq_R N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ and $\bar{b}_2 \in N_2^{<\omega}$. Then:*

$$\mathbf{tp}_{(R\text{-}Mod,\subseteq_R)}(\bar{b}_1/M; N_1) = \mathbf{tp}_{(R\text{-}Mod,\subseteq_R)}(\bar{b}_2/M; N_2) \textit{ if and only if } qf\text{-}tp(\bar{b}_1/M, N_1) = qf\text{-}tp(\bar{b}_2/M, N_2)$$

Proof sketch. The forward direction is trivial, so let us sketch the backward direction. By the amalgamation property we may assume that $N_1 = N_2 = N$. Define $f : \langle \bar{b}_1 M \rangle \to \langle \bar{b}_2 M \rangle$ as $f(\Sigma_{i=1}^n r_i b_{1,i} + \Sigma_{i=1}^k s_i m_i) = \Sigma_{i=1}^n r_i b_{2,i} + \Sigma_{i=1}^k s_i m_i$ where $\langle \bar{b}_\ell M \rangle$ is the submodule generated by $\bar{b}_\ell M$ inside N for $\ell \in \{1, 2\}$, $r_i, s_i \in R$ for all i and $m_i \in M$ for all i. Using that the quantifier free types are equal, it follows that f is an isomorphism. Then the result follows by applying amalgamation a couple of times. $\square$

Since we can witness that two Galois-types are different by a quantifier free formula, we obtain.

Corollary 8.3.3. *$(R$-Mod$, \subseteq_R)$ is $(< \aleph_0)$-tame.*

The above corollary also follows from the general theory of AECs [Vas17c, 3.7], since $(R\text{-Mod}, \subseteq_R)$ is a universal class in the sense of [Tar54] (see [Ch. 3, 2.1] for the definition).

An analogous argument to the one given in [Ch. 7, 4.8] can be used to show the following.

Proposition 8.3.4. *Let λ be an infinite cardinal. If $E \in (R\text{-}Mod, \subseteq_R)_\lambda$ is injective and $U \in R\text{-}Mod$ is universal in $(R\text{-}Mod, \subseteq_R)_\lambda$, then $E \oplus U$ is universal over E.*

The next fact from [Ekl71] will be useful.

Fact 8.3.5 ([Ekl71, Proposition 3])**.** Let λ be an infinite cardinal with $\lambda \geq |R| + \aleph_0$. $\lambda^{<\gamma_R} = \lambda$ if and only if there is an injective universal model in $(R\text{-Mod}, \subseteq_R)_\lambda$.

With it we will be able to show that $(R\text{-Mod}, \subseteq_R)$ is stable.

Lemma 8.3.6. *Let R be a ring and λ be an infinite cardinal with $\lambda \geq |R| + \aleph_0$. If $\lambda^{<\gamma_R} = \lambda$, then $(R\text{-}Mod, \subseteq_R)$ is λ-stable.*

Proof. By Fact 8.3.5 there is U an injective universal model in $(R\text{-Mod}, \subseteq_R)_\lambda$. Build $\{N_i : i < \omega\}$ by induction such that N_i is equal to $(i+1)$-many direct copies of U.

Since U is injective and injective objects are closed under finite direct sums, it follows that N_i is injective for every $i < \omega$. Moreover, by Proposition 8.3.4 it follows that N_{i+1} is universal over N_i for every $i < \omega$. Let $N = \bigcup_{i<\omega} N_i$. Observe that N is a (λ, ω)-limit model, so by Fact 8.2.11 it follows that $(R\text{-Mod}, \subseteq_R)$ is λ-stable. □

From the above theorem and Fact 8.2.11 it follows that there is a (λ, α)-limit model for every $\alpha < \lambda^+$ limit ordinal and cardinal λ such that $\lambda^{<\gamma_R} = \lambda$. The next lemma characterizes limit models in $(R\text{-Mod}, \subseteq_R)$.

Lemma 8.3.7. *Let R be a ring, λ be an infinite cardinal with $\lambda \geq \gamma_R + |R| + \aleph_0$ and $\alpha < \lambda^+$ be a limit ordinal. If M is a (λ, α)-limit model in $(R\text{-}Mod, \subseteq_R)$ and $\mathrm{cf}(\alpha) \geq \gamma_R$, then M is injective.*

Proof. By [Ekl71, Lemma 2] it is enough to show that if $\mathbb{E} = \{r_\delta x = a_\delta : \delta < \beta\}$ is a system of equations in one free variable x with $\beta < \gamma_R$ and $r_\delta \in R$, $a_\delta \in M$ for every $\delta < \beta$ and $\mathbb{E}$ has a solution in an extension of M, then $\mathbb{E}$ has a solution in M.

Let $\mathbb{E}$ be a system of equations as in the previous paragraph, $M' \in (R\text{-Mod}, \subseteq_R)_\lambda$ be an extension of M with $b \in M'$ realizing $\mathbb{E}$ and $\{M_i : i < \alpha\}$ be a witness to the fact that M is a (λ, α)-limit model. Since $\beta < \gamma_R$ and $\mathrm{cf}(\alpha) \geq \gamma_R$, there is $i < \alpha$ such that $\{a_\delta : \delta < \beta\} \subseteq M_i$. Since M_{i+1} is universal over M_i there is $f : M' \underset{M_i}{\longrightarrow} M$. It is clear that $f(b) \in M$ realizes $\mathbb{E}$. □

Using the above lemma, we can obtain an equivalence in Lemma 8.3.6

Corollary 8.3.8. *Let R be a ring and λ be an infinite cardinal with $\lambda \geq (|R| + \aleph_0)^+$. $\lambda^{<\gamma_R} = \lambda$ if and only if $(R\text{-}Mod, \subseteq_R)$ is λ-stable.*

Proof. The forward direction is Lemma 8.3.6 and the backward direction follows from the existence of limit models, Lemma 8.3.7 and Fact 8.3.5. □

Doing a similar proof to that of Lemma 8.3.7 and using the equivalence between saturation and model-homogeneity one can obtain the next result.

Lemma 8.3.9. *Let $\lambda \geq (|R| + \aleph_0)^+$. If M is λ-saturated in $(R\text{-}Mod, \subseteq_R)$, then M is injective.*

Since a ring R is noetherian if and only if $\gamma_R \leq \aleph_0$ (by Fact 8.2.23), the next result follows from the results we just obtained in this section.

Corollary 8.3.10. *If R is a left noetherian ring, then:*

1. *$(R\text{-}Mod, \subseteq_R)$ is λ-stable for every $\lambda \geq |R| + \aleph_0$.*
2. *There is a (λ, α)-limit model in $(R\text{-}Mod, \subseteq_R)$ for every $\lambda \geq |R| + \aleph_0$ and $\alpha < \lambda^+$ limit ordinal.*
3. *Every limit model in $(R\text{-}Mod, \subseteq_R)$ is injective.*

Moreover, the analogous of [Ch. 7, 4.9] can also be carried out in this context. Since the proof of the proposition is basically the same as that of [Ch. 7, 4.9] we omit it.

Proposition 8.3.11. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, ω)-limit model in $(R\text{-}Mod, \subseteq_R)$ and N is a $(\lambda, (|R| + \aleph_0)^+)$-limit model in $(R\text{-}Mod, \subseteq_R)$, then M is isomorphic to $N^{(\aleph_0)}$.*

With this we obtain a new characterization of left noetherian rings via superstability.[6]

Theorem 8.3.12. *For a ring R the following are equivalent.*

1. *R is left noetherian.*
2. *The class of left R-modules with embeddings is superstable.*
3. *For every $\lambda \geq |R| + \aleph_0$, there is $\chi \geq \lambda$ such that the class of left R-modules with embeddings has uniqueness of limit models of cardinality χ.*
4. *For every $\lambda \geq |R| + \aleph_0$, the class of left R-modules with embeddings has uniqueness of limit models of cardinality λ.*
5. *For every $\lambda \geq (|R| + \aleph_0)^+$, the class of left R-modules with embeddings has a superlimit of cardinality λ.*

[6] Conditions (4) through (6) of the theorem below were motivated by [GrVas17, 1.3].

6. *For every $\lambda \geq |R| + \aleph_0$, the class of left R-modules with embeddings is λ-stable.*

7. *Every limit model in the class of left R-modules with embeddings is Σ-injective.*

Proof. (1) $\Rightarrow$ (4) Let $\lambda \geq |R| + \aleph_0$. By Corollary 8.3.10.(2) there is (λ, α)-limit models for every $\alpha < \lambda^+$ limit ordinal. So we only need to show uniqueness of limit models. Let M and N be two limit models of cardinality λ. By Corollary 8.3.10.(3) M and N are injective and since M embeds into N and vice versa by Fact 8.2.10, it follows from Fact 8.2.26 that M is isomorphic to N.

(4) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) We will use the equivalence given in Fact 8.2.23.(3), so let M be an injective module. Since M is injective, it is absolutely pure so by Fact 8.2.20 it is enough to show that M is Σ-pure-injective. Let $\chi \geq (\|M\| + |R| + \aleph_0)^+$ such that $(R\text{-Mod}, \subseteq_R)$ has uniqueness of limit models of cardinality χ. Let N be a $(\chi, (|R| + \aleph_0)^+)$-limit model such that $M \subseteq_R N$. By Proposition 8.3.11 $N^{(\aleph_0)}$ is a (χ, ω)-limit model, then by uniqueness of limit models and Lemma 8.3.7 (using that $(|R|+\aleph_0)^+ \geq \gamma_R$) $N^{(\aleph_0)}$ is injective. Then N is Σ-pure-injective. Since M is injective, it follows that M is Σ-pure-injective by Fact 8.2.21.

(1) $\Rightarrow$ (5) Let $\lambda \geq (|R| + \aleph_0)^+$. By Corollary 8.3.10.(1) $(R\text{-Mod}, \subseteq_R)$ is λ-stable, so let M be a $(\lambda, (|R| + \aleph_0)^+)$-limit model. Then by condition (4) together with Fact 8.2.17.(2) M is saturated. So by Fact 8.2.17.(4) it is enough to show that an increasing chain of saturated models in $(R\text{-Mod}, \subseteq_R)_\lambda$ is saturated. Let $\{M_i : i < \delta\}$ be an increasing chain of saturated models in $(R\text{-Mod}, \subseteq_R)_\lambda$ and $\delta < \lambda^+$ be a limit ordinal.

Using that $\gamma_R \leq \aleph_0$ and that M_i is injective for every $i < \delta$ (by Lemma 8.3.9), one can show that $\bigcup_{i<\delta} M_i$ is an injective module. Moreover, $\bigcup_{i<\delta} M_i$ embeds into M_0, because M_0 saturated, and M_0 embeds into $\bigcup_{i<\delta} M_i$. Then by Fact 8.2.26 M_0 is isomorphic to $\bigcup_{i<\delta} M_i$. Hence, $\bigcup_{i<\delta} M_i$ is saturated.

(5) $\Rightarrow$ (2) Let $\lambda \geq |R| + \aleph_0$ and $\chi = (\lambda^+)^{\gamma_R}$. By Lemma 8.3.6 $(R\text{-Mod}, \subseteq_R)$ is χ-stable. Therefore there are (χ, α)-limit models for every $\alpha < \chi^+$ limit ordinal. So we only need to show uniqueness of limit models. Let M and N be two limit models of cardinality χ. Let L be a superlimit of size χ, then by Fact 8.2.17.(3) M is isomorphic to L and N is isomorphic to L. Hence M is isomorphic to N.

(1) $\Rightarrow$ (6) By Corollary 8.3.10.(1).

(6) $\Rightarrow$ (1) Assume for the sake of contradiction that R is not noetherian, then it follows that $\gamma_R > \aleph_0$. Let $\lambda = \beth_\omega(|R| + \aleph_0)$. Since $\text{cf}(\lambda) = \omega$, we have by Königs lemma that $\lambda^{<\gamma_R} > \lambda$. Then by Corollary 8.3.8 it follows that $(R\text{-Mod}, \subseteq_R)$ is not λ-stable. A contradiction to the hypothesis.

(1) $\Rightarrow$ (7) Let M be a (λ, α)-limit model for $\alpha < \lambda^+$ limit ordinal. Let $\mu \geq (\|M\| + |R| + \aleph_0)^+$ and N be a $(\mu, (|R| + \aleph_0)^+)$-limit model such that $M \subseteq_R N$, it exists by Corollary 8.3.10.(2). By Proposition 8.3.11 $N^{(\aleph_0)}$ is a (μ, ω)-limit model,

then by Corollary 8.3.10.(3) $N^{(\aleph_0)}$ is injective. So N is Σ-pure-injective. Therefore, since M is injective by Corollary 8.3.10.(3), it follows that M is Σ-pure-injective by Fact 8.2.21. Hence M is Σ-injective by Fact 8.2.20.

(7) $\Rightarrow$ (3) Let $\lambda \geq |R| + \aleph_0$ and let $\chi \geq \lambda$ such that $(R\text{-Mod}, \subseteq_R)$ is χ-stable, this exist by Corollary 8.3.8. There are clearly limit models of cardinality χ, so we only need to show uniqueness of limit models. Let M and N be two limit models of cardinality χ. By hypothesis M and N are injective and since M embeds into N and vice versa by Fact 8.2.10, it follows from Fact 8.2.26 that M is isomorphic to N. $\square$

This is not the first result where noetherian rings and superstability have been related. As it was mentioned in the introduction, Shelah noticed that superstability of the theory of modules implies that the ring is left noetherian in [Sh54, §8]. The precise equivalence he noticed is similar to that of (6) implies (1) of the above theorem. For a countable ω-stable ring a proof is given in [BaMc82, 9.1]. The equivalence between (1) and (6) is new. Another paper that relates both notions is [GrSh86]. In it, it is shown (for integral domains) that superstability (in the sense of [GrSh86, 1.2]) of the class of torsion divisible modules implies that the ring is noetherian.

Remark 8.3.13. Compared to [GrVas17, 1.3], the above theorem improves the bounds where the *nice propertis* show up from $\beth_{(2^{|R|+\aleph_0})^+}$ to $|R| + \aleph_0$ or $(|R| + \aleph_0)^+$ in the class of modules with embeddings. In the introduction of [GrVas17] is asked if this bounds can be improved.

Remark 8.3.14. Since $(R\text{-Mod}, \subseteq_R)$ has amalgamation, joint embedding, no maximal models and is $(< \aleph_0)$-tame (by Corollary 8.3.3) . Therefore, we could have simply quoted the main theorem of [GrVas17] and [Vas18] to obtain (5),(6) imply (2) of the above theorem. We decided to provide the proofs for those directions to make the paper more transparent and since the proofs in our case are easier than in the general case.

8.4 A new characterization of pure-semisimple rings

It is possible to obtain a similar proof for Theorem 8.4.28 (without conditions (7) and (8)) as the one presented for Theorem 8.3.12. The reason we do not do this is because there is a deep theory when one considers theories of modules with pure embeddings. What we will do is to study superstability for any first-order theory of modules with pure embeddings and as a simple corollary we will obtain Theorem 8.4.28.

Recall that for T a first-order theory of modules (not necessarily complete) $\mathbf{K}^T = (Mod(T), \leq_p)$. In [Ch. 7, 3.4], it is shown that if T is a theory of modules then $\mathbf{K}^T$

is an AEC. Most interesting results do not hold for all AECs, so we will assume, as in [Ch. 7], the next hypothesis throughout this section.

Hypothesis 8.4.1. Let R be a ring and T be a theory of modules with an infinite model such that:

1. $\mathbf{K}^T$ has joint embedding.
2. $\mathbf{K}^T$ has amalgamation.

These may seem like *adhoc* hypothesis, but there are many natural theories satisfying them. This is the case if T is a complete theory, but many other examples are given in [Ch. 7, 3.10]. For the proof of the main theorem, we will only use the well-known result that $\mathbf{K}^{\mathbf{Th}_R} = (R\text{-Mod}, \leq_p)$ satisfies the above hypothesis.

Since the theory of modules has *pp*-quantifier elimination (see for example [Zie84, 1.1]), one can show the following.

Fact 8.4.2 ([Ch. 7, 3.14])**.** Let $M, N_1, N_2 \in \mathbf{K}^T$, $M \leq_p N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ and $\bar{b}_2 \in N_2^{<\omega}$. Then:

$$\mathbf{tp}(\bar{b}_1/M; N_1) = \mathbf{tp}(\bar{b}_2/M; N_2) \text{ if and only if } pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2).$$

Moreover, if $N_1 \equiv N_2$ one can substitute the *pp*-types by the first-order types ([Ch. 7, 3.13]).

8.4.1 The theory $\tilde{T}$

In [Ch. 7, 3.16] it is shown that $\mathbf{K}^T$ is λ-stable if $\lambda^{|T|} = \lambda$. Then it follows from Fact 8.2.11 that there exist limit models of cardinality λ in $\mathbf{K}^T$ for every cardinal λ such that $\lambda^{|T|} = \lambda$. More importantly and key to the naturality of the theory we will introduce in this section is the following result.

Fact 8.4.3 ([Ch. 7, 4.3])**.** If M and N are limit models in $\mathbf{K}^T$, then M and N are elementary equivalent.

Let us introduce the main notion of this subsection.

Notation 8.4.4. For T a theory of modules, let $\tilde{M}_T$ be the $(2^{|T|}, \omega)$-limit model of $\mathbf{K}^T$ and $\tilde{T} = Th(\tilde{M}_T)$.

It is natural to ask which structures of $\mathbf{K}^T$ satisfy the complete first-order theory $\tilde{T}$. It follows from Fact 8.4.3 that limit models do, we record this for future reference.

Corollary 8.4.5. *If M is a limit model in $\mathbf{K}^T$, then M is a model of $\tilde{T}$.*

The next lemma gives another class of structures satisfying $\tilde{T}$.

Lemma 8.4.6. *Let $\lambda \geq |T|^+$. If M is a λ-saturated model in $\mathbf{K}^T$, then M is a model of $\tilde{T}$.*

Proof sketch. The proof is similar to that of [Ch. 7, 4.3], by using the equivalence between λ-saturation and λ-model-homogeneity. $\square$

Moreover, $\tilde{T}$ is closed upward under pure extensions.

Lemma 8.4.7. *If M is a model of $\tilde{T}$, $N \in \mathbf{K}^T$ and $M \leq_p N$, then $M \preceq N$. In particular, N is a model of $\tilde{T}$.*

Proof. Let $\lambda = \|N\|^{|T|}$, then by [Ch. 7, 3.16] $\mathbf{K}^T$ is λ-stable. So let $N^* \in \mathbf{K}^T_\lambda$ such that $N \leq_p N^*$ and N^* is a (λ, ω)-limit model. By Fact 8.4.3 $N^* \equiv M$. Then $M \preceq N \preceq N^*$ by [Pre88, 2.25]. Whence $M \preceq N$ and N is a model of $\tilde{T}$. $\square$

The next result shows the naturality of $\tilde{T}$, the proof is similar to that of the above lemma so we omit it.

Corollary 8.4.8. *If $M \in \mathbf{K}^T$, then there is N a model of $\tilde{T}$ such that $M \leq_p N$. Moreover, if T' is a complete first-order theory with this property, then $\tilde{T} = T'$.*

The following lemmas show that there is a close relationship between the class $\mathbf{K}^T$ and the first-order theory $\tilde{T}$. This is useful since complete first-order theories of modules are very well understood (see for instance [Pre88]).

Lemma 8.4.9. *For T a theory of modules and $\lambda \geq |T|$, the following are equivalent.*

1. *$\tilde{T}$ is λ-stable.*
2. *$\mathbf{K}^T$ is λ-stable.*

Proof. $\Rightarrow$: Let $M \in \mathbf{K}^T_\lambda$ and $\{p_i : i < \alpha\}$ be an enumeration without repetitions of $\mathbf{gS}(M)$. Fix $\mu = \lambda^{|T|}$ and $N \in \mathbf{K}^T_\mu$ a (μ, ω)-limit model such that $M \leq_p N$. Then there is $\{a_i : i < \alpha\} \subseteq N$ such that $p_i = \mathbf{tp}(a_i/M; N)$ for every $i < \alpha$.

Let $\Phi : \mathbf{gS}(M) \to S^{Th(N)}(M)$ be defined by $\phi(\mathbf{tp}(a_i/M; N)) = tp(a_i/M, N)$. By Fact 8.4.2 it follows that Φ is a well-defined injective function, so $|\mathbf{gS}(M)| \leq |S^{Th(N)}(M)|$. Finally, since N is a model of $\tilde{T}$ by Corollary 8.4.5 and $\tilde{T}$ is λ-stable by hypothesis, it follows that $|S^{Th(N)}(M)| \leq \|M\|$. Hence $|\mathbf{gS}(M)| \leq \|M\|$.

$\Leftarrow$: Let $A \subseteq N$ of size λ with $N \vDash \tilde{T}$ and N is λ^+-saturated in $\tilde{T}$. Let $\{p_i : i < \alpha\}$ be an enumeration without repetitions of $S^{\tilde{T}}(A)$. Let $\{a_i : i < \alpha\} \subseteq N$ such that $p_i = tp(a_i/A, N)$ for every $i < \alpha$.

Let M be the structure obtained by applying downward Löwenheim-Skolem-Tarski to A in N, observe that $\|M\| = \lambda$ because $\lambda \geq |T|$. Let $\Psi : S^{\tilde{T}}(A) \to \mathbf{gS}(M)$ be defined by $\phi(tp(a_i/A,N)) = \mathbf{tp}(a_i/M;N)$. By Fact 8.4.2 it follows that Φ is a well-defined injective function and doing a similar argument to the one above it follows that $|S^{\tilde{T}}(A)| \leq \|M\| = |A|$. $\square$

Lemma 8.4.10. *For T a theory of modules and $\lambda \geq |T|^+$, the following are equivalent.*

1. *M is a model of $\tilde{T}$ and M is λ-saturated in $\tilde{T}$.*
2. *M is λ-saturated in $\mathbf{K}^T$.*

Proof. $\Rightarrow$: Let $L \in \mathbf{K}^T$, $L \leq_p M$, $\|L\| < \lambda$ and $p \in \mathbf{gS}(L)$. Let L^* be a $(\|L\|^{|T|}, \omega)$-limit model such that $L \leq_p L^*$ and $a \in L^*$ with $p = \mathbf{tp}(a/L;L^*)$.

Realize that L^* is a model of $\tilde{T}$ by Fact 8.4.3, so $tp(a/L,L^*) \in S^{\tilde{T}}(L)$. Then since M is λ-saturated in $\tilde{T}$ there is $b \in M$ such that $tp(a/L,L^*) = tp(b/L,M)$. Therefore, since $L^* \equiv M$ and Fact 8.4.2, we conclude that $\mathbf{tp}(a/L;L^*) = \mathbf{tp}(b/L;M)$.

$\Leftarrow$: By Lemma 8.4.6 M is a model of $\tilde{T}$. Let $A \subseteq M$ and $p \in S^{\tilde{T}}(A)$ with $|A| < \lambda$. Let N be an elementary extension of M and $a \in N$ such that $p = tp(a/A,N)$. Let M^* be the structure obtained by applying downward Löwenheim-Skolem-Tarski to A in M, observe that $\|M^*\| < \lambda$ because $\lambda \geq |T|^+$.

Realize that $M^* \leq_p M \leq_p N$, so $\mathbf{tp}(a/M^*;N) \in \mathbf{gS}(M^*)$. Then since M is λ-saturated in $\mathbf{K}^T$, there is $b \in M$ such that $\mathbf{tp}(a/M^*;N) = \mathbf{tp}(b/M^*;M)$. Therefore, since $M \equiv N$ and Fact 8.4.2, we conclude that $tp(a/M^*,N) = tp(b/M^*,M)$. Hence $b \in M$ realizes p. $\square$

The following result was pointed out to us by an anonymous referee.

Lemma 8.4.11. *For T a theory of modules, $\lambda \geq |T|$ and $\alpha < \lambda^+$ a limit ordinal, the following are equivalent.*

1. *M is a (λ,α)-limit model in $(Mod(\tilde{T}), \preceq)$.*
2. *M is a (λ,α)-limit model in $\mathbf{K}^T$.*

Proof. $\Rightarrow$: Let $\{M_i : i < \alpha\}$ be a witness to the fact that M is a (λ,α)-limit model in $(Mod(\tilde{T}), \preceq)$. Observe that $\{M_i : i < \alpha\}$ is chain of models in $\mathbf{K}^T$, so it is enough to show that M_{i+1} is universal over M_i for every $i < \alpha$ in $\mathbf{K}^T$. This follows easily from Lemma 8.4.7.

$\Leftarrow$: Let $\{M_i : i < \alpha\}$ be a witness to the fact that M is a (λ,α)-limit model in $\mathbf{K}^T$. $\mathbf{K}^T$ is λ-stable, by Fact 8.2.11, so let N be a (λ,ω)-limit model over M_0. Since M_1 is universal over M_0, there is $f : N \underset{M_0}{\longrightarrow} M_1$. Then by Lemma 8.4.7 it follows that $\{M_i : 0 < i < \alpha\}$ is an elementary chain of models of $\tilde{T}$. Using Lemma 8.4.7 once again, one can show that M_{i+1} is universal over M_i for every $i < \alpha$ in $(Mod(\tilde{T}), \preceq)$. Hence $\{M_i : 0 < i < \alpha\}$ is a witness to the fact that M is a (λ,α)-limit model in $(Mod(\tilde{T}), \preceq)$. $\square$

Given ϕ, ψ *pp*-formulas in one free variable such that $\mathbf{Th}_R \vdash \psi \to \phi$ and a module M, $Inv(M, \phi, \psi)$ is the size of $\phi(M)/\psi(M)$ if $|\phi(M)/\psi(M)|$ is finite and infinity otherwise.

Lemma 8.4.12. *Let T be a theory of modules. If $\mathbf{K}^T$ is closed under direct sums, then $\tilde{T}$ is closed under direct sums.*

Proof. Recall that $\tilde{M}_T$ is the $(2^{|T|}, \omega)$-limit model of $\mathbf{K}^T$.

$\boxed{\text{Claim}}$ $Inv(\tilde{M}_T, \phi, \psi) = 1$ or ∞ for every ϕ, ψ *pp*-formulas in one free variable such that $\mathbf{Th}_R \vdash \psi \to \phi$.

Proof of Claim: Let ϕ, ψ *pp*-formulas such that $\mathbf{Th}_R \vdash \psi \to \phi$ and assume for the sake of contradiction that $Inv(\tilde{M}_T, \phi, \psi) = k > 1$ for $k \in \mathbb{N}$. Since $\mathbf{K}^T$ is closed under direct sums, $\tilde{M}_T \oplus \tilde{M}_T \in \mathbf{K}^T$ and by Fact 8.2.10 there is $f : \tilde{M}_T \oplus \tilde{M}_T \to \tilde{M}_T$ pure embedding. Then:

$$k^2 = Inv(\tilde{M}_T \oplus \tilde{M}_T, \phi, \psi) = Inv(f[\tilde{M}_T \oplus \tilde{M}_T], \phi, \psi) \leq Inv(\tilde{M}_T, \phi, \psi) = k$$

The first equality and last inequality follow from [Pre88, 2.23]. Clearly the above inequality gives us a contradiction. $\dagger_{\text{Claim}}$

Let N_1, N_2 be models of $\tilde{T}$. To show that $N_1 \oplus N_2$ is a model of $\tilde{T}$, by [Pre88, 2.18] it is enough to show that $Inv(N_1 \oplus N_2, \phi, \psi) = Inv(\tilde{M}_T, \phi, \psi)$ for every ϕ, ψ *pp*-formulas in one free variable such that $\mathbf{Th}_R \vdash \psi \to \phi$. Since $Inv(N_1 \oplus N_2, \phi, \psi) = Inv(N_1, \phi, \psi) Inv(N_2, \phi, \psi)$ (by [Pre88, 2.23]), the result follows from the above claim. $\square$

8.4.2 Superstability in classes closed under direct sums

In this section we will characterize superstability in classes of modules with pure embeddings closed under direct sums. Several examples of classes satisfying this hypothesis are given in [Ch. 7, 3.10].

Remark 8.4.13. Given T a theory of modules, if $\mathbf{K}^T$ is closed under direct sums then $\mathbf{K}^T$ satisfies Hypothesis 8.4.1 by [Ch. 7, 3.8]. Nevertheless, we keep Hypothesis 8.4.1 as an assumption to make the presentation smoother.

In [Ch. 7, §4] limit models on classes of the form $\mathbf{K}^T$ were studied. Below we record the two assertions we will use in this paper.

Fact 8.4.14 ([Ch. 7, 4.5])**.** Assume $\lambda \geq |T|^+$. If M is a (λ, α)-limit model in $\mathbf{K}^T$ and $\mathrm{cf}(\alpha) \geq |T|^+$, then M is pure-injective.

Fact 8.4.15 ([Ch. 7, 4.9])**.** Assume $\lambda \geq |T|^+$ and $\mathbf{K}^T$ is closed under direct sums. If M is a (λ, ω)-limit model and N is a $(\lambda, |T|^+)$-limit model, then M is isomorphic to $N^{(\aleph_0)}$.

With this we are ready to obtain the next result.

Lemma 8.4.16. *Assume $\mathbf{K}^T$ is closed under direct sums. If there exists $\mu \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality μ, then every (λ, α)-limit model is Σ-pure-injective for every $\lambda \geq |T|$ and $\alpha < \lambda^+$ limit ordinal.*

Proof. Let $M \in \mathbf{K}^T_\lambda$ be a (λ, α)-limit model. Fix $N \in \mathbf{K}^T_\mu$ be a (μ, ω)-limit model and $N^* \in \mathbf{K}^T_\mu$ be a $(\mu, |T|^+)$-limit model. By Fact 8.4.15 we have that $N \cong (N^*)^{(\omega)}$.

Then by uniqueness of limit models of size μ we have that $N \cong N^*$. Hence $(N^*)^{(\omega)} \cong N^*$. Moreover, N^* is pure-injective by Fact 8.4.14. Therefore, N^* is Σ-pure-injective.

Finally, observe that by Fact 8.4.3 N^* is elementary equivalent to M, hence M is Σ-pure-injective by Fact 8.2.21. □

Observe that in the above proof we did not use the full-strength of uniqueness of limit models of size μ, but the weaker statement that the (μ, ω)-limit model is isomorphic to the $(\mu, |T|^+)$-limit model. We record it as a corollary for future reference.

Corollary 8.4.17. *Assume $\mathbf{K}^T$ is closed under direct sums. If there exists $\mu \geq |T|^+$ such that the (μ, ω)-limit model is isomorphic to the $(\mu, |T|^+)$-limit model, then every (λ, α)-limit model is Σ-pure-injective for every $\lambda \geq |T|$ and $\alpha < \lambda^+$ limit ordinal.*

Lemma 8.4.18. *Assume $\mathbf{K}^T$ is closed under direct sums. If there exists $\mu \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality μ, then $\mathbf{K}^T$ is λ-stable for every $\lambda \geq |T|$.*

Proof. Since $\mathbf{K}^T$ has uniqueness of limit models of size μ, by Lemma 8.4.16 $\tilde{M}_T$ is Σ-pure-injective. Then by Fact 8.2.22 $Th(\tilde{M}_T) = \tilde{T}$ is λ-stable for every $\lambda \geq |T|$. Therefore, it follows from Lemma 8.4.9 that $\mathbf{K}^T$ is λ-stable for every $\lambda \geq |T|$. □

The next result is easy to prove, but due to its importance we record it.

Corollary 8.4.19. *If M, N are limit models and M, N are pure-injective, then M is isomorphic to N.*

Proof. Since M,N are limit models, it follows from Fact 8.2.10 that there are $f : M \to N$ and $g : N \to M$ pure embeddings. Then by Fact 8.2.26 and the hypothesis that M, N are pure-injective, we conclude that M is isomorphic to N. □

The next lemma is one of the key assertions of the section. In it we show that if the class is closed under direct sums, uniqueness of limit models in *one* cardinal implies uniqueness of limit models in *all* cardinals.

Lemma 8.4.20. *Assume $\mathbf{K}^T$ is closed under direct sums. The following are equivalent.*

1. *For every $\lambda \geq |T|$, $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*
2. *$\mathbf{K}^T$ is superstable.*
3. *There exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*

Proof. (1) implies (2) and (2) implies (3) are clear, so we show (3) implies (1).

Let $\lambda \geq |T|$. By Lemma 8.4.18 it follows that $\mathbf{K}^T$ is λ-stable. Hence for every $\alpha < \lambda^+$ limit ordinal there is a (λ, α)-limit model by Fact 8.2.11. So we only need to show uniqueness of limit models. Let M and N two limit models of cardinality λ. By Lemma 8.4.16 we have that M and N are both pure-injective modules. Therefore, it follows from the above corollary that M is isomorphic to N. $\square$

We will give several additional equivalent conditions to the ones of Lemma 8.4.20, but before we do that let us characterize superlimits in classes of modules.

Lemma 8.4.21. *Assume $\mathbf{K}^T$ is λ-stable and $\lambda \geq |T|^+$. If M is a superlimit of size λ, then M is pure-injective. Moreover, if $\mathbf{K}^T$ is closed under direct sums, then M is Σ-pure-injective.*

Proof sketch. By Fact 8.2.17.(3) M is isomorphic to every (λ, α)-limit model for $\alpha < \lambda^+$ limit ordinal. Then by Fact 8.4.14 M is pure-injective. For the moreover part, observe that the existence of a superlimit and the stability assumption imply uniqueness of limit models. Therefore, by Lemma 8.4.16, M is Σ-pure-injective. $\square$

The following lemma can be proven using a similar technique to Fact 8.2.17.(2) and using [Ch. 7, 4.5].

Lemma 8.4.22. *If M is a $(\lambda, |T|^+)$-limit model in $\mathbf{K}^T$, then M is saturated in $\mathbf{K}^T$.*

The following theorem characterizes superstability in classes of modules closed under direct sums.[7]

Theorem 8.4.23. *Assume $\mathbf{K}^T$ is closed under direct sums. The following are equivalent.*

1. *$\mathbf{K}^T$ is superstable.*
2. *There exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*

[7]Conditions (3) through (6) of the theorem below were motivated by [GrVas17, 1.3].

3. *For every $\lambda \geq |T|$, $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*

4. *For every $\lambda \geq |T|^+$, $\mathbf{K}^T$ has a superlimit of cardinality λ.*

5. *For every $\lambda \geq |T|$, $\mathbf{K}^T$ is λ-stable.*

6. *For every $\lambda \geq |T|^+$, an increasing chain of λ-saturated models in $\mathbf{K}^T$ is λ-saturated in $\mathbf{K}^T$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By Lemma 8.4.20.

(2) $\Rightarrow$ (5) By Lemma 8.4.18.

(5) $\Rightarrow$ (2) By Lemma 8.4.9 $\tilde{T}$ is λ-stable for every $\lambda \geq |T|$. Then by [Pre88, 3.1] every model of $\tilde{T}$ is Σ-pure-injective. Let $\lambda = |T|^+$. By λ-stability and Fact 8.2.11 there are (λ, α)-limit models for every $\alpha < \lambda^+$ limit ordinal. The uniqueness of limit models of cardinality λ follows from the fact that limit models are pure-injective, since they are models of $\tilde{T}$ by Corollary 8.4.5, and by Corollary 8.4.19.

(5) $\Rightarrow$ (6) Let $\{M_i : i < \delta\} \subseteq \mathbf{K}^T$ be an increasing chain of λ-saturated models. By Lemma 8.4.10 every M_i is a model of $\tilde{T}$ and λ-saturated in $\tilde{T}$. Moreover, by Lemma 8.4.7, for every $i < j$ we have that $M_i \preceq M_j$. Therefore, $\{M_i : i < \delta\}$ is an increasing chain of λ-saturated models in $\tilde{T}$.

Then by hypothesis and Lemma 8.4.9 $\tilde{T}$ is superstable as a first-order theory. Hence, by [Har75], $\bigcup_{i<\delta} M_i$ is a λ-saturated model of $\tilde{T}$. Therefore, by Lemma 8.4.10, $\bigcup_{i<\delta} M_i$ is λ-saturated in $\mathbf{K}^T$.

(6) $\Rightarrow$ (2) Let $\lambda = 2^{|T|}$ and M be a $(2^{|T|}, |T|^+)$-limit model. By Fact 8.4.14 M is pure-injective and by Lemma 8.4.22 M is saturated. Consider $\{M^n : 0 < n < \omega\}$ an increasing chain in $\mathbf{K}_{2^{|T|}}$ where M^n denotes n-many direct copies of M. Observe that each M^n is pure-injective (because pure-injective modules are closed under finite direct sums) and that there are pure embeddings between M^n and M and vice versa by Fact 8.2.10. Therefore, by Fact 8.2.26, $M \cong M^n$ for every $n > 0$. So in particular, M^n is $\|M\|$-saturated for every $n > 0$.

Then by hypothesis $M^{(\aleph_0)}$ is $\|M\|$-saturated, so $M^{(\aleph_0)}$ is saturated. Since saturated models of the same cardinality are isomorphic, $M^{(\aleph_0)} \cong M$. On the other hand, by Fact 8.4.15, $M^{(\aleph_0)}$ is the $(2^{|T|}, \omega)$-limit model. Then by Corollary 8.4.17 every limit model is Σ-pure-injective. Therefore, by Corollary 8.4.19, $\mathbf{K}^T$ has uniqueness of limit models of size $2^{|T|}$.

(4) $\Rightarrow$ (2) Similar to (5) implies (2) of Theorem 8.3.12.

(3) $\Rightarrow$ (4) Let $\lambda \geq |T|^+$. By condition (5) $\mathbf{K}^T$ is λ-stable, so let M be a $(\lambda, |T|^+)$-limit model. By Lemma 8.4.22 M is $\|M\|$-saturated. Moreover, by condition (6) any increasing chain of $\|M\|$-saturated models is $\|M\|$-saturated. Therefore it follows from Fact 8.2.17.(3) that there is a superlimit of cardinality λ.

□

Remark 8.4.24. In [GrVas17, 1.3] cardinals μ_ℓ for $\ell \in \{1, ..., 7\}$ were introduced. In the introduction of [GrVas17] it is asked if it is possible to calculate the values of the μ_ℓ's for certain AECs. The above lemma shows that in classes of modules satisfying Hypothesis 8.4.1 and closed under direct sums $\mu_3 = \mu_7 = |T|$ and $\mu_4, \mu_5 \leq |T|^+$. We did not calculate the values of μ_1, μ_2 and μ_6 since they measure properties that are more technical than the ones presented above and which we have not introduced. We hope to study those properties in future work.

Remark 8.4.25. Let T be a theory of modules such that Hypothesis 8.4.1 holds. Then by Fact 8.4.2 $\mathbf{K}^T$ is $(< \aleph_0)$-tame and by hypothesis $\mathbf{K}^T$ has amalgamation, joint embedding and no maximal models. Therefore, we could have simply quoted [GrVas17, 1.3] to obtain (4),(5),(6) imply (2) of the above theorem. We decided to provide the proofs for those directions to make the paper more transparent and since the proofs in our case are easier than in the general case. An important difference between our methods and those of [GrVas17] and [Vas18] is that the results of Grossberg and Vasey do not use the hypothesis that the class is closed under direct sums, but only Hypothesis 8.4.1. We will come back to this in Subsection 4.4.

8.4.3 Algebraic characterizations of superstability and pure-semisimple rings

We begin by giving algebraic characterizations of superstability in classes of modules closed under direct sums.

Theorem 8.4.26. *Assume $\mathbf{K}^T$ is closed under direct sums. The following are equivalent.*

1. $\mathbf{K}^T$ *is superstable.*
2. *There exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*

7. *Every $M \in \mathbf{K}^T$ is pure-injective.*
8. *There exists $\lambda \geq |T|^+$ such that $\mathbf{K}^T$ has a Σ-pure-injective universal model in $\mathbf{K}^T_\lambda$.*
9. *Every limit model in $\mathbf{K}^T$ is Σ-pure-injective.*

Proof. (1) $\Rightarrow$ (2) By Theorem 8.4.23.
(2) $\Rightarrow$ (9) By Lemma 8.4.16.
(9) $\Rightarrow$ (7) Let M be an R-module and let $\mu = \|M\|^{|R|+\aleph_0}$.

By [Ch. 7, 3.16] $\mathbf{K}^T$ is μ-stable, so let N be a (μ, ω)-limit model such that $M \leq_p N$. Then by hypothesis N is Σ-pure-injective. As $M \leq_p N$, by Fact 8.2.21 it follows that M is Σ-pure-injective. Hence M is pure-injective.

(7) $\Rightarrow$ (8) Let $\lambda = 2^{|T|}$, then by [Ch. 7, 3.16] $\mathbf{K}^T$ is λ-stable. Let M be a (λ, ω)-limit model. M is universal in $\mathbf{K}^T_\lambda$ by Fact 8.2.10 and M is Σ-pure-injective by condition (7) and closure under direct sums.

(8) $\Rightarrow$ (2) Let $\lambda \geq |T|^+$ and M be a Σ-pure-injective universal model in $\mathbf{K}^T_\lambda$. As in (6) to (2) of Theorem 8.4.23, consider $\{M^n : 0 < n < \omega\}$ an increasing chain in $\mathbf{K}^T_\lambda$. Observe that M^{n+1} is universal over M^n for every $n > 0$ by [Ch. 7, 4.8]. Hence the chain witnesses that $M^{(\aleph_0)}$ is a (λ, ω)-limit model in $\mathbf{K}^T$. So $\mathbf{K}^T$ is λ-stable by Fact 8.2.11. We are left to show uniqueness of limit models of size λ. Let N_1 and N_2 be two limit models of cardinality λ. Since $M^{(\aleph_0)}$, N_1 and N_2 are elementary equivalent by Fact 8.4.3 and M is Σ-pure-injective by hypothesis, then N_1 and N_2 are pure-injective by Fact 8.2.21. So from Corollary 8.4.19 we conclude that N_1 is isomorphic to N_2. Therefore, $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.

$\square$

Remark 8.4.27. In [Ch. 7, 4.10], stability is characterized by the existence of a pure-injetive universal model. The equivalence between (1) and (8) is another instance of how the existence of a universal model can be used to characterize an stability-like property.

The next theorem gives many equivalent conditions to the notion of pure-semisimple ring and relates it to superstability. It follows directly from Theorem 8.4.23 and Theorem 8.4.26 as a ring R is pure-semisimple if and only if every $M \in \mathbf{K}^{\mathbf{Th}_R}$ is pure-injective (Definition 11.3.25).[8]

Theorem 8.4.28. *For a ring R the following are equivalent.*

1. *R is left pure-semisimple.*
2. *$\mathbf{K}^{\boldsymbol{Th}_R}$ is superstable.*
3. *There exists $\lambda \geq (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\boldsymbol{Th}_R}$ has uniqueness of limit models of cardinality λ.*
4. *For every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\boldsymbol{Th}_R}$ has uniqueness of limit models of cardinality λ.*
5. *For every $\lambda \geq (|R| + \aleph_0)^+$, $\mathbf{K}^{\boldsymbol{Th}_R}$ has a superlimit of cardinality λ.*
6. *For every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\boldsymbol{Th}_R}$ is λ-stable.*

[8] Conditions (4) through (7) of the theorem below were motivated by [GrVas17, 1.3].

7. *For every $\lambda \geq (|R| + \aleph_0)^+$, an increasing chain of λ-saturated models in $\mathbf{K}^{\boldsymbol{Th}_R}$ is λ-saturated in $\mathbf{K}^{\boldsymbol{Th}_R}$.*

8. *There exists $\lambda \geq (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\boldsymbol{Th}_R}$ has a Σ-pure-injective universal model in $\mathbf{K}^{\boldsymbol{Th}_R}_\lambda$.*

9. *Every limit model in $\mathbf{K}^{\boldsymbol{Th}_R}$ is Σ-pure-injective.*

As it was mentioned in the introduction, Shelah noticed that superstability of the class of modules and pure-semisimple rings are equivalent in Theorem 8.7 of [Sh54] (without mentioning pure-semisimple rings). The precise equivalence Shelah noticed is similar to the equivalence between (1) and (6) of the above theorem. Shelah does not prove that pure-semisimplicty implies superstability of the theory of modules ((2) to (1) of his Theorem 8.7). The notion of superstability studied in [Sh54, §8] is not first-order superstability, but superstability with respect to positive primitive formulas. This is equivalent to our notion of superstability in the class of modules by Fact 8.4.2. For the direction that Shelah provides a proof ((1) to (5) of his Theorem 8.7), his technique is different from ours as he proceeds by contradiction and builds a tree of formulas.

One can add one more equivalent condition to the above theorem.

Lemma 8.4.29. *For a ring R the following are equivalent.*

1. *R is left pure-semisimple.*

3. *There exists $\lambda \geq (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\boldsymbol{Th}_R}$ has uniqueness of limit models of cardinality λ.*

10. *For all theories of modules T and for all $\lambda \geq |R|+\aleph_0$, if $\mathbf{K}^T$ satisfies Hypothesis 8.4.1 and it is closed under direct sums, then $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*

Proof. (1) implies (10) follows from condition (7) of Theorem 8.4.26. Moreover, it is clear that (10) implies (3). □

Assuming that the ring is left coherent, Theorem 8.4.26 can further be used to obtain a new characterization of left noetherian rings.

Lemma 8.4.30. *Let R be a left coherent ring and K_{abs} be the class of absolutely pure left R-modules. R is left noetherian if and only if $\mathbf{K}_A = (K_{abs}, \leq)$ is superstable.*

Proof. Since R is left coherent, by [Pre09, 3.4.24], there is T a first-order theory such that $Mod(T) = K_{\text{abs}}$. Observe that for absolutely pure modules embeddings are the same as pure embeddings. Moreover, absolutely pure modules are closed under direct sums and because of it have joint embedding and amalgamation with respect to pure embeddings by [Ch. 7, 3.8]. Therefore, we can use the results of Theorem 8.4.26. More precisely, we will show the equivalence with condition (7) of Theorem 8.4.26, i.e., we will show that R is left noetherian if and only if every absolutely pure module is pure-injective.

If R is left noetherian, then every absolutely pure module is injective by Fact 8.2.23.(5). Hence every absolutely pure module is pure-injective.

If every absolutely pure module is pure-injective, then it is clearly injective. Thus R is left noetherian by Fact 8.2.23.(5). $\square$

Remark 8.4.31. The above assertion is still true without the assumption that R is left coherent. That case lies outside of the scope of the present paper as in that case the class of absolutely pure modules is not first-order axiomatizable. That result is presented in [Ch. 11] .

Remark 8.4.32. Assuming that the ring is right coherent, Theorem 8.4.26 can also be used to characterize left perfect rings. The proof is similar to that of Lemma 8.4.30, but we do not present it as we have not introduced many of the notions needed to setup the proof. Moreover, a proof without the assumption that the ring is right coherent is given in [Ch. 9, 3.15]. That case is significantly more intricate as the class of flat modules is not first-order axiomatizable and is not closed under pure-injective envelopes.

We think that Theorem 8.3.12, Theorem 8.4.28 and Lemma 8.4.30 hint to the the fact that limit models and superstability could shed light in the understanding of algebraic concepts. They also provide further evidence of the naturality of the notion of superstability.

8.4.4 Superstable classes

In this section we will characterize superstability in classes of modules without assuming that the class is closed under direct sums. As in previous subsections we assume Hypothesis 8.4.1. In this section we assume the reader has some familiarity with first-order model theory.

Recall that given T' a complete first-order theory, $\kappa(T')$ is the least cardinal such that every type of T' does not fork over a set of size less than $\kappa(T')$. In this case, nonforking refers to first-order nonforking. Since it is well-known that $\kappa(T') \leq |T'|^+$ for stable theories, the following improves [Ch. 7, 4.7].

Lemma 8.4.33. *Let $\lambda \geq |T|$ and $\alpha, \beta < \lambda^+$ be limit ordinals. If M is a (λ, α)-limit model in $\mathbf{K}^T$, N is a (λ, β)-limit model in $\mathbf{K}^T$ and $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) \geq \kappa(\tilde{T})$, then M is isomorphic to N.*

Proof. By Lemma 8.4.11 M is a (λ, α)-limit model in $(Mod(\tilde{T}), \preceq)$ and N is a (λ, β)-limit model in $(Mod(\tilde{T}), \preceq)$. Then it follows from [GVV16, 1.6] that M and N are both saturated models of cardinality λ in $\tilde{T}$. Therefore, we conclude that M is isomorphic to N. □

Recall the following notions introduced in [Vas18, §2]. Given an AEC $\mathbf{K}$ and μ a cardinal, $\mathrm{Stab}(\mathbf{K}) = \{\mu : \mathbf{K} \text{ is } \mu\text{-stable}\}$ and $\underline{\kappa}(\mathbf{K}_\mu, \leq^{\mathrm{univ}}_{\mathbf{K}})$ is the set of regular cardinals χ such that whenever $\{M_i : i < \chi\}$ is an increasing chain in $\mathbf{K}_\mu$ with M_{i+1} is universal over M_i for every $i < \chi$ and $p \in \mathbf{gS}(\bigcup_{i<\chi} M_i)$, then there is $i < \chi$ such that p does not split over M_i. Since we will not use the notion of splitting, we will not introduce it. It is somehow similar to first-order splitting, the definition is presented in [Vas18, 2.3].

In [Ch. 7, 4.12] it is asked to describe the spectrum of limit models for classes of the form $\mathbf{K}^T$ where T is a theory of modules. The case when the ring is countable is studied in [Ch. 7, 4.12]. The next result provides a partial solution to that question.

Theorem 8.4.34. *Let $\lambda \geq |T|^+$ be a regular cardinal. Let M be a (λ, α)-limit model in $\mathbf{K}^T$, then:*

1. *If $\mathrm{cf}(\alpha) \geq \kappa(\tilde{T})$, then M is isomorphic to the (λ, λ)-limit model.*
2. *If $\mathrm{cf}(\alpha) < \kappa(\tilde{T})$, then M is not isomorphic to the (λ, λ)-limit model.*

Proof. (1) follows from Lemma 8.4.33 and the fact that $\kappa(\tilde{T}) \leq |T|^+$. So we prove (2). Assume for the sake of contradiction that M is isomorphic to the (λ, λ)-limit model in $\mathbf{K}^T$ and let $\tilde{\mathbf{K}} := (Mod(\tilde{T}), \preceq)$.

Since $\mathrm{cf}(\alpha) < \kappa(\tilde{T})$, by [Vas18, 4.8] $\mathrm{cf}(\alpha) \notin \underline{\chi}(\tilde{\mathbf{K}}) = \bigcup_{\mu \in \mathrm{Stab}(\tilde{\mathbf{K}})} \underline{\kappa}(\tilde{\mathbf{K}}_\mu, \leq^{\mathrm{univ}}_{\tilde{\mathbf{K}}})$. Since M is a limit model in $\mathbf{K}^T$, by Fact 8.2.11, $\mathbf{K}^T$ is λ-stable so by Lemma 8.4.9 $\tilde{\mathbf{K}}$ is λ-stable. Hence $\mathrm{cf}(\alpha) \notin \underline{\kappa}(\tilde{\mathbf{K}}_\lambda, \leq^{\mathrm{univ}}_{\tilde{\mathbf{K}}})$. Then by definition of $\underline{\kappa}$ there is $\{L_i : i < \mathrm{cf}(\alpha)\}$ an increasing chain in $\tilde{\mathbf{K}}_\lambda$ with L_{i+1} universal over L_i and $p \in \mathbf{gS}(\bigcup_{i<\mathrm{cf}(\alpha)} L_i)$ such that p splits over L_i in $\tilde{\mathbf{K}}$ for every $i < \mathrm{cf}(\alpha)$.

Observe that $L = \bigcup_{i<\mathrm{cf}(\alpha)} L_i$ is a $(\lambda, \mathrm{cf}(\alpha))$-limit model in $\mathbf{K}^T$ by Lemma 8.4.11. Then doing a back-and-forth argument $L \cong M$. And since by hypothesis M is isomorphic to the (λ, λ)-limit model, L is isomorphic to it. By Fact 8.2.17.(1) it follows that L is a λ-saturated model in $\mathbf{K}^T$. So by Lemma 8.4.10 L is λ-saturated in $\tilde{\mathbf{K}}$. Then by [Vas18, 4.12] there is $i < \mathrm{cf}(\alpha)$ such that p does not split over L_i in $\tilde{\mathbf{K}}$. This contradicts the choice of the $L_i's$. □

Moreover, the result of the above theorem gives a positive solution above $|T|^+$ to Conjecture 2 of [BoVan] in the case when T is a theory of modules satisfying Hypothesis 8.4.1.

Corollary 8.4.35. *Let T be a theory of modules such that $\mathbf{K}^T$ satisfies Hypothesis 8.4.1. Assume that $\lambda \geq |T|^+$ is a regular cardinal such that $\mathbf{K}^T$ is λ-stable. Then*

$$\Delta_\lambda := \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) = \alpha \text{ and the } (\lambda, \alpha)\text{-limit model is isomorphic to the } (\lambda, \lambda)\text{-limit model}\}$$

is an end segment of regular cardinals.

We finish this section by presenting a similar result to Theorem 8.4.23, but without the assumption that T is closed under direct sums.

Theorem 8.4.36. *Assume $\mathbf{K}^T$ satisfies Hypothesis 8.4.1. The following are equivalent.*

1. *$\mathbf{K}^T$ is superstable.*
2. *For every $\lambda \geq 2^{|T|}$, $\mathbf{K}^T$ has uniqueness of limit models of cardinality λ.*
3. *For every $\lambda \geq 2^{|T|}$, $\mathbf{K}^T$ has a superlimit of cardinality λ.*
4. *For every $\lambda \geq 2^{|T|}$, $\mathbf{K}^T$ is λ-stable.*
5. *For every $\lambda \geq 2^{|T|}$, an increasing chain of λ-saturated models in $\mathbf{K}^T$ is λ-saturated in $\mathbf{K}^T$.*

Proof sketch. (1) $\Rightarrow$ (4) By (1) and Lemma 8.4.11 $(Mod(\tilde{T}), \preceq)$ has uniqueness of limit models in a tail of cardinals. Then by Fact 8.2.11 $\tilde{T}$ is stable in a tail of cardinals. Since $\tilde{T}$ is a first-order theory, the tail has to begin at most in $2^{|T|}$.

(4) $\Rightarrow$ (2) By Lemma 8.4.9 $\tilde{T}$ is superstable, then by [Sh:a] (see [GrVas17, 1.1.(3)]) $\kappa(\tilde{T}) = \aleph_0$. The result follows from Fact 8.2.11 and Lemma 8.4.33.

(2) $\Rightarrow$ (1) Clear.

(4) $\Rightarrow$ (5) Similar to (5) implies (6) of Theorem 8.4.23.

(5) $\Rightarrow$ (4) The idea is to prove that $\tilde{T}$ is superstable and then by Lemma 8.4.9 the result would follow. To prove that $\tilde{T}$ is superstable, by [Sh:a] (see [GrVas17, 1.1.(2)]), it is enough to show that an elementary increasing chain of λ-saturated models in $\tilde{T}$ is λ-saturated. The proof is similar to (5) implies (6) of Theorem 8.4.28 by using Lemma 8.4.10.

(2) $\Rightarrow$ (3) Similar to (3) implies (4) of Theorem 8.4.23.

(3) $\Rightarrow$ (4) Assume for the sake of contradiction that (4) fails, then by Lemma 8.4.9 $\tilde{T}$ is not superstable. Then by [Sh:a] (see [GrVas17, 1.1.(3)]) $\kappa(\tilde{T}) > \aleph_0$. Let $\lambda = (2^{|T|})^+$. Observe that $\mathbf{K}^T$ is λ-stable since $\lambda^{|T|} = \lambda$, so by Fact 8.2.17.(3) and (3) $\mathbf{K}^T$ has uniqueness of limit models in λ. Now, by Theorem 8.4.34.(2), we know that the (λ, ω)-limit model is not isomorphic to the (λ, λ)-limit model, which clearly gives us a contradiction. $\square$

Remark 8.4.37. Compared to [GrVas17, 1.3], the above theorem improves the bounds where the *nice properties* show up from $\beth_{(2^{|T|})^+}$ to $2^{|T|}$ in the case of classes of modules satisfying Hypothesis 8.4.1. It is worth pointing out that to obtain (3), (5) imply (1) we could have simply quoted [GrVas17, 1.3]. We decided to provide the proofs for those direction to make the paper more transparent and to show the deep connection between $\mathbf{K}^T$ and $\tilde{T}$.

Besides the difference in the bounds between Theorem 8.4.23 and Theorem 8.4.36. The techniques are also quite different, while the proof of the first theorem relies more on algebraic notions, the proof of the second theorem relies heavily on model theoretic methods.

Chapter 9

On superstability in the class of flat modules and perfect rings

This chapter is based on [Ch. 9].

Abstract

We obtain a characterization of left perfect rings via superstability of the class of flat left modules with pure embeddings.

Theorem 9.0.1. *For a ring R the following are equivalent.*

1. *R is left perfect.*
2. *The class of flat left R-modules with pure embeddings is superstable.*
3. *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that the class of flat left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*
4. *Every limit model in the class of flat left R-modules with pure embeddings is Σ-cotorsion.*

A key step in our argument is the study of limit models in the class of flat modules. We show that limit models with chains of long cofinality are cotorsion and that limit models are elementarily equivalent.

We obtain a new characterization via limit models of the rings characterized in [Roth02]. We show that in these rings the equivalence between left perfect rings and superstability can be refined. We show that the results for these rings can be applied to extend [Sh820, 1.2] to classes of flat modules not axiomatizable in first-order logic.

9.1 Introduction

An *abstract elementary class* (AEC for short) is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is class of structures and $\leq_{\mathbf{K}}$ is a partial order on K extending the substructure relation. AECs are closed under directed limits and every subset of a model in the class is contained in a small model in the class. Shelah introduced them in [Sh88] to capture the semantic structure of non-first-order theories. Some interesting algebraic examples are: abelian groups with embeddings, torsion-free groups with pure embeddings, R-modules with embeddings, R-modules with pure embeddings and first-order axiomatizable classes of modules with pure embeddings. In this paper, we focus on the class of flat modules with pure embeddings. This is an AEC because flat modules are closed under pure submodules and directed limits. This class was already considered in [LRV1a, §6].

Superstable theories were introduced by Shelah in [Sh1] as part of his project to find dividing lines on the class of complete first-order theories. This project is still central in current research in model theory. Extensions of superstability for non-first-order theories were first studied in Grossberg's PhD thesis and published in [GrSh86]. In the context of AECs, superstability was first considered in [Sh394], but it was not until the work of Grossberg and Vasey ([GrVas17], [Vas18]) that it was fully grasped. In [GrVas17, 1.3] and [Vas18], it was shown (under extra hypotheses that are satisfied by the class of flat modules[1]) that superstability is a well-behaved concept and many conditions that were believed to characterize superstability were found to be equivalent. Grosberg's and Vasey's work builds on significant results of Boney, Shelah, Villaveces, and VanDieren.[2] Due to this and the important role that limit models play in this paper, we say that an AEC is *superstable* if it has uniqueness of limit models on a tail of cardinals.[3] This particular definition of superstability appears for the first time in [GrVas17].

A ring R is *left perfect* if every flat left R-module is a projective module. Left

[1]The hypothesis are amalgamation, joint embedding, no maximal models and tameness.

[2]A more detailed account of the development of the notion of superstability in AECs can be consulted in the introduction of [GrVas17].

[3]For a complete first-order T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory, i.e., T is λ-stable for every $\lambda \geq 2^{|T|}$.

perfect rings were introduced by Bass in [Bas60]. They play a significant role in homological algebra (see [Lam91, §8]). Xu was the first to notice a relation between perfect rings and cotorsion modules in [Xu96, 3.3.1].

In this paper, we provide further evidence that the concept of superstability has algebraic significance. In the context of AECs this was first noticed in [Ch. 8]. Prior to it, there were a few papers [Sh54], [BaMc82] and [GrSh86] where notherian rings, artinian rings and superstability were related.

More precisely, we characterize left perfect rings via superstability of the class of flat left modules with pure embeddings. The main theorem of the paper is the following.

Theorem 9.3.15. *For a ring R the following are equivalent.*

1. *R is left perfect.*
2. *The class of flat left R-modules with pure embeddings is superstable.*
3. *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that the class of flat left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*
4. *Every limit model in the class of flat left R-modules with pure embeddings is Σ-cotorsion.*

In order to obtain the above equivalence, we study the limit models in the class of flat modules. We show that limit models with chains of long cofinality are cotorsion (Theorem 9.3.5), show that limit models are elementarily equivalent (Lemma 9.3.10) and characterize limit models with chains of countable cofinality (Lemma 9.3.13).

Merging the main theorem of this paper together with the characterization of noetherian rings via superstability obtained in [Ch. 8, 3.12]; we obtain a characterization of artinian rings via superstability (Corollary 9.3.17).

In contrast to previous results on limit models with chains of long cofinality [Ch. 5, 4.10], [Ch. 7, 4.5], limit models with chains of long cofinalities in this case might not be pure-injective. This happens precisely because the class of flat modules is not necessarily closed under pure-injective envelopes. We obtain the following.

Theorem 9.4.10. *For a ring R the following are equivalent.*

1. *Every (λ, α)-limit model in the class of flat modules (with pure embeddings) with $\lambda \geq (|R| + \aleph_0)^+$ and $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$ is pure-injective.*
2. *The pure-injective envelope of every flat left R-module is flat.*

Since the rings characterized in [Roth02] are precisely those that satisfy the second condition of the above theorem, the result gives a new characterization of such rings. In these rings we characterize the Galois-types and the stability cardinals of the class of flat modules with pure embeddings. As a simple corollary we obtain a result of Shelah regarding universal torsion-free abelian groups with respect to pure embeddings [Sh820, 1.2] (see Lemma 9.4.6 and the remark below it). Moreover, by using that flat cotorsion modules are the same as pure-injective modules in this special case, we are able to lower the bound in Theorem 9.3.15 where the tail of cardinals where uniqueness of limit models begins to $|R| + \aleph_0$ (Theorem 9.4.12).

The class of flat modules is not first-order order axiomatizable [SaEk71, Theo. 4], but it is axiomatizable in $\mathbb{L}_{\infty,\omega}$ [HeRo09, §2]. Due to this, the results of this paper lie outside of the scope of first-order model theory and hint to the importance of the development of non-first-order methods.

The paper is divided into four sections. Section 2 presents necessary background. Section 3 studies limit models in the class of flat modules with pure embeddings and provides a new characterization of left perfect rings via superstability of the class of flat modules. Section 4 studies limit models in the class of flat modules under an additional assumption, characterizes this assumption via limit models and provides a refinement of the main theorem under the additional hypothesis.

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9.2 Preliminaries

We present the basic concepts of abstract elementary classes that are used in this paper. These are further studied in [Bal09, §4 - 8] and [Gro2X, §2, §4.4]. An introduction from an algebraic point of view is given in [Ch. 6, §2]. Regarding the background on module theory, we give a brief survey of the concepts we will use in this paper and present a few concepts throughout the text. The main module theoretic ideas used in this paper are studied in detail in [Xu96].

9.2.1 Abstract elementary classes

Abstract elementary classes (AECs) were introduced by Shelah in [Sh88, 1.2]. Among the requirements we have that an AEC is closed under directed limits and that every set is contained in a small model in the class. The reader can consult the definition in [Bal09, 4.1].

Notation 9.2.1.

- Given a model M, we will write $|M|$ for its underlying set and $\|M\|$ for its cardinality.
- If λ is a cardinal and $\mathbf{K}$ is an AEC, then $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$.
- Let $M, N \in \mathbf{K}$. If we write "$f : M \to N$", we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. In particular, $\mathbf{K}$-embeddings are always monomorphisms.

In [Sh300] Shelah introduced a notion of semantic type. The original definition was refined and extended by many authors who following [Gro02] call these semantic types Galois-types (Shelah recently named them orbital types). We present here the modern definition and call them Galois-types throughout the text. We follow the notation of [Ch. 3, 2.5].

Definition 9.2.2. Let $\mathbf{K}$ be an AEC.

1. Let $\mathbf{K}^3$ be the set of triples of the form $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence of elements from N.
2. For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E_{\text{at}}^{\mathbf{K}} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exist $\mathbf{K}$-embeddings $f_\ell : N_\ell \underset{A}{\to} N$ for $\ell \in \{1, 2\}$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$ and $N \in \mathbf{K}$.
3. Note that $E_{\text{at}}^{\mathbf{K}}$ is a symmetric and reflexive relation on $\mathbf{K}^3$. We let $E^{\mathbf{K}}$ be the transitive closure of $E_{\text{at}}^{\mathbf{K}}$.
4. For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_{E^{\mathbf{K}}}$. We call such an equivalence class a *Galois-type*. Usually, $\mathbf{K}$ will be clear from the context and we will omit it.
5. For $M \in \mathbf{K}$, $\mathbf{gS}_{\mathbf{K}}(M) = \{\mathbf{tp}_{\mathbf{K}}(b/M; N) : M \leq_{\mathbf{K}} N \in \mathbf{K} \text{ and } b \in N\}$.
6. For $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N)$ and $C \subseteq A$, $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N) \restriction_C := [(\mathbf{b}, C, N)]_E$.

Definition 9.2.3. An AEC $\mathbf{K}$ is λ-*stable* if for any $M \in \mathbf{K}_\lambda$, $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$.

Recall the following notion that was isolated by Grossberg and VanDieren in [GrVan06].

Definition 9.2.4. $\mathbf{K}$ is $(<\kappa)$-*tame* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}(M)$, there is $A \subseteq |M|$ such that $|A| < \kappa$ and $p \restriction_A \neq q \restriction_A$.

Before introducing the concept of limit model we recall the concept of universal extension.

Definition 9.2.5. M is λ-*universal over* N if and only if $N \leq_{\mathbf{K}} M$ and for any $N^* \in \mathbf{K}_{\leq\lambda}$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\rightarrow} M$. M is *universal over* N if and only if $\|N\| = \|M\|$ and M is $\|M\|$-*universal over* N.

With this we are ready to introduce limit models, they were originally introduced in [KolSh96].

Definition 9.2.6. Let λ be an infinite cardinal and $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that $M_0 := N$, M_{i+1} is universal over M_i for each $i < \alpha$ and $M = \bigcup_{i<\alpha} M_i$.

M is a (λ, α)-limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. M is a λ-limit model if there is a limit ordinal $\alpha < \lambda^+$ such that M is a (λ, α)-limit model. We say that M is a limit model if there is an infinite cardinal λ such that M is a λ-limit model.

Observe that if M is a λ-limit model, then M has cardinality λ. The next fact gives conditions for the existence of limit models.

Fact 9.2.7 ([Sh:h, §II], [GrVan06, 2.9])**.** Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation and no maximal models. If $\mathbf{K}$ is λ-stable, then for every $N \in \mathbf{K}_\lambda$ and limit ordinal $\alpha < \lambda^+$ there is M a (λ, α)-limit model over N. Conversely, if $\mathbf{K}$ has a λ-limit model, then $\mathbf{K}$ is λ-stable

The key question regarding limit models is the uniqueness of λ-limit models for a fixed cardinal λ. When the lengths of the cofinalities of the chains of the limit models are equal, one can show that the limit models are isomorphic by a back-and-forth argument.[4] Therefore, the question is what happens when the cofinalities of the chains of the limit models are different. This has been studied thoroughly in the context of abstract elementary classes [ShVi99], [Van06], [GVV16], [Bon14], [Van16], [BoVan] and [Vas19].

Definition 9.2.8. $\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has λ-limit models and if given M, N λ-limit models, M and N are isomorphic.

[4]Hence, for a fixed cardinal λ and a fixed limit ordinal $\alpha < \lambda^+$, there is a unique (λ, α)-limit model.

In [GrVas17, 1.3] and [Vas18] it was shown that for AECs that have amalgamation, joint embedding, no maximal models and are tame, the definition below is equivalent to every other definition of superstability considered in the context of AECs. Since the class of flat modules with pure embeddings satisfies these properties (see Fact 9.3.1), we introduce the following as the definition of superstability.

Definition 9.2.9. $\mathbf{K}$ is a *superstable* AEC if and only if $\mathbf{K}$ has uniqueness of limit models on a tail of cardinals.

Remark 9.2.10. For a complete first-order theory T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory, i.e., T is λ-stable for every $\lambda \geq 2^{|T|}$. The forward direction follows from Fact 9.2.7 and the backward direction from [GVV16, 1.6].

Finally, recall the standard notion of a universal model.

Definition 9.2.11. Let $\mathbf{K}$ be an AEC and λ be a cardinal. $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is a $\mathbf{K}$-embedding $f : N \to M$.

The following fact will be useful.

Fact 9.2.12 ([Ch. 5, 2.10])**.** Let $\mathbf{K}$ be an AEC with the joint embedding property. If M is a λ-limit model, then M is a universal model in $\mathbf{K}_\lambda$.

9.2.2 Module Theory

All rings considered in this paper are associative with an identity element. Recall that a left R-module F is *flat* if $(-) \otimes F$ is an exact functor. M is a *pure submodule* of N, denoted by $\leq_p$, if for every L right R-module $L \otimes M \to L \otimes N$ is a monomorphism.

Notation 9.2.13. Given a ring R, let $\mathbf{K}^{\mathcal{F}} = (K_{flat}, \leq_p)$ where K_{flat} is the class of flat left R-modules and $\leq_p$ denotes the pure submodule relation.

We assume the reader is familiar with pure-injective modules (see for example [Pre88, §2]) and focus on cotorsion modules. Cotorsion modules were introduced by Harrison in [Har59].

Definition 9.2.14. A left R-module M is *cotorsion* if and only if $Ext^1(F, M) = 0$ for every flat module F, or equivalently, every short exact sequence $0 \to M \to N \to F \to 0$ with F a flat module splits.

It is easy to check that a pure-injective module is a cotorsion module. The following generalization of Bumby's result [Bum65] will be useful.

Fact 9.2.15 ([GKS18, 2.5])**.** Let M, N be flat cotorsion modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then M and N are isomorphic.

Similar to the notion of pure-injective envelope there is the notion of cotorsion envelope. These are thoroughly studied by Xu in [Xu96, §1, 3.4].

Definition 9.2.16. Let M be a module, $M \hookrightarrow_i C(M)$ is the *cotorsion envelope* of M if and only if

1. If $\phi : M \to C$ and C is a cotorsion module, then there is $f : C(M) \to C$ such that $\phi = f \circ i$.
2. If an endomorphism $f : C(M) \to C(M)$ is such that $i = f \circ i$, then f is an automorphism.

The existence of a cotorsion envelope for every module is a deep result that is equivalent to *the Flat Cover Conjecture*. The Flat Cover Conjecture was asserted by Enoch in [Eno81] and proved twenty years later by Bican, El Bashir and Enochs in [BEE01]. We will use that there are cotorsion envelopes a few times in the text.

An easy assertion that we will use is the following.

Fact 9.2.17 ([Xu96, 3.4.2])**.** If $M \hookrightarrow_i C(M)$ is a cotorsion envelope, then $C(M)/M$ is flat and $M \leq_p C(M)$. Moreover, if M is flat, then $C(M)$ is flat.

We will also work with the following class of modules.

Definition 9.2.18. A left R-module M is Σ-*cotorsion* if and only if $M^{(I)}$ is a cotorsion module for every index set I.

Left perfect rings were introduced by Bass in [Bas60]. They play a significant role in homological algebra and have been thoroughly studied, see for example [Lam91, §8].

Definition 9.2.19. A ring R is *left perfect* if every flat left R-module is a projective module.

Below we give some equivalent conditions that characterize left perfect rings. Further equivalent conditions that mention cotorsion modules are given in [GuHe05].

Fact 9.2.20 ([Xu96, 3.3.1])**.** For a ring R the following are equivalent.

1. R is left perfect.
2. Every flat left R-modules is cotorsion.
3. Every left R-module is cotorsion.

9.3 The main case

In this section, we will work with the class of flat modules with pure embeddings. We obtain a characterization of left perfect rings via superstability of the class of flat left modules with pure embeddings (Theorem 9.3.15). This is obtained by understanding the limit models of the class.

The next result follows from [LRV1a, §6].

Fact 9.3.1. Let R be ring and $\mathbf{K}^{\mathcal{F}} = (K_{flat}, \leq_p)$ where K_{flat} is the class of flat left R-modules.

1. $\mathbf{K}^{\mathcal{F}}$ is an AEC with $\text{LS}(\mathbf{K}^{\mathcal{F}}) = |R| + \aleph_0$.
2. $\mathbf{K}^{\mathcal{F}}$ has joint embedding, amalgamation and no maximal models.
3. There is a cardinal $\theta_0 \geq |R| + \aleph_0$ such that if $\lambda^{\theta_0} = \lambda$, then $\mathbf{K}^{\mathcal{F}}$ is λ-stable.
4. There is a cardinal $\theta_1 \geq |R| + \aleph_0$ such that $\mathbf{K}^{\mathcal{F}}$ is θ_1-tame for Galois-types of finite length.

Proof. Observe that in the class of flat modules, N/M is flat if and only if M is a pure submodule of N (see for example [Ste75, 11.1]). Therefore we can use the results obtained in [LRV1a, §6]. Since $\mathbf{K}^{\mathcal{F}}$ is a flat-like category in the sense of [LRV1a, 6.11] and $\mathbf{K}^{\mathcal{F}}$ is closed under pure submodules, then by [LRV1a, 6.20, 6.21] it follows that $\mathbf{K}^{\mathcal{F}}$ is an AEC with amalgamation and moreover it has a stable independence notion. Then by [LRV19, 8.16] it follows that (3) and (4) hold. That joint embedding and no maximal models hold, follows from the fact that flat modules are closed under direct sums. $\square$

Notation 9.3.2. Fix θ_0 and θ_1 as the least cardinals such that (3) and (4) of Fact 9.3.1 hold.

A natural question to ask is the value of θ_0 and θ_1. We say more about this in the next section under additional assumptions, but for now we focus on proving the main theorem of the paper (Theorem 9.3.15).

Since $\mathbf{K}^{\mathcal{F}}$ has joint embedding, amalgamation and no maximal models, from Fact 9.3.1 and Fact 9.2.7, it follows that $\mathbf{K}^{\mathcal{F}}$ has a (λ, α)-limit model if $\lambda^{\theta_0} = \lambda$ and $\alpha < \lambda^+$ is a limit ordinal. As hinted by previous results [Ch. 5], [Ch. 7], limit models with chains of long cofinality are easier to understand than limit models with chains of small cofinality so we study these first.

Before we characterize these limit models, we need to carefully work with some of the ideas of [GuHe06] and [GuHe07]. Recall the following definition.

Definition 9.3.3 ([GuHe06, Def. 1])**.** Let I be a directed system, $(n_i)_{i\in I} \in \mathbb{N}^I$, $\bar{A} = (A_{ij})_{i\leq j}$ with A_{ij} a $n_i \times n_j$ matrix with coefficients in R and $(\bar{x}_i)_{i\in I}$ with $\bar{x}_i$ an n_i-tuple of variables. Given M a left R-module and $\mathbf{b} = (\mathbf{b}_{ij})_{i\leq j} \in \Pi_{i\leq j} M^{n_i}$, we associate the system:

$$\Omega^{\bar{A}}_{\mathbf{b}}(\bar{x}_i)_{i\in I} := \{\bar{x}_i - A_{ij}\bar{x}_j = \mathbf{b}_{ij}\}_{i\leq j}.$$

We call a system of linear equations *divisible of size* λ if it is of the form $\Omega^{\bar{A}}_{\mathbf{b}}(\bar{x}_i)_{i\in I}$, for every $i \leq j \leq k$ it holds that $A_{jk}A_{ij} = A_{ik}$, for every $i \in I$ it holds that $A_{ii} = \mathrm{id}$ and $|I| = \lambda$.

The next assertion is a minor improvement of [GuHe06, Cor. 3].

Fact 9.3.4. A left R-module M is cotorsion if and only if every finitely solvable divisible system of linear equations in M of size at most $|R| + \aleph_0$ is solvable in M.

Proof. The forward direction is [GuHe06, Cor. 3]. For the backward direction, recall that to show that M is cotorsion it is enough to show, by [BEE01, Prop. 2], that $Ext^1(F, M) = 0$ for every flat module F of cardinality at most $|R| + \aleph_0$. Then remember that in Lazard's Theorem the index set I to get a flat module F as a direct limit of finitely generated free R-modules is contained in $\{(J, N) : J \subseteq_{\mathrm{fin}} F \times \mathbb{Z}$ and $N \leq R^{(F\times\mathbb{Z})}$ finitely generated$\}$ (see for example [Osb00, 8.16]). This set has size at most $|R| + \aleph_0$ if $|F| \leq |R| + \aleph_0$. Then by repeating the argument given in [GuHe06, p. 3,4] one can obtain the result. $\square$

With this we are able to characterize limit models of big cofinality.

Theorem 9.3.5. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, α)-limit model in $\mathbf{K}^{\mathcal{F}}$ and $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$, then M is a cotorsion module.*

Proof. Fix $\{M_\beta : \beta < \alpha\}$ a witness to the fact that M is a (λ, α)-limit model. By Fact 9.3.4 it is enough to show that every finitely solvable divisible system of linear equations in M of size at most $|R| + \aleph_0$ is solvable in M. Let $\Omega^{\bar{A}}_{\mathbf{b}}(\bar{x}_i)_{i\in I}$ be a divisible system of linear equations satisfying these hypotheses.

Consider $C(M)$ the cotorsion envelope of M, observe that $\Omega^{\bar{A}}_{\mathbf{b}}(\bar{x}_i)_{i\in I}$ is finitely solvable in $C(M)$ because $M \leq_p C(M)$. Since $C(M)$ is cotorsion, by Fact 9.3.4 $\Omega^{\bar{A}}_{\mathbf{b}}(\bar{x}_i)_{i\in I}$ is solvable in $C(M)$. Let $(\bar{c}_i)_{i\in I} \in \Pi_i C(M)^{n_i}$ be a solution.

Since $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$ and $|I| \leq |R| + \aleph_0$, there is an ordinal $\beta < \alpha$ such that $\{\mathbf{b}_{ij} : i \leq j\} \subseteq M_\beta$. Observe that $C(M) \in \mathbf{K}^{\mathcal{F}}$ and $M_\beta \leq_p C(M)$ by Fact 9.2.17. Then applying the downward Löwenheim-Skolem-Tarski axiom to $M_\beta \cup \{\bar{c}_i : i \in I\}$ in $C(M)$ we obtain $M^* \in \mathbf{K}^{\mathcal{F}}_\lambda$ such that $M_\beta \leq_p M^*$ and $\{\bar{c}_i : i \in I\} \subseteq M^*$. Then there is $f : M^* \underset{M_\beta}{\longrightarrow} M_{\beta+1}$, because $M_{\beta+1}$ is universal over M_β. Since $\{\mathbf{b}_{ij} : i \leq j\}$ is fixed by the choice of M_β, it is easy to see that $\{f(\bar{c}_i) : i \in I\} \subseteq M_{\beta+1} \leq_p M$ is a solution to $\Omega^{\bar{A}}_{\mathbf{b}}(\bar{x}_i)_{i\in I}$ in M. Therefore, M is cotorsion. $\square$

Remark 9.3.6. The reader might wonder if the limit model above is pure-injective instead of just cotorsion. This is not the case as the class of flat modules is not necessarily closed under pure-injective envelopes. We will study this with more detail in the next section.

Since limit models are universal models by Fact 9.2.12, the following follows from Fact 9.2.15.

Corollary 9.3.7. *Let λ be a cardinal. If M, N are λ-limit models in $\mathbf{K}^{\mathcal{F}}$ and cotorsion modules, then M and N are isomorphic.*

Putting together the last two assertions we obtain.

Corollary 9.3.8. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If $M \in \mathbf{K}^{\mathcal{F}}$ is a (λ, α)-limit model and $N \in \mathbf{K}^{\mathcal{F}}$ is a (λ, β)-limit model such that $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) \geq (|R| + \aleph_0)^+$, then M and N are isomorphic.*

Remark 9.3.9. Conjecture 2 of [BoVan] asserts that for an AEC $\mathbf{K}$ and $\lambda \geq \mathrm{LS}(\mathbf{K})$ a regular cardinal such that $\mathbf{K}$ is λ-stable, the regular ordinals α less than λ^+ such that the (λ, α)-limit model is isomorphic to the (λ, λ)-limit model is an end segment of regular cardinals. Observe that the above corollary shows that the conjecture is true for $\mathbf{K}^{\mathcal{F}}$ if R is a countable ring.

Recall that two models are elementarily equivalent if they satisfy the same first-order sentences. Surprisingly, one can still obtain that every two limit models in $\mathbf{K}^{\mathcal{F}}$ are elementarily equivalent. The proof is basically the same as that of [Ch. 7, 4.3] so we omit it.

Lemma 9.3.10. *If M, N are limit models in $\mathbf{K}^{\mathcal{F}}$, then M and N are elementarily equivalent.*

We do not think that the previous result is as fundamental as the same result for classes axiomatizable in first-order logic, see [Ch. 8, §4.1], but anyhow this will be useful when characterizing left perfect rings.

The next step will be to characterize limit models with chains of countable cofinality. In order to do that we will need the following remark.

Remark 9.3.11. If M, N are flat modules, M is a cotorsion module and $M \leq_p N$, then M is a direct summand of N. This follows from the fact that N/M is a flat module and the definition of cotorsion module.

Using the above remark together with the fact that flat modules are closed under pure submodules and that cotorsion modules are closed under finite direct sums, we can construct universal extensions and characterize limit models of countable cofinality as in [Ch. 7, 4.8, 4.9].

Lemma 9.3.12. *Let λ be a cardinal. If $M \in \mathbf{K}^{\mathcal{F}}_{\lambda}$ is cotorsion and $U \in \mathbf{K}^{\mathcal{F}}_{\lambda}$ is a universal model in $\mathbf{K}^{\mathcal{F}}_{\lambda}$, then $M \oplus U$ is universal over M.*

Proof. Let $N \in \mathbf{K}^{\mathcal{F}}_{\lambda}$ with $M \leq_p N$. By Remark 9.3.11 there is an M' such that $N = M \oplus M'$. Since $\mathbf{K}^{\mathcal{F}}$ is closed under pure submodules $M' \in \mathbf{K}^{\mathcal{F}}_{\leq\lambda}$ and by universality of U there is $f : M' \to U$. Then observe that $g : M \oplus M' \to M \oplus U$ given by $g(m + m') = m + f(m')$ is as required. □

Lemma 9.3.13. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If $M \in \mathbf{K}^{\mathcal{F}}$ is a (λ, ω)-limit model and $N \in \mathbf{K}^{\mathcal{F}}$ is a $(\lambda, (|R| + \aleph_0)^+)$-limit model, then M and $N^{(\aleph_0)}$ are isomorphic.*

Proof. Let N be a $(\lambda, (|R| + \aleph_0)^+)$-limit model. By Theorem 9.3.5 N is a cotorsion module. Then using the above lemma $\{N^i : 0 < i < \omega\}$ is a witness to the fact that $N^{(\aleph_0)}$ is a (λ, ω)-limit model. Therefore, M is isomorphic to $N^{(\aleph_0)}$. □

Let us recall the following results from [ŠaŠt20]. They extended to uncountable rings the results of [GuHe07].

Fact 9.3.14. Let R be a ring.

1. If N is Σ-cotorsion and $M \leq_p N$, then M is Σ-cotorsion.
2. If N is Σ-cotorsion and M is elementarily equivalent to N, then M is Σ-cotorsion.
3. ([ŠaŠt20, 3.8]) M is Σ-cotorsion if and only if $M^{(|R|+\aleph_0)}$ is a cotorsion module.

Proof. (1) and (2) follow from [ŠaŠt20, 3.3] and using that the definable subcategory generated by a module is closed under pure submodules and elementarily equivalent modules. □

The next theorem is the main theorem of the paper.

Theorem 9.3.15. *For a ring R the following are equivalent.*

1. *R is left perfect.*
2. *The class of flat left R-modules with pure embeddings is superstable.*
3. *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that the class of flat left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*
4. *Every limit model in the class of flat left R-modules with pure embeddings is Σ-cotorsion.*

Proof. (1) $\Rightarrow$ (2) By Fact 9.3.1.(3) there is a $\theta_0 \geq |R| + \aleph_0$ such that $\mathbf{K}^{\mathcal{F}}$ is λ-stable if $\lambda^{\theta_0} = \lambda$. Let λ_0 be the least λ such that $\mathbf{K}^{\mathcal{F}}$ is λ_0-stable, we claim that for every $\lambda \geq \lambda_0$, $\mathbf{K}^{\mathcal{F}}$ has uniqueness of limit models of size λ.

By Fact 9.2.20 every flat module is a cotorsion module. Then by Corollary 9.3.7 there is at most one λ-limit model for each λ up to isomorphisms. To finish the proof, we show by induction that for every $\lambda \geq \lambda_0$, $\mathbf{K}^{\mathcal{F}}$ is λ-stable.

The base step follows from the choice of λ_0, so we do the induction step.

Suppose λ is an infinite cardinal and that $\mathbf{K}^{\mathcal{F}}$ is μ-stable for every $\mu \in [\lambda_0, \lambda)$. Let $\mathrm{cf}(\lambda) = \kappa$ and $\{\lambda_i : i < \kappa\}$ be a continuous increasing sequence of cardinals such that $\lambda_i < \lambda$ for each $i < \kappa$ and $sup_{i<\kappa}\lambda_i = \lambda^-$.[5] Using the hypothesis that $\mathbf{K}^{\mathcal{F}}$ is μ-stable for every $\mu \in [\lambda_0, \lambda)$, one can build $\{M_i : i < \kappa\}$ strictly increasing and continuous chain such that:

1. M_{i+1} is $\|M_{i+1}\|$-universal over M_i.
2. $M_i \in \mathbf{K}^{\mathcal{F}}_{\lambda_i}$.

Let $M = \bigcup_{i<\kappa} M_i$. By construction M is universal in $\mathbf{K}^{\mathcal{F}}_{\lambda}$.[6] Since R is left perfect, M is a cotorsion module. Then using Lemma 9.3.12, as in Lemma 9.3.13, one can show that $\{M^i : 0 < i < \omega\}$ witnesses that $M^{(\aleph_0)}$ is a (λ, ω)-limit model in $\mathbf{K}^{\mathcal{F}}$. Hence $\mathbf{K}^{\mathcal{F}}$ is λ-stable by Fact 9.2.7.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) We show that if N is the $(\lambda, (|R| + \aleph_0)^+)$-limit model, then N is Σ-cotorsion. This is enough by Lemma 9.3.10 and Fact 9.3.14.(2).

Consider $\{N^{(\gamma)} : 0 < \gamma \leq |R| + \aleph_0\} \subseteq \mathbf{K}^{\mathcal{F}}_{\lambda}$, we show by induction on $0 < \gamma \leq |R| + \aleph_0$ that:

1. $N^{(\gamma)}$ is cotorsion.
2. $N^{(\gamma+1)}$ is universal over $N^{(\gamma)}$.

Before we do the proof, observe that this is enough since by taking $\gamma = |R| + \aleph_0$ we have that $N^{(|R|+\aleph_0)}$ is cotorsion. Then by Fact 9.3.14 N is Σ-cotorsion.

<u>Base:</u> N is cotorsion by Theorem 9.3.5, so (1) holds. Moreover, $N \oplus N$ is universal over N by Lemma 9.3.12.

<u>Induction step:</u> If $\gamma = \beta + 1$, then $N^{(\beta+1)}$ is cotorsion because $N^{(\beta)}$ is cotorsion by induction hypothesis, N is cotorsion and cotorsion modules are closed under finite direct sums. As for (2), this follows from Lemma 9.3.12.

If γ is a limit ordinal, then consider $\{N^{(\beta)} : 0 < \beta < \gamma\}$. It is clear that it is an increasing and continuous chain in $\mathbf{K}^{\mathcal{F}}_{\lambda}$ such that $\bigcup_{\beta<\gamma} N^{(\beta)} = N^{(\gamma)}$. Moreover,

[5] For θ a cardinal, we define $\theta^- = \mu$ if $\theta = \mu^+$ and $\theta^- = \theta$ otherwise.

[6] A similar construction is presented in [Ch. 7, 3.18]

by induction hypothesis $N^{(\beta+1)}$ is universal over $N^{(\beta)}$ for $\beta < \gamma$. Therefore, $\{N^{(\beta)} : 0 < \beta < \gamma\}$ witnesses that $N^{(\gamma)}$ is a (λ, γ)-limit model. Then by uniqueness of limit models of size λ, $N^{(\gamma)}$ is isomorphic to N. We know that N is cotorsion by Theorem 9.3.5, hence $N^{(\gamma)}$ is a cotorsion module. That $N^{(\gamma+1)}$ is universal over $N^{(\gamma)}$ then follows from Lemma 9.3.12.

(4) $\Rightarrow$ (1) Let $M \in \mathbf{K}^{\mathcal{F}}$, by Fact 9.2.20 it is enough to show that M is cotorsion. Let $\mu \geq \|M\| + (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\mathcal{F}}$ is μ-stable, which exists by Fact 9.3.1.(3). Then fix $P \in \mathbf{K}^{\mathcal{F}}$ a $(\mu, (|R| + \aleph_0)^+)$-limit model. Observe that by Fact 9.2.12 there is $f : M \to P$ a pure embedding. Since P is Σ-cotorsion by hypothesis and Σ-cotorsion modules are closed under pure submodules by Fact 9.3.14, we conclude that M is a cotorsion module. □

Remark 9.3.16. It was pointed out to us by Baldwin that in [Gar80, p. 159] the following is shown for right coherent rings: if R is a left perfect ring, then every projective left R-module is totally transcendental. This can be used to show (1) implies (2) of the above theorem in the particular case when the ring is right coherent. This case is even more special than the one we consider in the next section (see Hypothesis 9.4.1) since if a ring R is right coherent, then the class of flat left R-modules is first-order axiomatizable by [SaEk71, Theo. 4].

As a simple corollary we obtain a characterization of artinian rings via superstability.

Corollary 9.3.17. *For a ring R the following are equivalent.*

1. *R is right artinian.*
2. *$\mathbf{K}^{\mathcal{F}}$ is superstable and the class of right R-modules with embeddings is superstable.*

Proof. It is known that a ring is right artinian if and only if it is left perfect and right noetherian (see for example [SaEk71, Prop. 3]). Moreover, R is left perfect if and only if $\mathbf{K}^{\mathcal{F}}$ is superstable by the theorem above. And R is right noetherian if and only if the class of right R-modules with embeddings is superstable by [Ch. 8, 3.12]. □

9.4 A special case

In this section we study $\mathbf{K}^{\mathcal{F}}$ under Hypothesis 9.4.1 (see below). This allows us to characterize Galois-types, bound the values of θ_0, θ_1 and lower the bound in Theorem 9.3.15 where the tail of cardinals where uniqueness of limit models begins to $|R| + \aleph_0$.

We assume the next hypothesis throughout this section.

Hypothesis 9.4.1. The pure-injective envelope of every flat left R-module is flat.

These rings were characterized by Rothmaler in [Roth02]. Every first-order axiomatizable class of flat modules satisfies this hypothesis since M is an elementary substructure of its pure-injective envelope. Example 3.3 of [Roth02] shows that there are rings satisfying Hypothesis 9.4.1 such that $\mathbf{K}^{\mathcal{F}}$ is not first-order axiomatizable. This shows that the results in this section extend those obtained in [Ch. 7] for the class of flat modules.

One of the characterizations obtained in [Roth02] that will be useful in this section is the following.

Fact 9.4.2 ([Roth02])**.** For a ring R the following are equivalent.

1. Hypothesis 9.4.1, i.e., the pure-injective envelope of every flat left R-module is flat.

2. All flat cotorsion left R-modules are pure-injective.

Recall that ϕ is a positive primitive formula (*pp*-formula for short), if ϕ is an existentially quantified system of linear equations. For M a module, $\bar{a} \in M^{<\omega}$ and $B \subseteq M$ we define the *pp*-type of $\bar{a}$ over B in M, denoted by $pp(\bar{a}/B, M)$, to be the set of *pp*-formulas $\phi(\bar{x}, \bar{b})$ such that $\bar{b} \in B$ and M satisfies $\phi(\bar{a}, \bar{b})$. Recall the following result.

Fact 9.4.3 ([Zie84, 3.6])**.** Let M, N be pure-injective left R-modules, $A \subseteq M$ and $B \subseteq M$. If there is $f : A \to B$ a partial isomorphism[7], then there is $g : H^M(A) \cong H^N(B)$ such that g extends f.

One of the missing pieces in the previous section is that we did not characterize Galois-types. The next lemma characterizes them under Hypothesis 9.4.1. We obtain the same characterization as that of [Ch. 7, 3.14], but with a conceptually different proof. The argument of [Ch. 7, 3.14] can not be applied in this setting and vice versa.

Lemma 9.4.4. *Let $M, N_1, N_2 \in \mathbf{K}^{\mathcal{F}}$, $M \leq_p N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ and $\bar{b}_2 \in N_2^{<\omega}$. Then:*

$$\mathbf{tp}(\bar{b}_1/M; N_1) = \mathbf{tp}(\bar{b}_2/M; N_2) \text{ if and only if } \mathrm{pp}(\bar{b}_1/M, N_1) = \mathrm{pp}(\bar{b}_2/M, N_2).$$

Proof. The forward direction is clear, so we only prove the backward direction.

Assume $\mathrm{pp}(\bar{b}_1/M, N_1) = \mathrm{pp}(\bar{b}_2/M, N_2)$, then by amalgamation there is $N \in \mathbf{K}^{\mathcal{F}}$ and $f : N_1 \underset{M}{\longrightarrow} N$ with $N_2 \leq_p N$. Since $\mathbf{K}^{\mathcal{F}}$ is closed under pure-injective envelopes by

[7] f is a bijection between A and B and f preserves *pp*-formulas.

Hypothesis 9.4.1, $PE(N) \in \mathbf{K}^{\mathcal{F}}$. Moreover, $N \leq_p PE(N)$ so $\text{pp}(f(\bar{b}_1)/M, PE(N)) = \text{pp}(\bar{b}_2/M, PE(N))$. Then by Fact 9.4.3 there is

$$g : H^{PE(N)}(M \cup \{f(\bar{b}_1)\}) \cong_M H^{PE(N)}(M \cup \{\bar{b}_2\})$$

with $g \circ f(\bar{b}_1) = \bar{b}_2$.

Since flat modules are closed under pure submodules, we have that $H^{PE(N)}(M \cup \{f(\bar{b}_1)\})$ and $H^{PE(N)}(M \cup \{\bar{b}_2\})$ are flat. Then applying amalgamation a couple of times we get the desired result. □

As a corollary we obtain that $\theta_0 = \aleph_0$, this improves the results of [LRV1a, §6] (Fact 9.3.1.(4)) for classes of flat modules with pure embeddings under Hypothesis 9.4.1.

Corollary 9.4.5. $\mathbf{K}^{\mathcal{F}}$ *is* $(< \aleph_0)$*-tame.*

As in [Ch. 7, 3.16-3.19] one can obtain the following results.

Lemma 9.4.6.

1. *If* $\lambda^{|R|+\aleph_0} = \lambda$*, then* $\mathbf{K}^{\mathcal{F}}$ *is* λ*-stable.*
2. *If* $\lambda^{|R|+\aleph_0} = \lambda$ *or* $\forall \mu < \lambda(\mu^{|R|+\aleph_0} < \lambda)$*, then* $\mathbf{K}^{\mathcal{F}}_\lambda$ *has a universal model*

Proof sketch.

1. Let $M \in \mathbf{K}^{\mathcal{F}}_\lambda$ and $\{\mathbf{tp}(a_i/M; N) : i < \alpha\}$ be an enumeration without repetitions of $\mathbf{gS}(M)$. Then define $\Phi : gS(M) \to S^{Th(N)}_{pp}(M)$ by $\phi(\mathbf{tp}(a_i/M; N)) = pp(a_i/M, N)$. Using that $\lambda^{|R|+\aleph_0} = \lambda$ and pp-quantifier elimination the result follows.
2. If $\lambda^{|R|+\aleph_0} = \lambda$, then there are limit models of cardinality λ and limit models are universal models. If $\forall \mu < \lambda(\mu^{|R|+\aleph_0} < \lambda)$, the argument is similar to the induction step of (1) implies (2) of Theorem 9.3.15.

□

Remark 9.4.7. Recall that [Sh820, 1.2] asserts that there is a universal group of size λ in the class of torsion-free abelian groups with pure embeddings if $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$. Observe that the class of torsion-free abelian groups is the class of flat $\mathbb{Z}$-modules and it satisfies Hypothesis 9.4.1. Therefore, the above lemma generalizes [Sh820, 1.2] to classes of flat modules not axiomatizable in first-order logic.

Remark 9.4.8. Observe that the above lemma bounds θ_1 by $|R| + \aleph_0$.

In this case we get that *long* limit models are not only cotorsion modules, but they are pure-injective modules.

Lemma 9.4.9. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, α)-limit model in $\mathbf{K}^{\mathcal{F}}$ and $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$, then M is pure-injective.*

Proof. By Theorem 9.3.5 M is a cotorsion module. Then by Hypothesis 9.4.1 and Fact 9.4.2 it follows that M is pure-injective. □

It is not a coincidence that we had to use Hypothesis 9.4.1 to obtain the above result. The next result shows that both notions are equivalent.

Theorem 9.4.10. *For a ring R the following are equivalent.*

1. *Every (λ, α)-limit model in $\mathbf{K}^{\mathcal{F}}$ with $\lambda \geq (|R| + \aleph_0)^+$ and $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$ is pure-injective.*

2. *Hypothesis 9.4.1, i.e., the pure-injective envelope of every flat left R-module is flat.*

Proof. The backward direction is Lemma 9.4.9, so we show the forward direction. Let $M \in \mathbf{K}^{\mathcal{F}}$. Pick $\lambda \geq \|M\| + (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\mathcal{F}}$ is λ-stable, this is possible by Fact 9.3.1.(3). Then by Fact 9.2.7 there is N a $(\lambda, (|R| + \aleph_0)^+)$-limit model. From the assumption we have that N is pure-injective and since there is $f : M \to N$ a pure embedding, it follows that $PE(M) \cong PE(f[M]) \leq_p N$. Since $\mathbf{K}^{\mathcal{F}}$ is closed under pure submodules, we conclude that $PE(M) \in \mathbf{K}^{\mathcal{F}}$. □

Remark 9.4.11. Since Hypothesis 9.4.1 is one of the equivalent assertions of the main theorem of [Roth02, 2.3], the above theorem gives a new characterization of the rings studied in [Roth02].

To finish this section we show that under Hypothesis 9.4.1, one can lower the bound where the tail of uniqueness of limit cardinals begins to $|R| + \aleph_0$.

Theorem 9.4.12. *For a ring R satisfying Hypothesis 9.4.1 the following are equivalent.*

1. *R is left perfect.*

2. *The class of flat left R-modules with pure embeddings is superstable.*

3. *There is a $\lambda \geq (|R| + \aleph_0)^+$ such that the class of flat left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*

4. *Every limit model in the class of flat left R-modules with pure embeddings is Σ-pure-injective.*

5. *For every $\lambda \geq |R| + \aleph_0$, the class of flat left R-modules with pure embeddings has uniqueness of limit models of cardinality λ.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Follow from Theorem 9.3.15 and Fact 9.4.2.

(5) $\Rightarrow$ (2) Clear.

(1) $\Rightarrow$ (5) Since every limit model is a cotorsion module. Then by Corollary 9.3.7 there is at most one λ-limit model for each λ up to isomorphisms. Hence to finish the proof, it is enough to show that for every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\mathcal{F}}$ is λ-stable.

Let $\lambda \geq |R| + \aleph_0$ and $M \in \mathbf{K}^{\mathcal{F}}_{\lambda}$. Let $\{\mathbf{tp}(a_i/M; N) : i < \alpha\}$ be an enumeration without repetitions of $\mathbf{gS}(M)$. We can assume that they are all realized in a fixed N by amalgamation. Now, consider $\Phi : gS(M) \to S^{Th(N)}_{pp}(M)$ given by $\Phi(\mathbf{tp}(a_i/M; N)) = pp(a_i/M, N)$. By Lemma 9.4.4 it follows that Φ is a well-defined injective function. Since N is Σ-pure-injective by (1) and Hypothesis 9.4.1, $Th(N)$ is totally transcendental (see for example [Pre88, 3.2]). In particular, since complete theories of modules have *pp*-quantifier elimination we can conclude that $|S^{Th(N)}_{pp}(M)| = |S^{Th(N)}(M)| \leq \lambda$. Therefore, $|\mathbf{gS}(M)| \leq \lambda$.

$\square$

Remark 9.4.13. Recall that $\mathbb{Z}$ is not a perfect ring. Then by condition five of the above theorem we have that the class of torsion-free abelian groups with pure embeddings does not have uniqueness of limit models in any uncountable cardinal. This was shown in [Ch. 5, 4.26] using AEC methods and in [Ch. 7, 4.15] using group theoretic methods. The results of those papers show more in the case of torsion-free groups as a group theoretic description of limit models is provided and it is shown that the class does not have uniqueness of limit models in $\aleph_0$.

Chapter 10

A note on torsion modules with pure embeddings

This chapter is based on [Ch. 10].

Abstract

We study Martsinkovsky-Russell torsion modules [MaRu20] with pure embeddings as an abstract elementary class. We give a model-theoretic characterization of the pure-injective and the Σ-pure-injective modules relative to the class of torsion modules assuming that the ring is right semihereditary. Our characterization of relative Σ-pure-injective modules extends the classical charactetization of [GrJe76] and [Zim77, 3.6].

We study the limit models of the class and determine when the class is superstable assuming that the ring is right semihereditary. As a corollary, we show that the class of torsion abelian groups with pure embeddings is strictly stable, i.e., stable not superstable.

10.1 Introduction

Martsinkovsky-Russell torsion modules were introduced in [MaRu20] as a natural generalization of torsion modules to rings that are not necessarily commutative domains (Definition 10.2.3). We will denote them by $\mathfrak{s}$-*torsion modules* throughout this paper. For a commutative domain, they are precisely the R-torsion modules, i.e., those modules such that every element of the module can be annihilate by a non-zero element of the ring.

For most rings the class of $\mathfrak{s}$-torsion modules is not first-order axiomatizable in the language of modules. For example, it is folklore that the class of torsion abelian groups is not first-order axiomatizable. For this reason, we use non-elementary model-theoretic methods to analyse the class. More precisely, we will study the class of $\mathfrak{s}$-torsion modules with pure embedding as an abstract elementary class (AEC for short).

An AEC $\mathbf{K}$ is a pair $(K, \leq_{\mathbf{K}})$ where K is a class of structures and $\leq_{\mathbf{K}}$ is a partial order on K. Additionally, the partial order on K extends the substructure relation, $\mathbf{K}$ is closed under unions of chains, and every set can be closed to a small structure in the class. The class of $\mathfrak{s}$-torsion modules with pure embedding is an abstract elementary class with amalgamation, joint embedding, and no maximal models. Moreover, it was shown in [Ch. 11, 4.16] that the class is stable. In this paper, assuming that the ring is right semihereditary, we study its class of limit models and use them to determine when the class is superstable. Recall that a *limit model* is a universal model with some level of homogeneity (Definition 10.2.9) and an AEC is *superstable* if there is a unique limit model up to isomorphims on a tail of cardinals (Definition 10.2.12).[1]

A difficulty when trying to understand the class of $\mathfrak{s}$-torsion modules is that the class might not be closed under pure-injective envelopes, see [Ch. 6, 3.1] for the case of torsion abelian groups. Therefore, we begin by developing relative notions of pure-injectivity and Σ-pure-injectivity. The following result extends the classical result of [GrJe76] and [Zim77, 3.6] where they characterize Σ-pure-injective modules (see Remark 10.3.21).

Lemma 10.3.19. *Assume R is right semihereditary and M is $\mathfrak{s}$-torsion. M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective if and only if M has the low-pp descending chain condition.*

The study of limit models for the class of $\mathfrak{s}$-torsion modules and the characterization of superstability we obtain parallels that of previous results, [Ch. 8], [Ch. 9] and [Ch. 11], with the added difficulty that we have to deal with relative pure-injective modules instead of with pure-injective or cotorsion modules. More precisely, we obtain the following result.

Theorem 10.4.14. *Assume R is right semihereditary and R_R is not absolutely*

[1] A detailed account of the development of the notion of superstability in AECs can be consulted in the introductions of [GrVas17] and [Ch. 8].

pure. The following are equivalent.

1. *The class of $\mathfrak{s}$-torsion modules with pure embeddings is superstable.*
2. *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that the class of $\mathfrak{s}$-torsion modules with pure embeddings has uniqueness of limit models of cardinality λ.*
3. *Every limit model is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
4. *Every $\mathfrak{s}$-torsion module is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
5. *Every $\mathfrak{s}$-torsion module is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
6. *For every $\lambda \geq |R| + \aleph_0$, the class of $\mathfrak{s}$-torsion modules with pure embeddings has uniqueness of limit models of cardinality λ.*
7. *For every $\lambda \geq |R| + \aleph_0$, the class of $\mathfrak{s}$-torsion modules with pure embeddings is λ-stable.*

An important question that is left open is to determine if there is a ring satisfying any of the equivalent conditions of the above theorem (see Question 10.4.16 and Question 10.4.18). Nevertheless, the theorem is important as it allows us to show that certain classes are not superstable.

In particular, we use our results to show that the class of torsion abelian groups with pure embeddings is strictly stable, i.e., stable not superstable. Determining if the class is superstable was the original objective of this paper.

Theorem 10.5.6. *The class of torsion abelian groups with pure embeddings is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$. Hence, it is strictly stable.*

This paper is part of a program to understand AECs of modules: [Ch. 5], [Ch. 7], [Ch. 8], [Ch. 9], [Ch. 6], [Ch. 11]. Other papers that have studied AECs of modules include: [BCG+], [BET07], [ŠaTr12], [Sh820], [Bon20, §6][LRV1a, §6], [LRV2, §3].

The paper is divided into five sections. Section 2 has the preliminaries. Section 3 has new characterizations of relative pure-injective and Σ-pure-injective modules. Section 4 analyses the class of $\mathfrak{s}$-torsion modules with pure embeddings as an abstract elementary class. Section 5 shows how to use the previous results to show that the class of torsion abelian groups with pure embeddings is strictly stable.

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10.2 Preliminaries

In this section we briefly present the basic notions of module theory and abstract elementary classes that we will use in this paper. The module theoretic preliminaries include the definition of the class of $\mathfrak{s}$-torsion modules and assert some of its basic properties.

10.2.1 Module theory

All rings considered in this paper are associative with unity. We write ${}_RM$ to specify that M is a left R-module and M_R to specify that M is a right R-module. If we simply write M, we assume that M is a left R-module.

Given a ring R, $L_R = \{0, +, -\} \cup \{r\cdot : r \in R\}$ is the language of left R-modules. Recall that ϕ is a *positive primitive formula*, *pp*-formula for short, if ϕ is an existentially quantified system of linear equations. M is a *pure submodule* of N if *pp*-formulas are preserved between M and N and we denote it by $M \leq_p N$. The next family of *pp*-formulas was introduced in [Roth2].

Definition 10.2.1. A *pp*-formula $\psi(x)$ is *low* if and only if $\psi[{}_RR] = 0$.

Remark 10.2.2. It is easy to show that if $\psi_1(x), \psi_2(x)$ are low formulas and $r \in R$, then $\psi_1 + \psi_2(x) := \exists y \exists z(\psi_1(y) \wedge \psi_2(z) \wedge x = y + z)$ and $r\psi_1(x) := \exists y(\psi(y) \wedge x = ry)$ are low formulas.

Given $\mathbf{b} \in M^{<\omega}$ and $A \subseteq M$, the *pp*-type of $\mathbf{b}$ over A in M, denoted by $pp(\mathbf{b}/A, M)$, is the set of all *pp*-formulas that hold for $\mathbf{b}$ in M with parameters in A.

As mentioned in the introduction, in this paper we will study the class of $\mathfrak{s}$-torsion modules. These were introduced in [MaRu20] and studied from a model-theoretic perspective in [MaRo] and [Roth1]. Below we present their model-theoretic definition.

Definition 10.2.3. We say that M is an $\mathfrak{s}$*-torsion module* if and only if for every $m \in M$ there is a low formula $\psi(x)$ such that $M \vDash \psi[m]$. We denote the class of $\mathfrak{s}$-torsion modules by $K^{\mathfrak{s}\text{-Tor}}$.

Remark 10.2.4. Let R be a commutative domain. Recall that a module M is an R-torsion module if for every $m \in M$ there is an $r \neq 0 \in R$ such that $rm = 0$. Denote the class of R-torsion modules by $K^{R\text{-Tor}}$. It was shown in [MaRu20, 2.2] that $K^{\mathfrak{s}\text{-Tor}} = K^{R\text{-Tor}}$. In particular, the class of of $\mathfrak{s}$-torsion abelian groups is precisely the class of torsion abelian groups.

The following was introduced in [MaRu20, 2.1]. The description we present will appear in the forthcoming paper [MaRo].

Definition 10.2.5. For a left R-module M, let

$$\mathfrak{s}(M) = \{m \in M : M \vDash \psi[m] \text{ for some low formula } \psi\}.$$

Remark 10.2.6.

- $M \in K^{\mathfrak{s}\text{-Tor}}$ if and only if $\mathfrak{s}(M) = M$.
- $K^{\mathfrak{s}\text{-Tor}}$ is closed under pure submodules and direct sums.
- ([MaRu20, 2.19]) $\mathfrak{s}$ is a radical, i.e., for every M, N: $\mathfrak{s}(M)$ is a submodule of M, if $f : M \to N$ then $f(\mathfrak{s}(M)) \leq \mathfrak{s}(N)$, and $\mathfrak{s}(M/\mathfrak{s}(M)) = 0$.

Remark 10.2.7. It is important to notice that in general $\mathfrak{s}(\mathfrak{s}(M))$ might be different from $\mathfrak{s}(M)$, see [MaRu20, p. 69]. For this reason, for arbitrary rings it might be the case that $\mathfrak{s}(M)$ is not an $\mathfrak{s}$-torsion module.

10.2.2 Abstract elementary classes

We summarize the notions of abstract elementary classes that are used in this paper. A more detailed introduction to abstract elementary classes from an algebraic point of view is given in [Ch. 6, §2]. Abstract elementary classes were introduced by Shelah in [Sh88] to study those classes of structures that are axiomatizable in infinitary logics. An *abstract elementary class* $\mathbf{K}$ is a pair $(K, \leq_{\mathbf{K}})$ where K is a class of structures and $\leq_{\mathbf{K}}$ is a partial order on K. Additionally, the partial order on K extends the substructure relation, $\mathbf{K}$ is closed under unions of chains, and every set can be closed to a small structure in the class.

Given a model M, we write $|M|$ for its underlying set and $\|M\|$ for its cardinality. For an infinite cardinal λ, we denote by $\mathbf{K}_\lambda$ the models in $\mathbf{K}$ of cardinality λ. If we write $f : M \to N$ for $M, N \in K$, we assume that f is a $\mathbf{K}$-embedding unless specified otherwise. Recall that f is a $\mathbf{K}$-*embedding* if $f : M \cong f[M] \leq_{\mathbf{K}} N$. Finally, for $M, N \in K$ and $A \subseteq M$, we write $f : M \underset{A}{\to} N$ if f is a $\mathbf{K}$-embedding from M to N that fixes A point-wise.

We say that $\mathbf{K}$ has the *amalgamation property* if every span of models can be completed to a commutative square, $\mathbf{K}$ has the *joint embedding property* if every two models can be $\mathbf{K}$-embedded into a single model, and $\mathbf{K}$ has *no maximal models* if every model can be properly extended.

Shelah introduced a semantic notion of type in [Sh300], we call them *Galois-types* following [Gro02]. Intuitively, a Galois-type over a model M can be identified with an orbit of the group of automorphisms of the monster model which fixes M point-wise. The full definition can be consulted in [Ch. 6, 2.6]. We denote by $\mathbf{gS}(M)$ the set of all Galois-types over M and we say that $\mathbf{K}$ is $(<\aleph_0)$-*tame* if for every $M \in K$ and $p \neq q \in \mathbf{gS}(M)$, there is a finite subset A of M such that $p \restriction_A \neq q \restriction_A$.

Definition 10.2.8. $\mathbf{K}$ is λ-*stable* if $|\mathbf{gS}(M)| \leq \lambda$ for every $M \in \mathbf{K}_\lambda$. We say that $\mathbf{K}$ is *stable* if $\mathbf{K}$ is λ-stable for some $\lambda \geq \mathrm{LS}(\mathbf{K})$.

A model M is *universal over* N if and only if $\|N\| = \|M\| = \lambda$ and for every $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\rightarrow} M$.

Definition 10.2.9. Let λ be an infinite cardinal and $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that:

1. $M_0 = N$.
2. $M = \bigcup_{i<\alpha} M_i$.
3. M_{i+1} is universal over M_i for each $i < \alpha$.

M is a (λ, α)-*limit model* if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. M is a λ-*limit model* if there is a limit ordinal $\alpha < \lambda^+$ such that M is a (λ, α)-limit model.

Fact 10.2.10 ([Sh:h, §II], [GrVan06, 2.9])**.** Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation, and no maximal models. $\mathbf{K}$ is λ-stable if an only if $\mathbf{K}$ has a λ-limit model.

A model is *universal in* $\mathbf{K}_\lambda$ if it has cardinality λ and if every model in $\mathbf{K}$ of size λ can be $\mathbf{K}$-embedded into it. It is known that every λ-limit model is universal in $\mathbf{K}_\lambda$ if $\mathbf{K}$ has the joint embedding property [Ch. 5, 2.10].

We will also be interested in saturated models. Given $\lambda > \mathrm{LS}(\mathbf{K})$ we say that M is λ-*saturated* if every Galois-type over a $\mathbf{K}$-substructure of size strictly less than λ is realized in M. We have the following relation between saturated models and limit models.

Fact 10.2.11. Let $\mathbf{K}$ be an AEC with joint embedding, amalgamation, and no maximal models. If M is a (λ, α)-limit model and $\mathrm{cf}(\alpha) > \mathrm{LS}(\mathbf{K})$, then M is $\mathrm{cf}(\alpha)$-saturated.

We say that $\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has λ-limit models and if any two λ-limit models are isomorphic.

Definition 10.2.12. $\mathbf{K}$ is *superstable* if and only if $\mathbf{K}$ has uniqueness of limit models on a tail of cardinals.

Superstability was first introduced in [Sh394] and further developed in [GrVas17] and [Vas18]. There it is shown that for AECs that have amalgamation, joint embedding, no maximal models, and LS($\mathbf{K}$)-tameness, the definition above is equivalent to any other definition of superstability introduced for AECs. In particular, for a complete first-order theory T, $(Mod(T), \preceq)$ is superstable if and only if T is superstable as a first-order theory[2].

Finally, we say that $\mathbf{K}$ is *strictly stable* if $\mathbf{K}$ is stable and not superstable.

10.3 Relative pure-injective and Σ-pure-injective modules

In this section we extend classical results of pure-injective and Σ-pure-injective modules to our setting. The arguments are similar to the standard arguments, but we provide them to show that they come through in this non-first-order setting.

We assume the following hypothesis throughout the paper.

Hypothesis 10.3.1. $K^{\mathfrak{s}\text{-Tor}}$ is non-trivial, i.e., there is a non-zero module in $K^{\mathfrak{s}\text{-Tor}}$.

The following fact gives an algebraic condition that implies our hypothesis.

Fact 10.3.2 ([MaRu20, 2.32])**.** Assume $\mathfrak{s}$ is idempotent, i.e., $\mathfrak{s}(\mathfrak{s}(M)) = \mathfrak{s}(M)$. R_R is absolutely pure[3] if and only if $K^{\mathfrak{s}\text{-Tor}}$ is trivial.

Remark 10.3.3. Since we will soon assume that R is right semihereditary (Hypothesis 10.3.4) and in that case $\mathfrak{s}$ is idempotent (Proposition 10.3.7) for our purposes we could have simply assumed that R_R is not absolutely pure.

If R is a commutative domain, then $K^{\mathfrak{s}\text{-Tor}}$ is trivial if and only if R is a field.

We will assume the following hypothesis for the rest of this section.

Hypothesis 10.3.4. R is right semihereditary, i.e., finitely generated right submodules of projective modules are projective.

The only reason we assume that R is right semihereditary is because of the following fact.

Fact 10.3.5 ([MaRu20, 2.17])**.** If R is right semihereditary, then $\mathfrak{s}(M) \leq_p M$ for every left R-module M.

[2] T is superstable if and only if T is λ-stable for every $\lambda \geq 2^{|T|}$.

[3] M_R is absolutely pure if for every N_R, if $M_R \subseteq_R N_R$ then $M_R \leq_p N_R$

Remark 10.3.6. Instead of assuming that R is right semihereditary our hypothesis could have been that $\mathfrak{s}(M) \leq_p M$ for every left R-module M as this is all we will use. We decided to assume that R is right semihereditary as it is a more natural hypothesis. An interesting question is to determine if both statements are equivalent. In the case of commutative domains, it is well-known that a commutative domain is semihereditary if and only if it is a a Prüfer domain. In that case, it is known that both assertions are equivalent [Lam07, p. 117].

The next proposition follows directly from Fact 10.3.5, but we record it due to its importance.

Proposition 10.3.7.

1. *$\mathfrak{s}$ is idempotent, i.e., $\mathfrak{s}(\mathfrak{s}(M)) = \mathfrak{s}(M)$ for every left R-module M.*

2. *$\mathfrak{s}(M) \in K^{\mathfrak{s}\text{-}Tor}$ for every left R-module M.*

Recall that a module M is *pure-injective* if for every N_1, N_2, if $N_1 \leq_p N_2$ and $f : N_1 \to M$ is a homomorphism then there is a homomorphism $g : N_2 \to M$ extending f. Given a module M, *the pure-injective envelope of* M, denoted by $PE(M)$, is a pure-injective module with $M \leq_p PE(M)$ and it is minimum with respect to these properties, i.e., if N is pure-injective and there is $f : M \to N$ pure embedding then there is $g : PE(M) \to N$ pure embeddings extending f.

Let us recall the following notion and assertion.

Definition 10.3.8 ([Pre09, p. 145])**.** Let $M \leq_p N$. M is a *pure-essential submodule* of N, denoted by $M \leq^e N$, if and only if for every homomorphism $f : N \to N'$, if $f \circ i$ is a pure embedding where $i : M \hookrightarrow N$ is the inclusion, then f is a pure embedding.

Fact 10.3.9 ([Pre09, 4.3.15, 4.3.16])**.**

1. If $M \leq_p N_1 \leq_p N_2$ and $M \leq^e N_2$, then $M \leq^e N_1$.

2. $M \leq^e PE(M)$.

We now introduce a relative notion of pure injectivity and saturation.

Definition 10.3.10. Let M be an $\mathfrak{s}$-torsion module.

- M is *$K^{\mathfrak{s}\text{-}Tor}$-pure-injective* if and only if for every $N_1, N_2 \in K^{\mathfrak{s}\text{-Tor}}$, if $N_1 \leq_p N_2$ and $f : N_1 \to M$ is a homomorphism then there is a homomorphism $g : N_2 \to M$ extending f.

- M is *relatively pp-saturated in* N if and only if $M \leq_p N$ and if every *pp*-type over M which is realized in N is realized in M.

The following notion of partial homomorphism will also be useful.

Definition 10.3.11. For two modules M, N, $A \subseteq |M|$ and $B \subseteq |N|$. A function $f : A \to B$ is a *pp-(M, N)-homomorphism* if and only if for every $\bar{a} \in A$ and $\phi(\bar{x})$ *pp*-formula:

$$M \vDash \phi[\bar{a}] \Rightarrow N \vDash \phi[f(\bar{a})].$$

Observe that if $f : M \to N$ is a homomorphism then f is a *pp*-(M, N)-homomorphism as *pp*-formulas are preserved under homomorphism.

We now prove several equivalences of $K^{\mathfrak{s}\text{-Tor}}$-pure-injectivity. These extend classical characterizations of pure-injectivity, see Remark 10.3.21 and the detailed history presented right before Theorem 2.8 of [Pre88].

Lemma 10.3.12. *Assume $M \in K^{\mathfrak{s}\text{-}Tor}$. The following are equivalent.*

1. *M is relatively pp-saturated in N for every $N \in K^{\mathfrak{s}\text{-}Tor}$.*
2. *M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
3. *$M = \mathfrak{s}(PE(M))$*
4. *$M = \mathfrak{s}(N)$ for some pure-injective module N.*
5. *If $M \leq_p M^*$ and $M^* \in K^{\mathfrak{s}\text{-}Tor}$, then M is a direct summand of M^*.*

Proof. (1) $\Rightarrow$ (2): Let $N_1 \leq_p N_2$ and $f : N_1 \to M$ be a homomorphism. Let

$$\mathcal{P} = \{g : f \subseteq g,\ g \text{ is a } pp\text{-}(N_2, M)\text{-homomorphism, and } dom(g) = A\}.$$

It is clear that one can apply Zorn's lemma to $\mathcal{P}$, so let $g : A \to M$ be a maximal *pp*-(N_2, M)-homomorphism extending f. We show that $A = N_2$. Let $b \in N_2$ and $p = pp(b/A, N_2)$. Consider $q(x) = \{\phi(x, g(\bar{a})) : \phi(x, \bar{a}) \in p\}$. Clearly $q(x)$ is a $Th(M)$-type so there is M^* elementary extension of M and $m^* \in M^*$ such that $q(x) \subseteq pp(m^*/M, M^*)$.

Since $N_2 \in K^{\mathfrak{s}\text{-Tor}}$, there is ψ low such that $N_2 \vDash \psi(b)$. Hence $\psi \in q(x)$ and $m^* \in \mathfrak{s}(M^*)$. Let $q'(x) = pp(m^*/M, \mathfrak{s}(M^*))$. Then by (1), there is $m \in M$ realizing $q'(x)$ and it is clear that $g \cup \{(b, m)\}$ is a *pp*-(N_2, M)-homomorphism extending f. So by maximality of g we have that $b \in A$.

(2) $\Rightarrow$ (3): Let $N_1 = M$, $N_2 = \mathfrak{s}(PE(M))$ and $f = \mathrm{id}_M$. Then by (2) there is a $g : \mathfrak{s}(PE(M)) \to M$ extending f. Observe that by Fact 10.3.9 $M \leq^e \mathfrak{s}(PE(M))$ as $M \leq_p \mathfrak{s}(PE(M)) \leq_p PE(M)$ and $M \leq^e PE(M)$. Then it follows that g is a pure embedding, so $\mathfrak{s}(PE(M)) = M$.

(3) $\Rightarrow$ (4): Clear.

(4) $\Rightarrow$ (5): Let $M \leq_p M^*$ and $M^* \in K^{\mathfrak{s}\text{-Tor}}$. Then by (4) we have that $M = \mathfrak{s}(N) \leq_p N$ for N a pure-injective module. Since N is pure-injective, there is a homomorphism $g : M^* \to N$ with $g\restriction_M = \mathrm{id}_M$. One can check that $M^* = M \oplus ker(g)$ using that $g[M^*] \subseteq \mathfrak{s}(N) = M$.

(5) $\Rightarrow$ (1): Let $M \leq_p M^* \in K^{\mathfrak{s}\text{-Tor}}$ and $p = pp(a/M, M^*)$ for some $a \in M^*$. Then by (5) there is L such that $M^* = M \oplus L$. Let $\pi_1 : M^* = M \oplus L \to M$ be the projection onto the first coordinate. One can check that $\pi_1(a) \in M$ realizes p. □

Recall the following notion introduced in [FuSa01, XIII.§6].

Definition 10.3.13. Assume R is a Prüfer domain. M is torsion-ultracomplete if for every module N, if $M \leq_p N$ and $N/M \in K^{\mathfrak{s}\text{-Tor}}$, then M is a direct summand of N.

The next lemma together with Lemma 10.3.12 show that torsion-ultracomplete modules can be described model theoretically for Prüfer domains.

Lemma 10.3.14. *Assume R is a Prüfer domain. Let M be an $\mathfrak{s}$-torsion module. The following are equivalent.*

- *M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
- *M is torsion-ultracomplete.*

Proof. The forward direction follows from the fact that if $M, N/M \in K^{\mathfrak{s}\text{-Tor}}$, then $N \in K^{\mathfrak{s}\text{-Tor}}$. The backward direction is clear as quotient of torsion modules is torsion. □

The standard argument can be used to show the following proposition.

Proposition 10.3.15. *Assume M and N are $\mathfrak{s}$-torsion modules. If M and N are $K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then $M \oplus N$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective*

We turn our attention to Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective modules.

Definition 10.3.16. Let M be an $\mathfrak{s}$-torsion module. M is *Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective* if and only if $M^{(I)}$ is $K^{\mathfrak{s}\text{-Tor}}$-pure-injective for every index set I where $M^{(I)}$ denotes the direct sum of M indexed by I.

Let us now consider the following notion.

Definition 10.3.17. Let M be an $\mathfrak{s}$-torsion module. M has *the low-pp descending chain condition* if and only if for every $\{\phi_n(x)\}_{n\in\omega}$ such that $\phi_0(x)$ is low and $\phi_n(x)$ is a *pp*-formula for every $n \in \omega$, if $\{\phi_n[M]\}_{n\in\omega}$ is a descending chain in M, then there exists $n_0 \in \omega$ such that $\phi_{n_0}[M] = \phi_k[M]$ for every $k \geq n_0$.

We will soon see that the previous notion is actually equivalent to being Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective

The next result will be useful to characterize Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective modules.

Lemma 10.3.18. *Let M be an $\mathfrak{s}$-torsion module. If M has the low-pp descending chain condition, then M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*

Proof. Let $p = pp(b/M, N)$ for some $N \in K^{\mathfrak{s}\text{-Tor}}$. It is enough to show that there is a $\phi \in p$ such that for every $\psi \in p$ and $c \in M$, $M \vDash \phi(c) \to \psi(c)$. Such a ϕ exists by the hypothesis on M and the fact that there is a low formula $\theta \in p$ as N is an $\mathfrak{s}$-torsion module. □

The next result extends a classic characterization of Σ-pure-injectivity, see [Pre88, 2.11] and Remark 10.3.21.

Lemma 10.3.19. *Let M be an $\mathfrak{s}$-torsion module. The following are equivalent.*

1. *M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
2. *$M^{(\aleph_0)}$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
3. *M has the low-pp descending chain condition.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): Assume for the sake of contradiction that there is a family of *pp*-formulas $\{\phi_n(x)\}_{n\in\omega}$ such that $\phi_0(x)$ is low and $\phi_n[M] \supset \phi_{n+1}[M]$ for every $n \in \omega$. For each $n \in \omega$ pick $a_n \in \phi_n[M]\backslash\phi_{n+1}[M]$ and set $b_n = (a_0, a_1, \cdots, a_{n-1}, 0, \cdots) \in M^{(\aleph_0)}$.

Let $p(x) = \{\phi_n(x - b_n) : n \geq 1\} \cup \{\phi_0(x)\}$. Realize that $p(x)$ is a $Th(M^{(\aleph_0)})$-type so there is $M^* \succeq M^{(\aleph_0)}$ and $c \in M^*$ realizing $p(x)$. Observe that $c \in \mathfrak{s}(M^*)$, then by hypothesis and Lemma 10.3.12.(1) there is $d \in M^{(\aleph_0)}$ realizing $pp(c/M^{(\aleph_0)}, \mathfrak{s}(M^*))$. Then one can show that $M \vDash \phi_{m+1}[a_m]$ for some $m \in \omega$, contradicting the choice of a_m.

(3) $\Rightarrow$ (1): It is known, see for example [Pre88, 2.10], that $\phi[N^{(I)}] = \phi[N]^{(I)}$ for every *pp*-formula ϕ. Therefore, it follows from (3) that $M^{(I)}$ has the low-pp descending chain condition and $M^{(I)} \in K^{\mathfrak{s}\text{-Tor}}$ by Remark 10.2.6. Hence $M^{(I)}$ is $K^{\mathfrak{s}\text{-Tor}}$-pure-injective by Lemma 10.3.18. □

The next corollary will be very useful.

Corollary 10.3.20. *Let M and N be $\mathfrak{s}$-torsion modules .*

- *If N is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then N is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
- *If $M \leq_p N$ and N is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*

- *If M is elementarily equivalent to N and N is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then M is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*

Remark 10.3.21. Let Φ be a collection of *pp*-formulas. We say that M is a $\mathfrak{s}_\Phi$-torsion module if and only if M satisfies:

$$\forall x(\bigvee_{\phi\in\Phi}\phi),$$

and given a module M, let $\mathfrak{s}_\Phi(M) = \{m \in M : M \vDash \psi[m] \text{ for some } \psi \in \Phi\}$.

If Φ is such that for every M we have that $\mathfrak{s}_\Phi(M) \leq_p M$, then $\mathfrak{s}_\Phi$(-) is an idempotent radical and all the results we have proven so far hold for $\mathfrak{s}_\Phi$-torsion modules. In particular, by taking $\Phi := \{x = x\}$ it follows that the results in this section extend the classical characterizations of pure-injective and Σ-pure-injective modules of [Ste67], [Kie67], [War69], [GrJe76], and [Zim77]. Another example is given by taking the ring of integers and letting $\Phi = \{p^n x = 0 : n < \omega\}$ for a fixed prime number p, it is clear that the $\mathfrak{s}_\Phi$-torsion modules are precisely the abelian p-groups. As we do not know of any other interesting choice for Φ, we do not explore this idea any further.

10.4 $\mathfrak{s}$-torsion modules as an AEC

In this section we study the class of $\mathfrak{s}$-torsion modules with pure embeddings as an abstract elementary class. There are three reasons why we decided to study $\mathfrak{s}$-torsion modules with respect to pure embeddings instead than with respect to embeddings. Firstly, the class of $\mathfrak{s}$-torsion modules is defined with respect to all low *pp*-formulas and not only those low quantifier-free formulas. Secondly, the class of $\mathfrak{s}$-torsion modules is closed under pure submodules, but it is not necessarily closed under submodules. Finally, the original objective of this paper was to understand the class of torsion abelian groups with pure embeddings.

As in the previous section we are assuming *Hypothesis 10.3.1*, i.e., there is a non-zero module in $K^{\mathfrak{s}\text{-Tor}}$.

10.4.1 Basic properties

We begin by recalling some basic properties of the AEC of $\mathfrak{s}$-torsion modules with pure embeddings.

Fact 10.4.1. Let R be a ring and $\mathbf{K}^{\mathfrak{s}\text{-Tor}} = (K^{\mathfrak{s}\text{-Tor}}, \leq_p)$.

1. $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{\mathfrak{s}\text{-Tor}}) = |R| + \aleph_0$.

2. $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ has amalgamation, joint embedding, and no maximal models.

3. If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ is λ-stable.

Proof. (1) and (2) follow from [Ch. 11, 4.2.(4), 4.8] and (3) from [Ch. 11, 4.16]. □

We show next that $\mathbf{K}^{\mathfrak{s}\text{-Tor}} = (K^{\mathfrak{s}\text{-Tor}}, \leq_p)$ is nicely generated in $(R\text{-Mod}, \leq_p)$ in the sense of [Ch. 6, 4.1], i.e., if $N_1, N_2 \in K^{\mathfrak{s}\text{-Tor}}$ and $N_1, N_2 \leq_p N$ for some module N, then there is $L \in K^{\mathfrak{s}\text{-Tor}}$ such that $N_1, N_2 \leq_p L \subseteq N$.

Lemma 10.4.2. *$\mathbf{K}^{\mathfrak{s}\text{-Tor}} = (K^{\mathfrak{s}\text{-Tor}}, \leq_p)$ is nicely generated in $(R\text{-Mod}, \leq_p)$.*

Proof. If $N_1, N_2 \in K^{\mathfrak{s}\text{-Tor}}$ and $N_1, N_2 \leq_p N$ for some module N, then $L = N_1 + N_2 \in K^{\mathfrak{s}\text{-Tor}}$ and $N_1, N_2 \leq_p L \subseteq N$. □

The next result follows directly from the previous lemma, [Ch. 6, 4.5], and [Ch. 7, 3.7].

Corollary 10.4.3. *Let $N_1, N_2 \in K^{\mathfrak{s}\text{-Tor}}$, $M \leq_p N_1, N_2$, $\bar{b}_1 \in N_1^{<\omega}$ and $\bar{b}_2 \in N_2^{<\omega}$, then:*

$$\mathbf{tp}_{K^{\mathfrak{s}\text{-Tor}}}(\bar{b}_1/M; N_1) = \mathbf{tp}_{K^{\mathfrak{s}\text{-Tor}}}(\bar{b}_2/M; N_2) \text{ if and only if } pp(\bar{b}_1/M, N_1) = pp(\bar{b}_2/M, N_2).$$

In particular, $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ is $(< \aleph_0)$-tame.

The next result follows from the previous lemma and [Ch. 6, 4.6].

Corollary 10.4.4. *Let $\lambda \geq |R| + \aleph_0$. If $(R\text{-Mod}, \leq_p)$ is λ-stable, then $(K^{\mathfrak{s}\text{-Tor}}, \leq_p)$ is λ-stable.*

We can not prove anything else without extra assumptions on the ring.

10.4.2 Limit models and superstability

We characterize limit models algebraically and use them to characterize superstability.

We assume *Hypothesis 10.3.4* for the rest of this section, i.e., we assume that R is right semihereditary.

We begin by showing that saturated models are $K^{\mathfrak{s}\text{-Tor}}$-pure-injective.

Lemma 10.4.5. *If M is $(|R| + \aleph_0)^+$-saturated in $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$, then M is $K^{\mathfrak{s}\text{-Tor}}$-pure-injective.*

Proof. We use Lemma 10.3.12. Let $M \leq_p N \in K^{\mathfrak{s}\text{-Tor}}$ and $p = pp(b/M, N)$ for some $b \in N$. Given $\phi(x, \bar{y})$ a *pp*-formula, let $A_\phi = \{\bar{m} \in M : \phi(x, \bar{m}) \in p\}$ and let $\bar{m}_\phi$ be an element A_ϕ if $A_\phi \neq \emptyset$ and $\bar{m}_\phi = \bar{0}$ otherwise. Let $B = \bigcup_{\phi \in pp\text{-formula}} \bar{m}_\phi$ and M^* be the structure obtained by applying downward Löwenheim-Skolem to B in M. Observe that $\|M^*\| = |R| + \aleph_0$.

Let $q = \mathbf{tp}(b/M^*; N)$. Since M is $(|R| + \aleph_0)^+$-saturated there is $c \in M$ such that $q = \mathbf{tp}(c/M^*; M)$. Then $pp(c/M^*, M) = pp(b/M^*, N)$ by Lemma 10.4.3. Using that *pp*-formulas determine cosets [Pre88, 2.2] and the choices of the $\bar{m}_\phi$'s, it follows that c realizes p. □

It follows directly from the above result and Fact 10.2.11 that long limit models are $K^{\mathfrak{s}\text{-Tor}}$-pure-injective.

Corollary 10.4.6. *If M is a (λ, α)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and $cf(\alpha) \geq (|R| + \aleph_0)^+$, then M is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*

We would like to show that limit models with *long chains* are isomorphic. In order to do that, we obtain a couple of algebraic results regarding pure-injective modules. We begin by recalling the following fact.

Fact 10.4.7 ([GKS18, 2.5])**.** Let M, N be pure-injective modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then M and N are isomorphic.

Lemma 10.4.8. *Let M, N be any two modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then $PE(M)$ and $PE(N)$ are isomorphic.*

Proof. It is enough to show that there are $f' : PE(M) \to PE(N)$ and $g' : PE(N) \to PE(M)$ pure embeddings, as then the result follows directly from Fact 10.4.7. The existence of f' and g' follow from the minimality of $PE(M)$ and $PE(N)$ respectively. □

The next corollary follows directly from the previous result and Lemma 10.3.12.

Corollary 10.4.9. *Let M, N be $\mathfrak{s}$-torsion and $K^{\mathfrak{s}\text{-}Tor}$-pure-injective modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then M and N are isomorphic.*

Since λ-limit models are universal in $(\mathbf{K}^{\mathfrak{s}\text{-Tor}})_\lambda$, we obtain the following result.

Corollary 10.4.10. *Assume $\lambda \geq |R| + \aleph_0$. If M, N are λ-limit models in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and $K^{\mathfrak{s}\text{-}Tor}$-pure-injective, then M and N are isomorphic.*

Putting together the above assertion with Lemma 10.4.6, we obtain the promised result that limit models with long chains are isomorphic.

Corollary 10.4.11. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, α)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and N is a (λ, β)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ such that $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) \geq (|R| + \aleph_0)^+$, then M and N are isomorphic.*

Regarding limit models with lengths of countable cofinality, the standard argument can be used to obtain the following assertion by Lemma 10.3.12.(5) and Proposition 10.3.15. See for example [Ch. 7, 4.5, 4.6].

Lemma 10.4.12. *Assume $\lambda \geq (|R| + \aleph_0)^+$. If M is a (λ, ω)-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ and N is a $(\lambda, (|R| + \aleph_0)^+)$-limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$, then M and $N^{(\aleph_0)}$ are isomorphic.*

We also have that limit models are elementarily equivalent. The argument of [Ch. 7, 4.2] can be used in this setting as the class has the joint embedding property.

Lemma 10.4.13. *If M and N are limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$, then M and N are elementarily equivalent.*

This is all we need to characterize superstability in $\mathbf{K}^{\mathfrak{s}\text{-}\mathrm{Tor}}$.

Theorem 10.4.14. *Assume R is right semihereditary and R_R is not absolutely pure. The following are equivalent.*

1. *$\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is superstable.*
2. *There exists a $\lambda \geq (|R| + \aleph_0)^+$ such that $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ has uniqueness of limit models of cardinality λ.*
3. *Every limit model in $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
4. *Every $M \in K^{\mathfrak{s}\text{-}Tor}$ is Σ-$K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
5. *Every $M \in K^{\mathfrak{s}\text{-}Tor}$ is $K^{\mathfrak{s}\text{-}Tor}$-pure-injective.*
6. *For every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ has uniqueness of limit models of cardinality λ.*
7. *For every $\lambda \geq |R| + \aleph_0$, $\mathbf{K}^{\mathfrak{s}\text{-}Tor}$ is λ-stable.*

Proof. (1) ⇒ (2): Clear.

(2) ⇒ (3): The proof is similar to that of (2) to (3) of [Ch. 11, 3.15]. The reason that argument goes through in $\mathbf{K}^{\mathfrak{s}\text{-}\mathrm{Tor}}$ is because of Lemma 10.4.12, Lemma 10.4.6, Lemma 10.3.19, Lemma 10.4.13, and Corollary 10.3.20.

(3) ⇒ (4): Follows from Corollary 10.3.20 and the fact that limit models are universal.

(4) ⇒ (5): Follows from Corollary 10.3.20.

(5) ⇒ (6): By Corollary 10.4.10 for every cardinal $\lambda \geq |R| + \aleph_0$ there is at most one λ-limit model up to isomorphisms, so we only need to show existence. We show

that $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ is λ-stable for every $\lambda \geq |R| + \aleph_0$, this is enough by Fact 10.2.10. Let $\lambda \geq |R| + \aleph_0$ and $M \in \mathbf{K}^{\mathfrak{s}\text{-Tor}}_\lambda$.

Let $N \in \mathbf{K}^{\mathfrak{s}\text{-Tor}}$ and $\{a_i : i \leq \kappa\} \subseteq N$ such that $\{\mathbf{tp}(a_i/M; N) : i \leq \kappa\}$ is an enumeration without repetitions of $\mathbf{gS}(M)$. Let $\Delta := \{pp(a_i/M, N) : i \leq \kappa\}$ and observe that $|\mathbf{gS}(M)| \leq |\Delta|$ since $\Phi : \mathbf{gS}(M) \to \Delta$ given by $\Phi(\mathbf{tp}(a_i/M; N)) = pp(a_i/M, N)$ is injective by Lemma 10.4.3.

Since N is Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective by (5), it follows from Lemma 10.3.19 that N has the low-pp descending chain condition. Then it follows, as in Lemma 10.3.18, that for every $p \in \Delta$ there is $\psi_p \in p$ such that for every $\theta \in p$ and $c \in N$, $N \vDash \psi_p(c) \to \theta(c)$. Let $\Psi : \Delta \to \{\phi(x, \bar{m}) : \phi(x, \bar{y}) \text{ is a } pp\text{-formula and } \bar{m} \in M\}$ be given by $\Psi(p) = \psi_p$. It is easy to show that Ψ is injective and as $|\{\phi(x, \bar{m}) : \phi(x, \bar{y}) \text{ is a } pp\text{-formula and } \bar{m} \in M\}| = (|R| + \aleph_0)\lambda = \lambda$, we can conclude that $|\Delta| \leq \lambda$. Therefore, $|\mathbf{gS}(M)| \leq \lambda$.

(6) $\Rightarrow$ (1): Clear.

(6) $\Rightarrow$ (7): Clear.

(7) $\Rightarrow$ (4): Assume for the sake of contradiction that there is $M \in K^{\mathfrak{s}\text{-Tor}}$ which is not Σ-$K^{\mathfrak{s}\text{-Tor}}$-pure-injective. It follows from Lemma 10.3.19 that there is a set of formulas $\{\phi_n(x)\}_{n\in\omega}$ such that $\phi_0(x)$ is low and $\phi_n(x)$ is a *pp*-formula for every $n \in \omega$ such that $\phi_n[M] \supset \phi_{n+1}[M]$ for every $n \in \omega$.

Let $\lambda = \beth_\omega(|R| + \aleph_0)$. Observe that since $[\phi_n[M] : \phi_{n+1}[M]] \geq 2$, it follows that $[\phi_n[M^{(\lambda)}] : \phi_{n+1}[M^{(\lambda)}]] = \lambda$ for each $n \in \omega$ and $M^{(\lambda)} \in K^{\mathfrak{s}\text{-Tor}}$ by Remark 10.2.6. For every $n \in \omega$ pick $\{a_{n,\alpha} : \alpha < \lambda\} \subseteq M^{(\lambda)}$ a complete set of representatives of $\phi_n[M^{(\lambda)}]/\phi_{n+1}[M^{(\lambda)}]$.

Let $A = \bigcup_{n<\omega}\{a_{n,\alpha} : \alpha < \lambda\}$ and N be a structure obtained by applying downward Löwenheim-Skolem to A in $M^{(\lambda)}$. It is clear that $N \in \mathbf{K}^{\mathfrak{s}\text{-Tor}}_\lambda$. For every $\eta \in \lambda^\omega$, let $\Phi_\eta = \{\phi_{n+1}(x - \Sigma_{i=0}^n a_{i,\eta(i)}) : n < \omega\} \cup \{\phi_0(x)\}$. Φ_η is a $Th(M^{(\lambda)})$-type, so pick $M_\eta \succeq M^{(\lambda)}$ and $c_\eta \in M_\eta$ realizing Φ_η. It is clear that $c_\eta \in \mathfrak{s}(M_\eta)$ so consider $q_\eta = \mathbf{tp}(c_\eta/N; \mathfrak{s}(M_\eta))$.

Using Lemma 10.4.3 and that $\mathfrak{s}(M_\eta) \leq_p M_\eta$ for every $\eta \in \lambda^\omega$, it can be shown that if $\eta_1 \neq \eta_2 \in \lambda^\omega$ then $q_{\eta_1} \neq q_{\eta_2}$. Hence $|\mathbf{gS}(N)| \geq \lambda^{\aleph_0} > \lambda$ by the choice of λ and König's lemma. This contradicts our assumption that $\mathbf{K}^{\mathfrak{s}\text{-Tor}}$ was λ-stable. $\square$

Remark 10.4.15. The equivalence between (4) and (7) of the above theorem is a natural extension of a result of Garavaglia and Macintyre [Gar80, Theo 1].

Previous results that characterised superstability in classes of modules always corresponded to classical rings [Ch. 8], [Ch. 9], [Ch. 11]. In this case we do not know if that is the case. Moreover, we do not even know if there exists a ring such that the class of $\mathfrak{s}$-torsion modules is superstable. So we ask the following question.

Question 10.4.16. Is there a right semihereditary ring R such that R_R is not absolutely pure and R satisfies any of the equivalent conditions given in Theorem 10.4.14?

Remark 10.4.17. If there is R left pure-semsimple ring such that R is right semi-hereditary and R_R is not absolutely pure, then the above question would have a positive solution by Theorem 10.4.14.(5).[4] For this reason, we think of a ring satisfying any of the conditions given in Theorem 10.4.14 as a weak pure-semisimple ring.

As foreshadow by the remark, we think that the above question has a positive solution. Nevertheless, even if the above question has a negative solution Theorem 10.4.14 is still interesting as it can be used to show that certain classes are not superstable. An example of this is given in the next section.

A finer question would be to determine if there is a commutative domain satisfying any of the equivalent conditions given in Theorem 10.4.14. We ask the question in algebraic terms.

Question 10.4.18. Is there a Prüfer domain such that R is not a field, but every torsion module is torsion-ultracomplete?

Finally, a natural question is if any of the results presented in this section can be extended to rings that are not necessarily right semihereditary. We think it is unlikely. However, we think that if one studies the class of $\mathfrak{s}$-torsion modules with respect to other embeddings it is possible to obtain analogous results to the ones presented here for rings that are not necessarily right semihereditary.

10.5 Torsion abelian groups

In this section we apply our general results to the class of torsion abelian groups with pure embeddings. We show it is strictly stable, characterize its stability cardinals, and describe its limit models. We will denote the class of torsion abelian groups with pure embeddings by $\mathbf{K}^{Tor}$.

Remark 10.5.1. Recall that the class of $\mathfrak{s}$-torsion abelian groups is precisely the class of torsion abelian groups, i.e., those groups such that every element has finite order. Moreover, $\mathbb{Z}$ is semihereditary since it is a Prüfer domain. Therefore, we can use the results obtained in the previous section to study the class of torsion abelian groups.

[4] A few days before submitting this thesis Mike Prest informed us that the path algebra of a linear quiver A_n for every $n \geq 2$ is such an example. These rings are left and right hereditary and left and right pure-semisimple [Pre09, 4.5.18]. Moreover, they are not (right) absolutely pure as that would imply that they are semisimple rings which is not the case for these rings.

The following fact collects what is known of the class of torsion abelian groups with pure embeddings. They were first obtained in [Ch. 6, §4], but they also follow from the results of the previous section.

Fact 10.5.2. Let $\mathbf{K}^{Tor} = (K^{\mathrm{Tor}}, \leq_p)$.

- $\mathbf{K}^{Tor}$ is an AEC with $\mathrm{LS}(\mathbf{K}^{Tor}) = \aleph_0$ that has amalgamation, joint embedding, and no maximal models.
- If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{Tor}$ is λ-stable.
- $\mathbf{K}^{Tor}$ is $(< \aleph_0)$-tame.

We will use the following algebraic result to show that the class is not superstable. Given an abelian group G, we will denote its torsion part by the standard $t(G)$ instead of $\mathfrak{s}(G)$.

Remark 10.5.3 ([Fuc15, §10.3])**.** Let $B_n = \mathbb{Z}(p^n)^{(\lambda)}$ and $B = \bigoplus_n B_n$. The following holds:

$$g = (b_n)_{n\in\omega} \in t(PE(B)) \leq \Pi_n B_n \text{ if and only if the orders of } \{b_n\}_{n\in\omega} \text{ are bounded.}$$

Using the above characterization of $t(PE(B))$, it is easy to show that $\|t(PE(B))\| = \lambda^{\aleph_0}$ as $|B_n[p]| = |\{b \in B_n : pb = 0\}| = \lambda$ for every $n \in \omega$.

Lemma 10.5.4. $\mathbf{K}^{Tor}$ *is not superstable. Hence,* $\mathbf{K}^{Tor}$ *is strictly stable.*

Proof. Assume for the sake of contradiction that $\mathbf{K}^{Tor}$ is superstable. Let $\lambda = \beth_\omega$ and $B = \bigoplus_n B_n$ where $B_n = \mathbb{Z}(p^n)^{(\lambda)}$ for every $n < \omega$ as in Remark 10.5.3. Then by Theorem 10.4.14.(5) and Lemma 10.3.12.(3), it follows that $B = t(PE(B))$. This is a contradiction as $\|t(PE(B))\| = \lambda^{\aleph_0} > \lambda$ by König's lemma. $\square$

Remark 10.5.5. The previous result contrasts with the fact that the class of torsion abelian groups with embedding is superstable [Ch. 6, 4.8].

We are actually able to obtain a complete characterization of the stability cardinals.

Theorem 10.5.6. $\mathbf{K}^{Tor}$ *is* λ*-stable if and only if* $\lambda^{\aleph_0} = \lambda$.

Proof. The backward direction follows from Fact 10.5.2 so we show the forward direction. We divide the proof into two cases:

<u>Case 1:</u> $\lambda > \aleph_0$. Assume that $\mathbf{K}^{Tor}$ is λ-stable. Let M be a (λ, ω_1)-limit model and $B = \bigoplus_n B_n$ where $B_n = \mathbb{Z}(p^n)^{(\lambda)}$ for every $n < \omega$ as in Remark 10.5.3. Since M is a λ-limit model and B has size λ there is a pure embedding $f : B \to M$. Then

there is $g : PE(B) \to PE(M)$ pure embedding extending f by the minimality of $PE(B)$.

In particular, $g \restriction_{t(PE(B))}: t(PE(B)) \to t(PE(M))$ is injective. So $\|t(PE(B))\| \leq \|t(PE(M))\|$. Since $t(PE(M)) = M$ by Corollary 10.4.6 and Lemma 10.3.12 and $\|t(PE(B))\| = \lambda^{\aleph_0}$ by Remark 10.5.3, it follows that $\lambda = \lambda^{\aleph_0}$

Case 2: $\lambda = \aleph_0$. Assume for the sake of contradiction that $\mathbf{K}^{Tor}$ is ω-stable. Since $\mathbf{K}^{Tor}$ is $(< \aleph_0)$-tame by Fact 10.5.2, it follows from [BKV06, 3.6] that $\mathbf{K}^{Tor}$ is $\beth_\omega$-stable. This contradicts the previous case as $\beth_\omega^{\aleph_0} > \beth_\omega$ by König's lemma. $\square$

From the above results we can precisely describe the spectrum function for limit models.

Corollary 10.5.7. *If $\lambda^{\aleph_0} = \lambda$, then $\mathbf{K}^{Tor}$ has two non-isomorphic λ-limit models. Moreover, for every other λ, $\mathbf{K}^{Tor}$ has no λ-limit models.*

Proof. The first part follows from Corollary 10.4.11 and Theorem 10.4.14.(2). The moreover part follows from Theorem 10.5.6 and Fact 10.2.10. $\square$

We go one step further and give an algebraic description of the limit models. Recall that given $n \in \mathbb{N}$ and G an abelian group, $G[n]$ denotes the elements of order n in G and nG denotes the elements of the form ng for some g in G.

Lemma 10.5.8. *Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$ and $\alpha < \lambda^+$ be a limit ordinal. If M is a (λ, α)-limit model in $\mathbf{K}^{Tor}$, then:*

1. *If $cf(\alpha) \geq \omega_1$, then $M \cong t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)})) \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$.*
2. *If $cf(\alpha) = \omega$, then $M \cong t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)}))^{(\aleph_0)} \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$.*

Proof. (2) follows directly from (1) and Lemma 10.4.12, so we show (1). By Lemma 10.3.12 we have that $M = t(G)$ for some pure-injective group G. Since G is pure-injective, it follows from [EkFi72, §1], that:

$$G = \Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathbb{Z}_p^{(\beta_p)}) \oplus \mathbb{Q}^{(\delta)} \oplus (\bigoplus_p \mathbb{Z}(p^\infty)^{(\gamma_p)}),$$

for some specific $\alpha_{p,n}$, β_p, δ, γ_p described in [EkFi72, §1] for p a prime number and $n < \omega$.

Since

$$t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathbb{Z}_p^{(\beta_p)}) \oplus \mathbb{Q}^{(\delta)} \oplus (\bigoplus_p \mathbb{Z}(p^\infty)^{(\gamma_p)})) = t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})})) \oplus (\bigoplus_p \mathbb{Z}(p^\infty)^{(\gamma_p)})$$

we only need to determine $\alpha_{p,n}$ and γ_p for p a prime number and $n < \omega$.

By [EkFi72, 1.9] for every prime number p we have that $\gamma_p = dim_{\mathbb{F}_p}(D(G)[p])$ where $D(G)$ is the divisible part of G. Let p be a prime number. Since $\mathbb{Z}(p^\infty)^{(\lambda)}$ can be purely embedded in M, because M is universal in $\mathbf{K}^{Tor}_\lambda$, it can be purely embedded in G. Hence, $\gamma_p = \lambda$.

By [EkFi72, 1.5] for every prime number p and $n < \omega$ we have that $\alpha_{p,n} = dim_{\mathbb{F}_p}((p^{n-1}G)[p]/(p^nG)[p])$. Let p be a prime number and $n < \omega$. Since $\mathbb{Z}(p^n)^{(\lambda)}$ can be purely embedded in M, because M is universal in $\mathbf{K}^{Tor}_\lambda$, it can be purely embedded in G. Hence $\alpha_{p,n} = \lambda$.

Therefore, we can conclude that $M = t(\Pi_p PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)})) \oplus \bigoplus_p \mathbb{Z}(p^\infty)^{(\lambda)}$. $\square$

We finish by recording the following results for the class of abelian p-groups with pure embeddings. The proofs are similar to those for torsion abelian groups so we omit them. Recall that G is an abelian p-group if every element of G has order p^n for some $n \in \mathbb{N}$.

Lemma 10.5.9. *Let p be a fixed prime number and denote by $\mathbf{K}^{p\text{-}grp}$ the class of abelian p-groups with pure embeddings.*

1. *$\mathbf{K}^{p\text{-}grp}$ is strictly stable.*
2. *$\mathbf{K}^{p\text{-}grp}$ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$.*
3. *Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$ and $\alpha < \lambda^+$ be a limit ordinal. If M is a (λ, α)-limit model in $\mathbf{K}^{p\text{-}grp}$, then:*
   - *If $cf(\alpha) \geq \omega_1$, then $M \cong t(PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)})) \oplus \mathbb{Z}(p^\infty)^{(\lambda)}$.*
   - *If $cf(\alpha) = \omega$, then $M \cong t(PE(\bigoplus_n \mathbb{Z}(p^n)^{(\lambda)}))^{(\aleph_0)} \oplus \mathbb{Z}(p^\infty)^{(\lambda)}$.*

Chapter 11

Some stable non-elementary classes of modules

This chapter is based on [Ch. 11].

Abstract

Fisher [Fis75] and Baur [Bau75] showed independently in the seventies that if T is a complete first-order theory extending the theory of modules, then the class of models of T with pure embeddings is stable. In [Ch. 6, 2.12], it is asked if the same is true for any abstract elementary class $(K, \leq_p)$ such that K is a class of modules and $\leq_p$ is the pure submodule relation. In this paper we give some instances where this is true:

Theorem 11.0.1. *Assume R is an associative ring with unity. Let $(K, \leq_p)$ be an AEC such that $K \subseteq R$-Mod and K is closed under finite direct sums, then:*

- *If K is closed under direct summands and pure-injective envelopes, then $\mathbf{K}$ is λ-stable for every $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\lambda^{|R|+\aleph_0} = \lambda$.*

- *If K is closed under pure submodules and pure epimorphic images, then $\mathbf{K}$ is λ-stable for every λ such that $\lambda^{|R|+\aleph_0} = \lambda$.*

- *Assume R is Von Neumann regular. If $\mathbf{K}$ is closed under submodules and has arbitrarily large models, then $\mathbf{K}$ is λ-stable for every λ such that $\lambda^{|R|+\aleph_0} = \lambda$.*

As an application of these results we give new characterizations of noetherian rings, pure-semisimple rings, dedekind domains, and fields via superstability. Moreover, we show how these results can be used to show a link between being *good* in the stability hierarchy and being *good* in the axiomatizability hierarchy.

Another application is the existence of universal models with respect to pure embeddings in several classes of modules. Among them, the class of flat modules and the class of $\mathfrak{s}$-torsion modules.

11.1 Introduction

An abstract elementary class $\mathbf{K}$ (AEC for short) is a pair $\mathbf{K} = (K \leq_{\mathbf{K}})$ where K is a class of structures and $\leq_{\mathbf{K}}$ is a partial order on K extending the substructure relation. Additionaly, an AEC $\mathbf{K}$ is closed under directed colimits and satisfies an instance of the Downward Löwenheim-Skolem theorem. These were introduced by Shelah in [Sh88]. In this paper, we will study AECs of modules with respect to pure embeddings, i.e., classes of the form $(K, \leq_p)$ where K is a class of R-modules for a fixed ring R and $\leq_p$ is the pure submodule relation.

Fisher [Fis75] and Baur [Bau75, Theo 1] showed independently in the seventies that if T is a complete first-order theory extending the theory of modules, then $(Mod(T), \leq_p)$ is λ-stable for every λ such that $\lambda^{|R|+\aleph_0} = \lambda$. A modern proof can be consulted in [Pre88, 3.1]. After realizing that many other classes of modules with pure embeddings were stable such as abelian groups [Ch. 7, 3.16], torsion-free abelian groups [BET07, 0.3], torsion abelian groups [Ch. 6, 4.8], reduced torsion-free abelian groups [Sh820, 1.2], definable subclasses of modules [Ch. 7, 3.16] and flat R-modules [LRV1b, 4.3]. It was asked in [Ch. 6, 2.12] the following question.

Question 11.1.1. Let R be an associative ring with unity. If $(K, \leq_p)$ is an abstract elementary class such that $K \subseteq R$-Mod, is $(K, \leq_p)$ stable? Is this true if $R = \mathbb{Z}$? Under what conditions on R is this true?

In this paper, we show that many classes of modules are stable. The way we approach the problem is by showing that if the class has some nice algebraic properties then it has to be stable. This approach is new, covers most of the examples known to be stable[1] and can be used to give new examples.

Firstly, we study classes closed under direct sums, direct summands and pure-injective envelopes. These include absolutely pure modules, locally injective modules,

[1]The only set of examples that this approach does not cover is that of classes axiomatizable by complete first-order theories.

locally pure-injective modules, reduced torsion-free groups and definable subclasses of modules (see Example 11.3.2).

Theorem 11.3.7. *Assume* $\mathbf{K} = (K, \leq_p)$ *is an AEC with* $K \subseteq R$*-Mod for* R *an associative ring with unity such that* K *is closed under direct sums, directed summands and pure-injective envelopes. If* $\lambda^{|R|+\aleph_0} = \lambda$ *and* $\lambda \geq \mathrm{LS}(\mathbf{K})$*, then* $\mathbf{K}$ *is* λ*-stable.*

By characterizing the limit models in these classes (Lemma 11.3.9 and Lemma 11.3.10), we are able to obtain new characterizations of noetherian rings, pure-semisimple rings, dedekind domains and fields via superstability. An example of such a result is the next assertion which extends [Ch. 8, 4.30].

Theorem 11.3.18. *Let* R *be an associative ring with unity.* R *is left noetherian if and only if the class of absolutely pure left* R*-modules with pure embeddings is superstable.*

Moreover, the above result can be used to show a link between being *good* in the stability hierarchy and being *good* in the axiomatizability hierarchy. More precisely, if the class of absolutely pure modules with pure embeddings is superstable, then it is first-order axiomatizable (see Corollary 11.3.20).

The results for these classes of modules can also be used to partially solve Question 11.1.1 if one substitutes *stable* for *superstable*.

Lemma 11.3.25. *Let* R *be an associative ring with unity. The following are equivalent.*

1. *R is left pure-semisimple.*
2. *Every AEC* $\mathbf{K} = (K, \leq_p)$ *with* $K \subseteq R$*-Mod, such that* K *is closed under direct sums and direct summands, is superstable.*

Secondly, we study classes closed under direct sums, pure submodules and pure epimorphic images. These include flat modules, torsion abelian groups, $\mathfrak{s}$-torsion modules and any class axiomatized by an F-sentence (see Example 11.4.2).

Theorem 11.4.16. *Assume* $\mathbf{K} = (K, \leq_p)$ *is an AEC with* $K \subseteq R$*-Mod for* R *an associative ring with unity such that* K *is closed under direct sums, pure submodules and pure epimorphic images. If* $\lambda^{|R|+\aleph_0} = \lambda$*, then* $\mathbf{K}$ *is* λ*-stable.*

This result can be used to construct universal models with respect to pure embeddings. In particular, we obtain the next result which extends [Sh820, 1.2], [Ch. 9, 4.6] and [Ch. 6, 3.7].

Corollary 11.3.8. *Let* R *be an associative ring with unity. If* $\lambda^{|R|+\aleph_0} = \lambda$ *or* $\forall \mu < \lambda(\mu^{|R|+\aleph_0} < \lambda)$*, then there is a universal model in the class of flat* R*-modules with pure embeddings and in the class of* $\mathfrak{s}$*-torsion* R*-modules with pure embeddings of cardinality* λ*.*

Finally, we study classes of modules that are closed under pure submodules and that are contained in a well-understood class of modules which is closed under pure submodules and that admits intersections. The main examples for this case are subclasses of the class of torsion-free groups such as $\aleph_1$-free-groups and finitely Butler groups (see Example 11.5.2).

We use the results obtained for these classes of modules to provide a partial solution to Question 11.1.1.

Lemma 11.5.10. *Assume R is a Von Neumann regular ring. If K is closed under submodules and has arbitrarily large models, then $\mathbf{K} = (K, \leq_p)$ is λ-stable if $\lambda^{|R|+\aleph_0} = \lambda$.*

The paper is organized as follows. Section 2 presents necessary background. Section 3 studies classes closed under direct sums, direct summands and pure-injective envelopes. Section 4 studies classes closed under direct sums, pure submodules and pure epimorphic images. Section 5 studies classes of modules that are closed under pure submodules and that are contained in a well-understood class of modules which is closed under pure submodules and that admits intersection.

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11.2 Preliminaries

In this section, we recall the necessary notions from abstract elementary classes, independence relations and module theory that are used in this paper.

11.2.1 Abstract elementary classes

We briefly present the notions of abstract elementary classes that are used in this paper. These are further studied in [Bal09, §4 - 8] and [Gro2X, §2, §4.4]. An introduction from an algebraic perspective is given in [Ch. 6, §2].

Abstract elementary classes (AECs for short) were introduced by Shelah in [Sh88] to study those classes of structures axiomatized in $L_{\omega_1,\omega}(Q)$. An AEC $\mathbf{K}$ is a pair $(K, \leq_{\mathbf{K}})$ where K is a class of structures and $\leq_{\mathbf{K}}$ is a binary relation on K. Additionally, an AEC is closed under unions of chains and every set is contained in a small structure in the class. The reader can consult the definition in [Ch. 6, 2.2].

Given a model M, we will write $|M|$ for its underlying set and $\|M\|$ for its cardinality. Given λ a cardinal and $\mathbf{K}$ an AEC, we denote by $\mathbf{K}_\lambda$ the models in $\mathbf{K}$ of cardinality λ. Moreover, if we write "$f : M \to N$", we assume that f is a $\mathbf{K}$-embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. In particular, $\mathbf{K}$-embeddings are always monomorphisms.

Shelah introduced a notion of semantic type in [Sh300]. Following [Gro02], we call these semantic types Galois-types. Given $(\mathbf{b}, A, N)$, where $N \in \mathbf{K}$, $A \subseteq |N|$, and $\mathbf{b}$ is a sequence in N, the *Galois-type of* $\mathbf{b}$ *over* A *in* N, denoted by $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; N)$, is the equivalence class of $(\mathbf{b}, A, N)$ module $E^{\mathbf{K}}$; $E^{\mathbf{K}}$ is the transitive closure of $E^{\mathbf{K}}_{\text{at}}$ where $(\mathbf{b}_1, A_1, N_1)E^{\mathbf{K}}_{\text{at}}(\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, and there exist $\mathbf{K}$-embeddings $f_\ell : N_\ell \underset{A}{\to} N$ for $\ell \in \{1, 2\}$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$ and $N \in \mathbf{K}$.

If $M \in K$ and α is an ordinal, let $\mathbf{gS}^\alpha_{\mathbf{K}}(M) = \{\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/M; N) : M \leq_{\mathbf{K}} N \in \mathbf{K}$ and $\mathbf{b} \in N^\alpha\}$. When $\alpha = 1$, we write $\mathbf{gS}_{\mathbf{K}}(M)$ instead of $\mathbf{gS}^1_{\mathbf{K}}(M)$. We let $\mathbf{gS}^{<\infty}_{\mathbf{K}}(M) = \bigcup_{\alpha \in OR} \mathbf{gS}^\alpha_{\mathbf{K}}(M)$.

Since Galois-types are equivalence classes, they might not be determined by their finite restrictions. We say that $\mathbf{K}$ is fully $(< \aleph_0)$-*tame* if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{gS}^{<\infty}(M)$, there is $A \subseteq |M|$ such that $|A| < \aleph_0$ and $p \restriction_A \neq q \restriction_A$. This notion was isolated by Grossberg and VanDieren in [GrVan06].

We now introduce the main notion of this paper.

Definition 11.2.1. An AEC $\mathbf{K}$ is λ-*stable* if for any $M \in \mathbf{K}_\lambda$, $|\mathbf{gS}_{\mathbf{K}}(M)| \leq \lambda$.

Recall that a model M is *universal over* N if and only if $\|N\| = \|M\| = \lambda$ and for every $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \underset{N}{\to} M$. Let us recall the notion of limit model.

Definition 11.2.2. Let λ be an infinite cardinal and $\alpha < \lambda^+$ be a limit ordinal. M is a (λ, α)-*limit model over* N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that:

1. $M_0 = N$.
2. $M = \bigcup_{i<\alpha} M_i$.
3. M_{i+1} is universal over M_i for each $i < \alpha$.

M is a (λ, α)-limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α)-limit model over N. M is a λ-limit model if there is a limit ordinal $\alpha < \lambda^+$ such that M is a (λ, α)-limit model.

We say that $\mathbf{K}$ has *uniqueness of limit models of cardinality* λ if $\mathbf{K}$ has λ-limit models and if any two λ-limit models are isomorphic. We introduce the notion of superstability for AECs.

Definition 11.2.3. $\mathbf{K}$ is a *superstable* AEC if and only if $\mathbf{K}$ has uniqueness of limit models on a tail of cardinals.

Remark 11.2.4. In [GrVas17, 1.3] and [Vas18] was shown that for AECs that have amalgamation, joint embedding, no maximal models and are tame, the definition above is equivalent to every other definition of superstability considered in the context of AECs. In particular for a complete first-order theory T, $(Mod(T), \preceq)$ is superstable if and only if T is λ-stable for every $\lambda \geq 2^{|T|}$.

Finally, recall that a model $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is a $\mathbf{K}$-embedding $f : N \to M$. We say that $\mathbf{K}$ has a universal model of cardinality λ if there is a universal model in $\mathbf{K}_\lambda$. In [Ch. 5, 2.10], it is shown that if $\mathbf{K}$ is an AEC with the joint embedding property and M is a λ-limit model, then M is universal in $\mathbf{K}_\lambda$.

11.2.2 Independence relations

We recall the basic properties of independence relations on arbitrary categories. These were introduced and studied in detail in [LRV19].

Definition 11.2.5 ([LRV19, 3.4])**.** An independence relation on a category $\mathcal{C}$ is a set $\mathop{\smile}\limits^{|}$ of commutative squares such that for any commutative diagram:

$$\begin{tikzcd}
 & & E \\
B \arrow[r, "g_1"] \arrow[urr, bend left, "h_1"] & D \arrow[ur, "t"] & \\
A \arrow[u, "f_1"] \arrow[r, "f_2"'] & C \arrow[u, "g_2"'] \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}$$

we have that $(f_1, f_2, g_1, g_2) \in \mathop{\smile}\limits^{|}$ if and only if $(f_1, f_2, h_1, h_2) \in \mathop{\smile}\limits^{|}$.

We will be particularly interested in weakly stable independence relations. Recall that an independence relation $\mathop{\smile}\limits^{|}$ is *weakly stable* if it satisfies: symmetry [LRV19, 3.9], existence [LRV19, 3.10], uniqueness [LRV19, 3.13], and transitivity [LRV19, 3.15].

They also introduced the notion of a stable independence relation for any category $\mathcal{C}$ in [LRV19, 3.24]. As the definition is long and we will only study independence relations on AECs, we introduce the definition for AECs instead. For an AEC $\mathbf{K}$, an indepedence relation $\mathop{\smile}\limits^{|}$ is *stable* if it is weakly stable and satisfies local character [LRV19, 8.6] and the witness property [LRV19, 8.7].

11.2.3 Module Theory

We succinctly introduce the notions from module theory that are used in this paper. These are further studied in [Pre88].

All rings considered in this paper are associative with unity. A formula ϕ is a positive primitive formula (*pp*-formula for short), if ϕ is an existentially quantified finite system of linear equations. Given $\bar{b} \in M^{<\infty}$ and $M \subseteq N$, the *pp*-type of $\bar{b}$ over M in N, denoted by $\text{pp}(\bar{b}/M, N)$, is the set of *pp*-formulas with parameters in M that hold for $\bar{b}$ in N.

Given M and N R-modules, M is a *pure submodule* of N, denoted by $M \leq_p N$, if and only if M is a submodule of N and for every $\bar{a} \in M^{<\omega}$, $pp(\bar{a}/\emptyset, M) = pp(\bar{a}/\emptyset, N)$. Moreover, $f : M \to N$ is a *pure epimorphism* if f is an epimorphism and $ker(f) \leq_p M$.

Recall that a module M is *pure-injective* if for every N, if M is a pure submodule of N, then M is a direct summand of N. Given a module M, the *pure-injective envelope of* M, denoted by $PE(M)$, is a pure-injective module such that $M \leq_p PE(M)$ and it is minimum with respect to this property. Its existence follows from [Zie84, 3.6] and the fact that every module can be embedded into a pure-injective module.

The following property of pure-injective modules will be useful.

Fact 11.2.6 ([GKS18, 2.5])**.** Let M, N be pure-injective modules. If there are $f : M \to N$ a pure embedding and $g : N \to M$ a pure embedding, then M and N are isomorphic.

M is Σ-*pure-injective* if $M^{(\aleph_0)}$ is pure-injective. The next three properties of Σ-pure-injective modules will be useful.

Fact 11.2.7.

- If N is Σ-pure-injective and $M \leq_p N$, then M is Σ-pure-injective.
- If N is Σ-pure-injective and M is elementary equivalent to N, then M is Σ-pure-injective.
- ([Pre88, 3.2]) If M is Σ-pure-injective, then $(Mod(Th(M)), \leq_p)$ is λ-stable for every $\lambda \geq |Th(M)|$.

11.3 Classes closed under pure-injective envelopes

In this section we study classes closed under direct sums, direct summands and pure-injective envelopes. We show that they are always stable and we give an algebraic characterization of when they are superstable.

Hypothesis 11.3.1. Let $\mathbf{K} = (K, \leq_p)$ be an AEC with $K \subseteq R$-Mod for a fixed ring R such that:

1. K is closed under direct sums.
2. K is closed under direct summands.
3. K is closed under pure-injective envelopes, i.e., if $M \in K$, then $PE(M) \in K$.

Below we give some examples of classes of modules satisfying Hypothesis 11.3.1.

Example 11.3.2.

1. $(R\text{-AbsP}, \leq_p)$ where R-AbsP is the class of absolutely pure R-modules. A module M is absolutely pure if it is pure in every module containing it. Closure under direct sums and direct summands follows from [Pre09, 2.3.2, 2.3.5], while closure under pure-injective envelopes follows from [Pre09, 4.3.12].

2. $(R\text{-l-inj}, \leq_p)$ where R-l-inj is the class of locally injective R-modules (also called finitely injective modules). A module M is locally injective if given $\bar{a} \in M^{<\omega}$ there is an injective submodule of M containing $\bar{a}$. Closure under direct sums is clear while closure under direct summands follows from [RaRa73, 3.1]; and closure under pure-injective envelopes follows from the fact that locally injective modules are absolutely pure [RaRa73, 3.1] and [Pre09, 4.3.12].

3. $(R\text{-l-pi}, \leq_p)$ where R-l-pi is the class of locally pure-injective R-modules. A module M is locally pure-injective if given $\bar{a} \in M^{<\omega}$ there is a pure-injective pure submodule of M containing $\bar{a}$. Closure under direct sums, direct summands and pure-injective envelopes follow from [Zim02, 2.4].

4. $(\text{RTF}, \leq_p)$ where RTF is the class of reduced torsion-free abelian groups. A group G is reduced if it does not have non-trivial divisible subgroups. Closure under direct sums and direct summands are easy to check, while closure under pure-injective envelopes follows from [Fuc15, 6.4.3].

5. $(R\text{-Flat}, \leq_p)$ where R-Flat is the class of flat R-modules under the additional assumption that the pure-injective envelope of every flat modules is flat.[2] Closure under direct sums and direct summands are easy to check and we are assuming closure under pure-injective envelopes.

6. $(\chi, \leq_p)$ where χ is a definable category of modules in the sense of [Pre09, §3.4]. A class of modules is definable if it is closed under direct products, direct limits and pure submodules. Closure under pure-injective envelopes follows from [Pre09, 4.3.21].

[2]These rings were introduced in [Roth02] and this class was studied in detail in [Ch. 9, §3].

Remark 11.3.3. It is worth mentioning that none of the above examples are first-order axiomatizable with the exception of the last one.

11.3.1 Stability

We begin by showing some structural properties of the classes satisfying Hypothesis 11.3.1. The argument for the amalgamation property is due to T.G. Kucera.

Lemma 11.3.4. *If* $\mathbf{K}$ *satisfies Hypothesis 11.3.1, then* $\mathbf{K}$ *has joint embedding, amalgamation, no maximal models and* $|R| + \aleph_0 \leq \mathrm{LS}(\mathbf{K})$*.*

Proof. Joint embedding and no maximal models follow directly from closure under direct sums. So we show the amalgamation property.

Let $M \leq_p N_1, N_2$ be models of K. By minimality of the pure-injective envelope we obtain that $PE(M) \leq_p PE(N_1), PE(N_2)$ and observe that all of these models are in K by closure under pure-injective envelopes.

Let $L := PE(N_1) \oplus PE(N_2)$ which is in K by closure under direct sums. Now, as $PE(M)$ is pure-injective, there are N_1' and N_2' such that $PE(N_i) = PE(M) \oplus N_i'$ for $i \in \{1, 2\}$. Hence, $L = (PE(M) \oplus N_1') \oplus (PE(M) \oplus N_2')$. Define $f : N_1 \to L$ by $f(m + n_1) = (m, n_1, m, 0)$ for $m \in PE(M)$ and $n_1 \in N_1'$ and $g : N_2 \to L$ by $g(m + n_2) = (m, 0, m, n_2)$ for $m \in PE(M)$ and $n_2 \in N_2'$. One can show that f, g are pure embeddings such that $f \restriction_M = g \restriction_M$. $\square$

We characterize the Galois-types in term of the *pp*-types. The result is similar to [Ch. 7, 3.14], but the argument given there cannot be applied in this setting. As the argument given in [Ch. 9, 4.4] works in the more general setting of classes satisfying Hypothesis 11.3.1 we only sketch the proof.

Lemma 11.3.5. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1. Let* $M, N_1, N_2 \in \mathbf{K}$*,* $M \leq_p N_1, N_2$*,* $\bar{b}_1 \in N_1^{<\infty}$ *and* $\bar{b}_2 \in N_2^{<\infty}$*. Then:*

$$\mathbf{tp}(\bar{b}_1/M; N_1) = \mathbf{tp}(\bar{b}_2/M; N_2) \text{ if and only if } \mathrm{pp}(\bar{b}_1/M, N_1) = \mathrm{pp}(\bar{b}_2/M, N_2).$$

Proof. The forward direction is trivial so we show the backward direction. As K has the amalgamation property we may assume that $N_1 = N_2$ and since K is closed under pure-injective envelopes we may assume that $N_1 = N_2$ is pure-injective. Let $N = N_1 = N_2$. Then by [Zie84, 3.6] there is

$$f : H^N(M \cup \{\bar{b}_1\}) \cong_M H^N(M \cup \{\bar{b}_2\})$$

with $f(\bar{b}_1) = \bar{b}_2$.

As K is closed under direct summands, it follows that $H^N(M \cup \{\bar{b}_1\}), H^N(M \cup \{\bar{b}_2\}) \in K$, so applying the amalgamation property a few times the result follows. $\square$

An immediate corollary is that the classes satisfying Hypthesis 11.3.1 are tame.

Corollary 11.3.6. *If* $\mathbf{K}$ *satisfies Hypothesis 11.3.1, then* $\mathbf{K}$ *is fully* $(< \aleph_0)$*-tame.*

Our first theorem also follows from the above lemma.

Theorem 11.3.7. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1 and* $\lambda \geq \mathrm{LS}(\mathbf{K})$*. If* $\lambda^{|R|+\aleph_0} = \lambda$*, then* $\mathbf{K}$ *is* λ*-stable.*

Proof. Let $M \in \mathbf{K}_\lambda$ and $\{p_i : i < \alpha\}$ be an enumeration without repetitions of $\mathbf{gS}(M)$. Fix $N \in K$ an extension of M such that there is $\{a_i : i < \alpha\} \subseteq N$ with $p_i = \mathbf{tp}(a_i/M; N)$ for every $i < \alpha$. This can be done by amalgamation.

Let $\Phi : \mathbf{gS}(M) \to S^{Th(N)}_{pp}(M)$ be such that $\phi(\mathbf{tp}(a_i/M; N)) = pp(a_i/M, N)$. By Lemma 11.3.5 we have that Φ is a well-defined injective function, so $|\mathbf{gS}(M)| \leq |S^{Th(N)}_{pp}(M)|$. Then $|S^{Th(N)}_{pp}(M)| = |S^{Th(N)}(M)|$ by *pp*-quantifier elimination (see [Pre88, §2.4]). Hence $|\mathbf{gS}(M)| = |S^{Th(N)}(M)| \leq \lambda$ by the fact that every complete first-order theory of modules is λ-stable if $\lambda^{|R|+\aleph_0} = \lambda$ by [Pre88, 3.1]. $\square$

Then from [Ch. 7, 3.20] we can conclude the existence of universal models.

Corollary 11.3.8. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1 and* $\lambda \geq \mathrm{LS}(\mathbf{K})$*. If* $\lambda^{|R|+\aleph_0} = \lambda$ *or* $\forall \mu < \lambda(\mu^{|R|+\aleph_0} < \lambda)$*, then* $\mathbf{K}$ *has a universal model of cardinality* λ*.*

11.3.2 Limit models and superstability

Since K has joint embedding, amalgamation and no maximal models, it follows from [Sh:h, §II.1.16] that $\mathbf{K}$ has a (λ, α)-limit model if $\lambda^{|R|+\aleph_0} = \lambda$, $\lambda \geq \mathrm{LS}(\mathbf{K})$ and $\alpha < \lambda^+$ is a limit ordinal. We characterize limit models with chains of big cofinality. This extends [Ch. 7, 4.5] and [Ch. 9, 4.9] to any class satisfying Hypothesis 11.3.1.

Lemma 11.3.9. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1 and* $\lambda \geq \mathrm{LS}(\mathbf{K})^+$*. If* M *is a* (λ, α)*-limit model and* $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$*, then* M *is pure-injective.*

Proof. Fix $\{M_i : i < \alpha\}$ a witness to the fact that M is a (λ, α)-limit model. We show that every $p(x)$ M-consistent *pp*-type over $A \subseteq M$ with $|A| \leq |R| + \aleph_0$ is realized in M. This enough to show that M is pure-injective by [Pre88, 2.8].

Observe that p is a $PE(M)$-consistent *pp*-type as $M \preceq PE(M)$. Since $PE(M)$ is pure-injective, there is $a \in PE(M)$ realizing p. As $\mathrm{cf}(\alpha) \geq (|R| + \aleph_0)^+$, there is $i < \alpha$ such that $A \subseteq M_i$. Applying Downward Löwenheim-Skolem to $M_i \cup \{a\}$ in $PE(M)$ we obtain $N \in \mathbf{K}_\lambda$ with $M_i \leq_p N$ and $a \in N$. Then there is $f : N \underset{M_i}{\longrightarrow} M$ because M_{i+1} is universal over M_i. Hence $f(a) \in M$ realizes p. $\square$

Since K is closed under direct sums, the usual argument [Ch. 7, 4.9] can be use to characterize limit models of countable cofinality.

Lemma 11.3.10. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1 and* $\lambda \geq \mathrm{LS}(\mathbf{K})^+$*. If* M *is a* (λ, ω)*-limit model and* N *is a* $(\lambda, (|R| + \aleph_0)^+)$*-limit model, then* M *is isomorphic to* $N^{(\aleph_0)}$*.*

Moreover, any two limit models of $\mathbf{K}$ are elementarily equivalent. The proof is similar to that of [Ch. 7, 4.3] so we omit it.

Lemma 11.3.11. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1. If* M, N *are limit models of* $\mathbf{K}$*, then* M *and* N *are elementary equivalent.*

Therefore, it makes sense to introduce the following first-order theory:

Notation 11.3.12. For $\mathbf{K}$ satisfying Hypothesis 11.3.1, let $\tilde{M}_{\mathbf{K}}$ be the $(2^{\mathrm{LS}(\mathbf{K})}, \omega)$-limit model of $\mathbf{K}$ and $\tilde{T}_{\mathbf{K}} = Th(\tilde{M}_T)$.

In [Ch. 8, §4.1] a similar theory, called $\tilde{T}$ there, was introduced. There it was shown that there was a very close relation between the AEC $\mathbf{K}$ and $\tilde{T}_{\mathbf{K}}$. We do not think that this is the case when $\mathbf{K}$ satisfies Hypothesis 11.3.1 and is not first-order axiomatizable. We think that this is the case because there can be models of $\tilde{T}_{\mathbf{K}}$ that are not in $\mathbf{K}$. Nevertheless, stability transfers from $\tilde{T}_{\mathbf{K}}$ to $\mathbf{K}$. As the proof is similar to that of [Ch. 8, 4.9] we omit it.

Lemma 11.3.13. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1 and let* $\lambda \geq \mathrm{LS}(\mathbf{K})$*. If* $\tilde{T}_{\mathbf{K}}$ *is* λ*-stable, then* $\mathbf{K}$ *is* λ*-stable.*

Remark 11.3.14. In [Ch. 8, 4.9], it is shown that the converse is true if $\mathbf{K}$ is first-order axiomatizable. We do not think that the converse is true in this more general setting, but we do not have a counterexample.

We characterize superstability for classes satisfying Hypothesis 11.3.1. The next result extends [Ch. 8, 4.26] to classes not necessarily axiomatizable by a first-order theory and [Ch. 9, 4.12] to a different class than that of Example 3.2.(5).

Theorem 11.3.15. *Assume* $\mathbf{K}$ *satisfies Hypothesis 11.3.1. The following are equivalent.*

1. $\mathbf{K}$ *is superstable.*
2. *There is a* $\lambda \geq \mathrm{LS}(\mathbf{K})^+$ *such that* $\mathbf{K}$ *has uniqueness of limit models of cardinality* λ*.*
3. *Every limit model in* $\mathbf{K}$ *is* Σ*-pure-injective.*

4. *Every model in* $\mathbf{K}$ *is pure-injective.*

5. *For every* $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ *has uniqueness of limit models of cardinality* λ.

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Let $\lambda \geq \mathrm{LS}(\mathbf{K})^+$ such that $\mathbf{K}$ has uniqueness of limit models of size λ. Let M be a $(\lambda, (|R| + \aleph_0)^+)$-limit model in $\mathbf{K}$. It follows from Lemma 11.3.10 that $M^{(\aleph_0)}$ is the (λ, ω)-limit model. As $\mathbf{K}$ has uniqueness of limit models of size λ, we have that M is isomorphic to $M^{(\aleph_0)}$. Since M is pure-injective by Lemma 11.3.9, it follows that $M^{(\aleph_0)}$ is pure-injective. Hence M is Σ-pure-injective. Since limit models are elementarily equivalent by Lemma 11.3.11 and Σ-pure-injectivity is preserved under elementarily equivalence by Fact 11.2.7, it follows that every limit model is Σ-pure-injective.

(3) $\Rightarrow$ (4) Let $N \in \mathbf{K}$ and N' be a $(\|N\|^{|R|+\aleph_0}, \omega)$-limit model, this exist by Theorem 11.3.7. Then there is $f : N \to N'$ a pure embedding by [Ch. 5, 2.10]. Since N' is Σ-pure-injective and f is a pure embedding, it follows from Fact 11.2.7 that N is Σ-pure-injective. Hence every model in $\mathbf{K}$ is pure-injective.

(4) $\Rightarrow$ (5) Let M be a $(2^{\mathrm{LS}(\mathbf{K})}, \omega)$-limit model. By (4) and closure under direct sums we have that M is Σ-pure-injective, so $Th(M)$ is λ-stable for every $\lambda \geq |R| + \aleph_0$ by Fact 11.2.7. As $Th(M) = \tilde{T}_{\mathbf{K}}$ by definition, it follows from Lemma 11.3.13 that $\mathbf{K}$ is λ-stable for every $\lambda \geq \mathrm{LS}(\mathbf{K})$. Therefore, by [Sh:h, §II.1.16] there exist a λ-limit model for every $\lambda \geq \mathrm{LS}(\mathbf{K})$.

Regarding uniqueness, observe that given M and N λ-limit models, there are $f : M \to N$ and $g : N \to M$ pure embeddings by [Ch. 5, 2.10]. Since we have that M and N are pure-injective, it follows from Fact 11.2.6 that M and N are isomorphic.

(5) $\Rightarrow$ (1) Clear. $\square$

Remark 11.3.16. It can also be shown as in [Ch. 8, 4.26] that $\mathbf{K}$ is superstable if and only if there exists $\lambda \geq \mathrm{LS}(\mathbf{K})^+$ such that $\mathbf{K}$ has a Σ-pure-injective universal model of cardinality λ.

11.3.3 Characterizing several classes of rings

We will use the results of the preceding subsection to characterize noetherian rings, pure-semimple rings, dedekind domains and fields via superstability.

Recall that a module M is injective if it is a direct summand of every module containing it. The next result will be useful.

Fact 11.3.17 ([Pre09, 4.4.17])**.** Let R be a ring. The following are equivalent.

1. R is left noetherian.

2. The class of absolutely pure left R-modules is the same as the class of injective left R-modules.

3. Every direct sum of injective left R-modules is injective.

We begin by giving two new characterizations of noetherian rings. The equivalence between (1) and (2) extends [Ch. 8, 4.30]. Recall that R-AbsP is the class of absolutely pure R-modules and that R-l-inj is the class of locally injective R-modules, these were introduced in Example 11.3.2.

Theorem 11.3.18. *Let R be a ring. The following are equivalent.*

1. *R is left noetherian.*

2. *$(R\text{-}AbsP, \leq_p)$ is superstable.*

3. *$(R\text{-}l\text{-}inj, \leq_p)$ is superstable.*

Proof. Recall that absolutely pure modules and locally injective modules satisfy Hypothesis 11.3.1, so we can use the results from the previous subsection. More precisely, we use Theorem 11.3.15.(4) to show the equivalences.

(1) $\Rightarrow$ (2) If R is noetherian, then every absolutely pure modules is injective by Fact 11.3.17. Hence, every absolutely pure module is pure-injective. So the result follows from Theorem 11.3.15.

(2) $\Rightarrow$ (3) Every locally injective module is absolutely pure by [RaRa73, 3.1]. Then it follows that every locally injective module is pure-injective by (2). Hence, the class of locally injective R-modules is superstable.

(3) $\Rightarrow$ (1) We show that the direct sum of injective modules is injective, this is enough by Fact 11.3.17. Let $\{M_i : i \in I\}$ be a family of injective modules. As they are all locally injective, we have that $\oplus_{i\in I} M_i$ is locally injective. Moreover, as $(R\text{-l-inj}, \leq_p)$ is superstable, we have that $\oplus_{i\in I} M_i$ is also pure-injective by Theorem 11.3.15. Recall that locally injective modules are absolutely pure, so $\oplus_{i\in I} M_i$ is absolutely pure and pure-injective. Therefore, $\oplus_{i\in I} M_i$ is injective. Hence R is noetherian. □

We use the above result to study the class of injective R-modules with pure embeddings, we will denote it by $(R\text{-Inj}, \leq_p)$.

Corollary 11.3.19. *Let R be a ring. If $(R\text{-}Inj, \leq_p)$ is an AEC, then $(R\text{-}Inj, \leq_p)$ is superstable.*

Proof. If $(R\text{-Inj}, \leq_p)$ is an AEC then the direct sum of injective modules is an injective module because injective modules are closed under finite direct sums. Hence R is left noetherian. Then $(R\text{-Inj}, \leq_p)$ is superstable by Theorem 11.3.18 and using that in noetherian rings injective modules are the same as absolutely pure modules by Fact 11.3.17. □

The next corollary shows a connection between being *good* in the stability hierachy and being *good* in the axiomatizability hierachy.

Corollary 11.3.20. *Let R be a ring.*

1. *If $(R\text{-}AbsP, \leq_p)$ is superstable, then the class of absolutely pure left R-modules is first-order axiomatizable.*
2. *If $(R\text{-}l\text{-}inj, \leq_p)$ is superstable, then the class of locally injective left R-modules is first-order axiomatizable.*

Proof.

1. Since $(R\text{-AbsP}, \leq_p)$ is superstable, then by Theorem 11.3.18 R is left noetherian. Then R is left coherent, so it follows from [Pre09, 3.4.24] that absolutely pure modules are first-order axiomatizable
2. The proof is similar to that of (1), using that if R is noetherian then the class of absolutely pure modules is the same as the class of locally injective modules.

□

We turn our attention to pure-semisimple rings. A ring is *pure-semisimple* if and only if every R-module is pure-injective. These have been thoroughly studied [Cha60], [Aus74], [Aus76], [Sim77], [Z-H79], [Sim81], [Pre84], [Sim00], [Pre09, §4.5.1] and [Ch. 8]. Recall that R-l-pi is the class of locally pure-injective R-modules, these were introduced in Example 11.3.2. The equivalence between (1) and (2) of the next assertion was obtained in [Ch. 8, 4.28].

Theorem 11.3.21. *Let R be a ring. The following are equivalent.*

1. *R is left pure-semisimple.*
2. *$(R\text{-}Mod, \leq_p)$ is superstable.*
3. *$(R\text{-}l\text{-}pi, \leq_p)$ is superstable.*

Proof. Recall that R-modules and locally pure-injective R-modules satisfy Hypothesis 11.3.1. We use Theorem 11.3.15.(4) to show the equivalences. The equivalence between (1) and (2) and the direction (2) to (3) are straightforward. We show (3) to (1).

Let M be an R-module, then $PE(M)$ is locally pure-injective and $M \leq_p PE(M)$. Observe that $PE(M)^{(\aleph_0)}$ is locally pure-injective. Then $PE(M)^{(\aleph_0)}$ is pure-injective by hypothesis (3), so $PE(M)$ is Σ-pure-injective. Hence, M is pure-injective by Fact 11.2.7. Therefore, R is left pure-semisimple. □

We can obtain an analogous result to Corollary 11.3.19 by substituting the class of injective modules by that of pure-injective modules. We denote by R-pi the class of pure-injective R-modules.

Corollary 11.3.22. *Let R be a ring. If $(R\text{-}pi, \leq_p)$ is an AEC, then $(R\text{-}pi, \leq_p)$ is superstable.*

Proof. If $(R\text{-pi}, \leq_p)$ is an AEC, then every pure-injective module is Σ-pure-injective because pure-injective modules are closed under finite direct sums. Then doing an argument similar to that of the previous result, one can show that R is pure-semisimple. Thus, the class of pure-injective R-modules is the same as the class of R-modules. Therefore, $(R\text{-pi}, \leq_p)$ is superstable by Theorem 11.3.21. □

We also get a relation between being *good* in the stability hierarchy and being *good* in the axiomatizability hierarchy for locally pure-injective modules.

Corollary 11.3.23. *Let R be ring. If $(R\text{-}l\text{-}pi, \leq_p)$ is superstable, then the class of locally pure-injective left R-modules is first-order axiomatizable.*

Proof. Since $(R\text{-l-pi}, \leq_p)$ is superstable, then by Theorem 11.3.21 R is left pure-semisimple. Then the class of locally pure-injective R-modules is the same as the class of R-modules. Therefore, it is clearly first-order axiomatizable. □

Corollaries 11.3.20 and 11.3.23 may suggest that given an AEC of modules satisfying Hypothesis 11.3.1, it follows that if the class is superstable, then the class is first-order axiomatizable. This is not the case as witnessed by the next example.

Example 11.3.24. It was shown in [Ch. 9, 3.15] that $(R\text{-Flat}, \leq_p)$ is superstable if and only if R is left perfect. It is known that the class of flat left R-modules is first-order axiomatizable if and only if R is right coherent. Therefore, the ring R described in [Roth02, 3.3] is such that $(R\text{-Flat}, \leq_p)$ satisfies Hypothesis 11.3.1, $(R\text{-Flat}, \leq_p)$ is superstable and R-Flat is not first-order axiomatizable.

As mentioned in the introduction, the main focus of the paper is Question 11.1.1. The results of this section can be used to characterized those rings for which all AECs closed under direct sums and direct summands are superstable.

Lemma 11.3.25. *Let R be a ring. The following are equivalent.*

1. *R is left pure-semisimple.*

2. *Every AEC $\mathbf{K} = (K, \leq_p)$ with $K \subseteq R\text{-}Mod$, such that K is closed under direct sums and direct summands, is superstable.*

Proof. The backward direction follows from Theorem 11.3.21 as (R-Mod, $\leq_p$) satisfies (2). We show the forward direction.

Let K be a class satisfying (2), then $\mathbf{K}$ is closed under pure-injective envelopes as every module is pure-injective by the hypothesis on the ring. Hence, $\mathbf{K}$ satisfies Hypothesis 11.3.1. Therefore, $\mathbf{K}$ is superstable by Theorem 11.3.15.(4). □

The next well-known ring theoretic result follows from the above lemma, Theorem 11.3.18 and [Ch. 9, 3.15].

Corollary 11.3.26. *Assume R is an associative ring with unity. If R is left pure-semisimple, then R is left noetherian and left perfect.*

We finish this subsection by applying the technology developed in this section to integral domains. Given an integral domain R, we study the class of divisible R-modules, denoted by R-Div, and the class torsion-free R-modules, denoted by R-TF. A module M is a divisible R-module if for every $m \in M$ and $r \neq 0 \in R$, there is $n \in M$ such that $rn = m$. While a module M is a torsion-free R-module if for every $m \neq 0 \in M$ and every $r \neq 0 \in R$, $rm \neq 0$. It is easy to show that (R-Div, $\leq_p$) and (R-TF, $\leq_p$) both satisfy Hypothesis 11.3.1, this is the case as they are both definable classes in the sense of Example 11.3.2.(6).

Lemma 11.3.27. *Let R be an integral domain.*

1. *R is a dedekind domain if and only if (R-Div, $\leq_p$) is superstable.*
2. *R is a field if and only if (R-TF, $\leq_p$) is superstable.*

Proof.

1. ⇒: Since R is a dedekind domain, every divisible R-module is injective by [Rot09, 4.24]. As injective modules are pure-injective, (R-Div, $\leq_p$) is superstable by Theorem 11.3.15.

   ⇐: Recall that the class of h-divisible R-modules is contained in the class of divisible R-modules. Then every h-divisible R-module is pure-injective by Theorem 11.3.15. Therefore, R is a dedekind domain by [Sal07, 2.5].

2. ⇒: If R is a field, clearly R is a Prüfer domain. So the class of flat modules is the same as the class of torsion-free modules by [Rot09, 4.35]. Then (R-TF, $\leq_p$) is superstable since R is perfect and by [Ch. 9, 3.15].

   ⇐: It follows from Theorem 11.3.15 and [Sal07, 2.3] that R is a Prüfer domain. So, as before, the class of flat modules is the same as the class of torsion-free modules. Then R is left perfect by [Ch. 9, 3.15]. Therefore, R is a field by [Sal11, 2.3].

□

11.4 Classes closed under pure epimorphic images

In this section we study classes closed under direct sums, pure submodules and pure epimorphic images. We show that they are always stable. The proof is different to that of the previous section as we first show the existence of a weakly stable independence relation with local character and from it we obtain the stability cardinals.

Let us introduce the hypothesis for this section.

Hypothesis 11.4.1. Let $\mathbf{K} = (K, \leq_p)$ be an AEC with $K \subseteq R$-Mod for a fixed ring R such that:

1. K is closed under direct sums.
2. K is closed under pure submodules.
3. K is closed under pure epimorphic images.

Below we give some examples of classes of modules satisfying Hypothesis 11.4.1.

Example 11.4.2. Our main source of examples are F-classes. These were introduced in [PRZ94] and studied in detail in [HeRo09]. Let us recall that an F-class is a class of modules axiomatizable by formulas of the form:

$$\forall \bar{x}(\phi \to \bigvee \Psi).$$

Where ϕ is a *pp*-formula and Ψ is a collection of *pp*-formulas (possibly infinite) such that Ψ is closed under addition and $\psi[M] \subseteq \phi[M]$ for every $\psi \in \Psi$ and M an R-module.

It follows from [HeRo09, 2.3] that every F-class is closed under direct sums, pure submodules and pure epimorphic images. Moreover, it is clear that F-classes with pure embeddings are AECs. Therefore, every F-class satisfies Hypothesis 11.4.1.

Some interesting examples of F-classes are[3]:

1. $(R\text{-Flat}, \leq_p)$ where R-Flat is the class of flat left R-modules. A module M is flat if $(-) \otimes M$ is an exact functor.
2. $(\text{p-grp}, \leq_p)$ where p-grp is the class of abelian p-groups for p a prime number. A group G is a p-group if every element $g \neq 0$ has order p^n for some $n \in \mathbb{N}$.
3. $(\text{Tor}, \leq_p)$ where Tor is the class of torsion abelian groups. A group G is a torsion group if every element $g \neq 0$ has finite order.

[3]All of these examples are presented in [Roth1] and there it is explained why they are F-classes.

4. ($\mathfrak{s}$-Tor, $\leq_p$) where s-Tor is the class of $\mathfrak{s}$-torsion R-modules in the sense of [MaRu20]. A module M is an $\mathfrak{s}$-torsion module if it satisfies:

$$\forall x(x = x \to \bigvee_{\psi(R)=0,\, \psi \in pp\text{-formula}} \psi)$$

   This model theoretic description is obtained in [Roth1, 3.6].

5. $(\chi, \leq_p)$ where χ is a definable category of modules in the sense of [Pre09, §3.4].

Remark 11.4.3. It is worth mentioning that none of the above examples are first-order axiomatizable with the exception of the last one.

Remark 11.4.4. (R-AbsP, $\leq_p$) and (RTF, $\leq_p$) both satisfy Hypothesis 11.3.1, but do not satisfy Hypothesis 11.4.1. If either class satisfied Hypothesis 11.4.1, then they would be first-order axiomatizable by [Pre09, 3.4.7], which we know is not the case.

On the other hand, (R-Flat, $\leq_p$), (p-grp, $\leq_p$) and (Tor, $\leq_p$) satisfy Hypothesis 11.4.1, but do not satisfy Hypothesis 11.3.1. The case of flat modules is well-known and for torsion groups see [Ch. 6, 3.1].

Therefore, the classes of modules satisfying Hypothesis 11.3.1 are not contained in those satisfying Hypothesis 11.4.1 and vice versa. Definable classes satisfy both of the hypothesis, but there are non-definable classes as well (see Example 11.3.2.(5)).

11.4.1 Stability

We begin by recalling some important properties of pushouts in the category of R-modules with morphisms, we denote this category by R-Mod.

Remark 11.4.5.

- Given a pair of morphisms ($f_1 : M \to N_1, f_2 : M \to N_2$) in R-Mod, a *pushout* is a triple (P, g_1, g_2) with $g_1 \circ f_1 = g_2 \circ f_2$ that is a solution to the universal property that for every (Q, h_1, h_2) such that $h_1 \circ f_1 = h_2 \circ f_2$, there is a unique $t : P \to Q$ making the following diagram commute:

$$\begin{tikzcd}
 & & Q \\
N_1 \arrow[r, "g_1"] \arrow[urr, bend left, "h_1"] & P \arrow[ur, "t"] & \\
M \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\urcorner"] & N_2 \arrow[u, "g_2"'] \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}$$

- The pushout of a pair of morphisms $(f_1 : M \to N_1, f_2 : M \to N_2)$ in R-Mod is given by:

  $(P = N_1 \oplus N_2 / \{(f_1(m), -f_2(m)) : m \in M\},\ g_1 : n_1 \mapsto [(n_1, 0)],\ g_2 : n_2 \mapsto [(0, n_2)])$.

  Moreover, for every (Q, h_1, h_2) such that $h_1 \circ f_1 = h_2 \circ f_2$, we have that $t : P \to Q$ is given by $t([(n_1, n_2)]) = h_1(n_1) + h_2(n_2)$.

- ([Pre09, 2.1.13]) If $(f_1 : M \to N_1, f_2 : M \to N_2)$ are a pair of pure embeddings in R-Mod and (P, g_1, g_2) is a pushout, then g_1 and g_2 are pure embeddings.

The next result will be useful to study classes under Hypothesis 11.4.1.

Lemma 11.4.6. *Let $K \subseteq R$-Mod be closed under finite direct sums, pure submodules and isomorphisms, then the following are equivalent:*

1. *K is closed under pushouts of pure embeddings in R-Mod, i.e., if $M, N_1, N_2 \in K$, $f_1 : M \to N_1$ is a pure embedding, $f_2 : M \to N_2$ is a pure embeddings and P is the pushout of (f_1, f_2) in R-Mod, then $P \in K$.*

2. *K is closed under pure epimorphic images.*

Proof. $\Rightarrow$: Assume that the following is a pure-exact sequence:

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{g} C \longrightarrow 0$$

with $B \in K$. As $A \leq_p B$ and K is closed under pure submodules, it follows that $A \in K$. Then by hypothesis we have $B \oplus B/\{(a, -a) : a \in A\} \in K$ because this is the pushout of $(A \hookrightarrow B, A \hookrightarrow B)$.

Define $f : B/A \to B \oplus B/\{(a, -a) : a \in A\}$ by $f(b + A) = (b, -b) + \{(a, -a) : a \in A\}$. It is easy to check that f is a pure embeddings. As K is closed under pure submodules, this implies that $B/A \in K$. Hence $C \in K$.

$\Leftarrow$: Let $A \leq_p B, C$ be a span with $A, B, C \in K$. Observe that $B \oplus C/\{(a, -a) : a \in A\}$ is the pushout of $(A \hookrightarrow B, A \hookrightarrow C)$. Since K is closed under direct sums $B \oplus C \in K$ and it is straightforward to show that $\pi : B \oplus C \to B \oplus C/\{(a, -a) : a \in A\}$ is a pure epimorphism. Therefore, $B \oplus C/\{(a, -a) : a \in A\} \in K$. $\square$

Corollary 11.4.7. *If* $\mathbf{K}$ *satisfies Hypothesis 11.4.1, then* K *is closed under pushouts of pure embeddings in* R*-Mod , i.e., if* $M, N_1, N_2 \in K$*,* $f_1 : M \to N_1$ *is a pure embedding,* $f_2 : M \to N_2$ *is a pure embeddings and* P *is the pushout of* (f_1, f_2) *in* R*-Mod, then* $P \in K$*.*

From the corollary above and closure under direct sums it is clear that if a class satisfies Hypothesis 11.4.1, then it has joint embedding, amalgamation and no maximal models. We record this result for future reference.

Lemma 11.4.8. *If* $\mathbf{K}$ *satisfies Hypothesis 11.4.1, then* $\mathbf{K}$ *has joint embedding, amalgamation, no maximal models and* $\mathrm{LS}(\mathbf{K}) = |R| + \aleph_0$*.*

Our proof that $\mathbf{K}$ is stable under Hypothesis 11.4.1 is longer than that under Hypothesis 11.3.1. This is the case as we do not know if Galois-types and *pp*-types can be identified under Hypothesis 11.4.1.[4] The way we proceed is by defining an independence relation in the sense of Subsection 2.2 and showing that it is a weakly stable independence relation with local character.

Definition 11.4.9. Assume $\mathbf{K}$ is an AEC satisfying Hypothesis 11.4.1. $(f_1, f_2, h_1, h_2) \in \smile$ if and only if all the arrows of the outer square are pure embeddings and the unique map $t : P \to Q$ is a pure embedding:

$$\begin{tikzcd}
 & & Q \\
N_1 \arrow[r, "g_1"] \arrow[urr, bend left, "h_1"] & P \arrow[ur, "t"] & \\
M \arrow[u, "f_1"] \arrow[r, "f_2"'] \arrow[ur, phantom, "\urcorner"] & N_2 \arrow[u, "g_2"'] \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}$$

Remark 11.4.10. The definition given above is an instance of [LRV1b, 2.2] where their $\mathcal{K}$ is the category K with morphisms and $\mathcal{M}$ is the class of pure embeddings. Observe that $(\mathcal{K}, \mathcal{M})$ might not be cellular in the sense of [LRV1b] as $\mathcal{K}$ might not be cocomplete.

[4]For torsion groups and p-groups this can be done, see [Ch. 6, 3.4, 4.5].

Even without the hypothesis that $(\mathcal{K}, \mathcal{M})$ is cellular, one can show as in [LRV1b] that $\mathop{\smile\!\!\!\!\!\!|}$ is a weakly stable independence relation in $\mathbf{K}$ under Hypothesis 11.4.1. The key result is Corollary 11.4.7.

Fact 11.4.11 ([LRV1b, 2.7])**.** If $\mathbf{K}$ satisfies Hypothesis 11.4.1, then $\mathop{\smile\!\!\!\!\!\!|}$ is a weakly stable independence relation.

Notation 11.4.12. Given $\mathop{\smile\!\!\!\!\!\!|}$ an independence relation on an AEC, recall that one writes $M_1 \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$ if $M \leq_{\mathbf{K}} M_1, M_2 \leq_{\mathbf{K}} N$ and $(i_1, i_2, j_1, j_2) \in \mathop{\smile\!\!\!\!\!\!|}$ where $i_1 : M \to M_1, i_2 : M \to M_2, j_1 : M_1 \to N, j_2 : M_2 \to N$ are the inclusion maps.

The next result will be essential to describe the stability cardinals.

Theorem 11.4.13. *If $\mathbf{K}$ satisfies Hypothesis 11.4.1, then $\mathop{\smile\!\!\!\!\!\!|}$ has local character. More precisely, if $M_1, M_2 \leq_p N$, then there are $M_1', M_0 \in K$ such that $M_0 \leq_p M_1', M_2 \leq_p N$, $M_1 \leq_p M_1'$, $\|M_0\| \leq \|M_1\| + |R| + \aleph_0$ and $M_1' \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$.*

Proof. Let $M_1, M_2 \leq_p N$. We build two increasing continuous chains $\{M_{0,i} : i < \omega\}$ and $\{M_{1,i}' : i < \omega\}$ such that:

1. $M_{1,0}' = M_1$.
2. $M_{0,i} \leq_p M_{1,i+1}', M_2 \leq_p N$.
3. $\|M_{0,i}\|, \|M_{1,i}'\| \leq \|M_1\| + |R| + \aleph_0$.
4. If $\bar{a} \in M_{1,i}'$, $\phi(\bar{x}, \bar{y})$ is a *pp*-formula and there is $\bar{m} \in M_2$ such that $N \vDash \phi[\bar{a}, \bar{m}]$, then there is $\bar{l} \in M_{0,i}$ such that $N \vDash \phi[\bar{a}, \bar{l}]$.

$\boxed{\text{Construction}}$ Base: Let $M_{1,0}' = M_1$. For each $\bar{a} \in M_1$ and $\phi(\bar{x}, \bar{y})$ a *pp*-formula, if there is $\bar{m} \in M_2$ such that $N \vDash \phi[\bar{a}, \bar{m}]$ let $\bar{m}_\phi^{\bar{a}}$ be such a witness in M_2 and $\bar{0}$ otherwise. Let $M_{0,0}$ be the structure obtained by applying Downward Löwenheim-Skolem to $\bigcup\{\bar{m}_\phi^{\bar{a}} : \bar{a} \in M_1 \text{ and } \phi \text{ is a } pp\text{-formula}\}$ in M_2. It is easy to see that $M_{0,0}$ satisfies what is needed.

Induction step: Let $M_{1,i+1}'$ be the structured obtained by applying Downward Löwenheim-Skolem to $M_{0,i}$ in N. Construct $M_{0,i+1}$ as we constructed $M_{0,0}$, but replacing M_1 by $M_{1,i}'$.

$\boxed{\text{Enough}}$ Let $M_0 = \bigcup_{i<\omega} M_{0,i}$ and $M_1' = \bigcup_{i<\omega} M_{1,i}'$. Observe that $\|M_0\| \leq \|M_1\| + |R| + \aleph_0$ and we show that $M_1' \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$.

Recall that the pushout in R-Mod is given by:

$$
\begin{tikzcd}
M_1' \arrow[r] & M_1' \oplus M_2/\{(m,-m) : m \in M_0\} \\
M_0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & M_2 \arrow[u]
\end{tikzcd}
$$

Moreover, $t : M_1' \oplus M_2/\{(m,-m) : m \in M_0\} \to N$ is given by $t([(m,n)]) = m+n$. So we are left to show that t is a pure embedding.

We begin by proving that t is an embedding, so assume that $m_1 + n_1 = m_2 + n_2$ with $m_i \in M_1'$ and $n_i \in M_2$ for $i \in \{1,2\}$. Then $N \vDash x - y = z[m_1, m_2, n_2 - n_1]$, so by condition (3) of the construction there is $m \in M_0$ such that $N \vDash x-y = z[m_1, m_2, m]$. Hence $[(m_1, n_1)] = [(m_2, n_2)]$ in the pushout.

We show that t is pure. Let $\exists \bar{x}\phi(\bar{x}, y)$ be a *pp*-formula such that $N \vDash \exists \bar{x}\phi(\bar{x}, y)[m+n]$ with $m \in M_1'$ and $n \in M_2$. So $N \vDash \exists w \exists \bar{x}(\phi(\bar{x}, w) \wedge w = z + z')[m, n]$. Observe that this is a *pp*-formula, $m \in M_1'$ and $n \in M_2$, then by condition (3) of the construction there is $p \in M_0$ such that $N \vDash \exists w \exists \bar{x}(\phi(\bar{x}, w) \wedge w = z+z')[m, p]$. So $N \vDash \exists \bar{x}\phi(\bar{x}, y)[m+p]$. Then as $M_1' \leq_p N$ there is $\bar{m}^\star \in M_1'$ such that

$$N \vDash \phi[\bar{m}^\star, m + p]. \tag{11.4.1}$$

As solutions to *pp*-formulas form a subgroup, it is easy to get that $N \vDash \exists \bar{x}\phi(\bar{x}, y)[n-p]$. Then as $M_2 \leq_p N$ there is $\bar{n}^\star \in M_2$ such that

$$N \vDash \phi[\bar{n}^\star, n - p]. \tag{11.4.2}$$

So by adding equation (1) and (2) we obtain that:

$$N \vDash \phi[\bar{m}^\star + \bar{n}^\star, m + n]. \tag{11.4.3}$$

Therefore, $t : M_1' \oplus M_2/\{(m,-m) : m \in M_0\} \to N$ is a pure embedding. $\square$

As presented in [LRV19, 8.2], it is possible to interpret an independence relation $\mathop{\vert\!\!\!\smile}$ as a relation on Galois-types.

Definition 11.4.14. Given $M \leq_p N \in K$, $\bar{a} \in N$ and $B \subseteq N$, we say that $\mathbf{tp}(\bar{a}/B; N)$ does not fork over M if and only if there are $M_1, M_2, N' \in K$ such that $\bar{a} \in M_1$, $B \subseteq M_2$, $N \leq_p N'$, $M \leq_p M_1, M_2 \leq_p N'$ and $M_1 \underset{M}{\overset{N'}{\mathop{\vert\!\!\!\smile}}} M_2$

The next result has some of the properties that the independence relation defined in Definition 11.4.9 has when seen as a relation on Galois-types.

Lemma 11.4.15. *Assume* **K** *satisfies Hypothesis 11.4.1. Then:*

1. *(Uniqueness) If $M \leq_p N$, $p, q \in \mathbf{gS}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p = q$.*

2. *(Local character) If $p \in \mathbf{gS}(M)$, then there is $N \leq_p M$ such that p does not fork over N and $\|N\| \leq |R| + \aleph_0$.*

Proof. (1) follows from Fact 11.4.11 and [LRV19, 8.5]. As for (2), this follows from Theorem 11.4.13 and [LRV19, 8.5]. □

With this we obtain the main result of this section.

Theorem 11.4.16. *Assume $\mathbf{K}$ satisfies Hypothesis 11.4.1. If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}$ is λ-stable.*

Proof. Let $M \in \mathbf{K}_\lambda$ with $\lambda^{|R|+\aleph_0} = \lambda$. Assume for the sake of contradiction that $|\mathbf{gS}(M)| > \lambda$ and let $\{p_i : i < \lambda^+\}$ be an enumerations without repetitions of types in $\mathbf{gS}(M)$.

By Lemma 11.4.15, for every $i < \lambda^+$, there is $N_i \leq_p M$ such that p_i does not fork over N_i and $\|N_i\| = |R| + \aleph_0$. Then by the pigeon hole principle and using that $\lambda^{|R|+\aleph_0} = \lambda$, we may assume that there is an $N \in K$ such that $N_i = N$ for every $i < \lambda^+$. Therefore, by uniqueness, there are $i \neq j < \lambda^+$ such that $p_i = p_j$. This is clearly a contradiction. □

The following improves the results of [LRV1b], where it is shown that the class of flat modules with pure embeddings is stable, by giving a precise descriptions of the cardinals where the class is stable. It also extends [Ch. 9, 4.6] where the same result is proved for those rings such that the pure-injective envelope of every flat module is flat.

Corollary 11.4.17. *If $\lambda^{|R|+\aleph_0} = \lambda$, then $(R\text{-}Flat, \leq_p)$ is λ-stable*

Moreover, by Theorem 11.4.16 and [Ch. 7, 3.20] we can conclude the existence of universal models.

Corollary 11.4.18. *Assume $\mathbf{K}$ satisfies Hypothesis 11.4.1. If $\lambda^{|R|+\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{|R|+\aleph_0} < \lambda)$, then $\mathbf{K}$ has a universal model of cardinality λ.*

Remark 11.4.19. The above result applied to the class of flat modules extends [Ch. 9, 4.6] which in turned extended [Sh820, 1.2]. On the other hand, the above result applied to the class of $\mathfrak{s}$-torsion modules extends [Ch. 6, 4.6].

Another result that follows from having an independence relation is that classes satisfying Hypothesis 11.4.1 are tame.

Lemma 11.4.20. *If $\mathbf{K}$ satisfies Hypothesis 11.4.1, then $\mathbf{K}$ is $(|R| + \aleph_0)$-tame.*

Proof. Follows from Lemma 11.4.15 and [LRV19, 8.16]. □

Since K has joint embedding, amalgamation and no maximal models, it follows from [Sh:h, §II.1.16] that $\mathbf{K}$ has a (λ, α)-limit model if $\lambda^{|R|+\aleph_0} = \lambda$ and $\alpha < \lambda^+$ is a limit ordinal. For classes satisfying Hypothesis 11.4.1, we do not know how limit models look like in general or if there is even a general theory as the one under Hypothesis 11.3.1. For the specific class of flat modules, it was shown that long limit models are cotorsion modules in [Ch. 9, 3.5].

Since we were not able to characterize limit models, we are not able to characterize superstability for classes satisfying Hypothesis 11.4.1. Again, for the class of flat modules this was done in [Ch. 9]. There it was shown that the class of flat left R-modules is superstable if and only if R is left perfect.

We are not sure if it is possible to obtain a result as Theorem 11.3.15 for classes satisfying Hypothesis 11.4.1, but we think that characterizing superstability in the class of $\mathfrak{s}$-torsion R-modules will have interesting algebraic consequences.

11.4.2 Classes satisfying Hypothesis 11.3.1

We briefly study those classes that satisfy Hypotheses 11.3.1 and 11.4.1. Recall that definable classes and Example 11.3.2.(5) are examples of classes satisfying both hypotheses.

Lemma 11.4.21. *If* $\mathbf{K}$ *satisfies Hypotheses 11.3.1 and 11.4.1, then* $\mathop{\smile\kern-0.65em\raise0.5ex\hbox{$|$}}$ *has the* $(< \aleph_0)$*-witness property. Moreover,* $\mathop{\smile\kern-0.65em\raise0.5ex\hbox{$|$}}$ *is a stable independence relation.*

Proof. By Corollary 11.3.6 we have $\mathbf{K}$ is fully $(< \aleph_0)$-tame. Then it follows from [LRV19, 8.8, 8.9] that $\mathop{\smile\kern-0.65em\raise0.5ex\hbox{$|$}}$ has the $(< \aleph_0)$-witness property. The moreover part follows from Fact 11.4.11 and Theorem 11.4.13. □

A natural question to ask is if the above results follows from Hypothesis 11.4.1.

Question 11.4.22. If $\mathbf{K}$ satisfy Hypothesis 11.4.1, is $\mathop{\smile\kern-0.65em\raise0.5ex\hbox{$|$}}$ a stable independence relation?

Remark 11.4.23. In the case of p-groups and torsion groups this is the case by [Ch. 6, 3.4, 4.5], Lemma 11.3.5 and doing a similar argument as that of Lemma 11.4.21.

The next assertion follows from the previous lemma and [LRV1b, 3.1]. For the notions not defined in this paper, the reader can consult [LRV1b].

Corollary 11.4.24. *Pure embeddings are cofibrantly generated in the class of R-modules, i.e., they are generated from a set of morphisms by pushouts, transfinite composition and retracts.*

Proof. Observe that the class of left R-modules with pure embeddings satisfies Hypotheses 11.3.1 and 11.4.1, then by Lemma 11.4.21 $\downarrow$ is a stable independence relation. Since R-Mod with pure embeddings is an accessible cellular category which is retract-closed, coherent and $\aleph_0$-continuous. Therefore, pure embeddings are cofibrantly generated by [LRV1b, 3.1]. $\square$

Remark 11.4.25. The main result of [LPRV20] is that the above result holds in locally finitely accessible additive categories. Their proof is very different from our proof as they use categorical methods.

11.5 Classes that admit intersections

In this section we study classes that admit intersections and their subclasses. We use the ideas of this section to provide a partial solution to Question 11.1.1 for AECs of torsion-free abelian groups. Moreover, we give a condition that implies a positive solution to Question 11.1.1.

Definition 11.5.1. Let $\mathbf{K} = (K, \leq_p)$ and $\mathbf{K}^\star = (K^\star, \leq_p)$ be a pair of AECs with $K, K^\star \subseteq R$-Mod for a fixed ring R. We say $\mathbf{K}^\star$ is *closed below* $\mathbf{K}$ if the following hold:

1. $K^\star \subseteq K$.
2. K and $K^\star$ are closed under pure submodules.
3. $\mathbf{K}$ admits intersections, i.e., for every $N \in K$ and $A \subseteq |N|$ we have that $cl^N_{\mathbf{K}}(A) = \bigcap\{M \leq_p N : A \subseteq |M|\} \in K$ and $cl^N_{\mathbf{K}}(A) \leq_p N$.[5]

Example 11.5.2. The following classes are all closed below the class of torsion-free groups with pure embeddings:

1. $(TF, \leq_p)$ where TF is the class of torsion-free groups. A group G is torsion-free if every element has infinite order.
2. $(RTF, \leq_p)$ where RTF is the class of reduced torsion-free abelian groups. A group G is reduced if it does not have non-trivial divisible subgroups.

[5] Classes admitting intersections were introduced in [BaSh08, 1.2] and studied in detail in [Vas17c, §2].

3. ($\aleph_1$-free, $\leq_p$) where $\aleph_1$-free is the class of $\aleph_1$-free groups. A group G is $\aleph_1$-free if every countable subgroups is free.

4. ($B_0, \leq_p$) where B_0 is the class of finitely Butler groups. A group G is a finitely Butler group if G is torsion-free and every pure subgroup of finite rank is a pure subgroup of a finite rank completely decomposable group (see [Fuc15, §14.4] for more details).

5. (TF-l-cyc, $\leq_p$ p) where TF-l-cyc is the class of torsion-free locally cyclic groups. A group G is locally cyclic if every finitely generated subgroup is cyclic.

Remark 11.5.3. It is worth pointing out that the second, third and fifth example are not first-order axiomatizable. While the fourth one is probably not first-order axiomatizable.

Remark 11.5.4. The class of $\aleph_1$-free groups is closed below the class of torsion-free groups, but does not satisfy Hypothesis 11.3.1 or Hypothesis 11.4.1. This is the case as it does not have the amalgamation property. We showed that if a class satisfied either of the hypotheses then it had the amalgamation property (Lemma 11.3.4 and Lemma 11.4.8).

(R-Mod, $\leq_p$) satisfies Hypothesis 11.3.1 and Hypothesis 11.4.1, but it is not closed below any class of modules for most rings. For example, if $R = \mathbb{Z}$, this is the case as the class of abelian groups with pure embeddings does not admit intersections.

Therefore, there are classes studied in this section that do not satisfy Hypotheses 11.3.1 or 11.4.1 and there are classes satisfying those hypotheses that can not be handled with the methods of this section.

11.5.1 Stability

The proof of the next result is straightforward so we omit it.

Proposition 11.5.5. *If $\mathbf{K}^\star$ is closed below $\mathbf{K}$, then $\mathbf{K}^\star$ admits intersections. Moreover, for every $N \in \mathbf{K}^\star$ and $A \subseteq N$ we have that $cl^N_{\mathbf{K}}(A) = cl^N_{\mathbf{K}^\star}(A)$.*

With it we can show that there is a close relation between Galois-types in $\mathbf{K}$ and $\mathbf{K}^\star$.

Lemma 11.5.6. *Assume $\mathbf{K}^\star$ is closed below $\mathbf{K}$. Let $A \subseteq N_1, N_2 \in K^\star$, $\bar{a} \in N_1^{<\infty}$ and $\bar{b} \in N_2^{<\infty}$, then:*

$$\mathbf{tp}_{\mathbf{K}}(\bar{a}/A; N_1) = \mathbf{tp}_{\mathbf{K}}(\bar{b}/A; N_2) \textit{ if and only if } \mathbf{tp}_{\mathbf{K}^\star}(\bar{a}/A; N_1) = \mathbf{tp}_{\mathbf{K}^\star}(\bar{b}/A; N_2)$$

Proof. The backward direction is obvious so we prove the forward direction. Since $\mathbf{K}$ admits intersection, by [Vas17c, 2.18], there is $f : cl^{N_1}_{\mathbf{K}}(\bar{a} \cup A) \cong_M cl^{N_2}_{\mathbf{K}}(\bar{b} \cup A)$ with $f(\bar{a}) = \bar{b}$. Then using the proposition above we have that $cl^{N_1}_{\mathbf{K}}(\bar{a} \cup A) = cl^{N_1}_{\mathbf{K}^\star}(\bar{a} \cup A)$ and $cl^{N_2}_{\mathbf{K}}(\bar{b} \cup A) = cl^{N_2}_{\mathbf{K}^\star}(\bar{b} \cup A)$. So the result follows from the fact that $\mathbf{K}^\star$ admits intersections and [Vas17c, 2.18]. $\square$

From that characterization we obtain the following.

Corollary 11.5.7. *Assume $\mathbf{K}^\star$ is closed below $\mathbf{K}$.*

1. *Let $\lambda \geq \mathrm{LS}(\mathbf{K}^\star)$. If $\mathbf{K}$ is λ-stable, then $\mathbf{K}^\star$ is λ-stable.*
2. *Let λ be an infinite cardinal. If $\mathbf{K}$ is $(< \lambda)$-tame, then $\mathbf{K}^\star$ is $(< \lambda)$-tame.*

Using the above result we are able to answer Question 11.1.1 in the case of AECs of torsion-free abelian groups closed under pure submodules and with arbitrary large models.

Lemma 11.5.8. *If $\mathbf{K} = (K, \leq_p)$ is an AEC closed under pure submodules and with arbitrary large models such that $K \subseteq TF$, then $\mathbf{K}$ is λ-stable for every infinite cardinal λ such that $\lambda^{\aleph_0} = \lambda$.*

Remark 11.5.9. The above result applies in particular to reduced torsion-free groups, $\aleph_1$-free groups and finitely Butler groups. The result for reduced torsion-free groups is in [Sh820, 1.2], for $\aleph_1$-free groups is in [Ch. 6, 2.9] and for finitely Butler groups is in [Ch. 5, 5.9].

We see the next result as a first approximation to Question 11.1.1. Recall that a ring R is Von Neumann regular if and only if for every $r \in R$ there is an $s \in R$ such that $r = rsr$ if and only if every left R-modules is absolutely pure (see for example [Pre09, 2.3.22]).

Lemma 11.5.10. *Assume R is a Von Neumann regular ring. If K is closed under submodules and has arbitrarily large models, then $\mathbf{K} = (K, \leq_p)$ is λ-stable for every infinite cardinal λ such that $\lambda^{|R|+\aleph_0} = \lambda$.*

Proof. We show that $\mathbf{K}$ is closed below $(R\text{-Mod}, \leq_p)$. Observe that the only things that need to be shown are that $(R\text{-Mod}, \leq_p)$ admits intersections and that K is closed under pure submodules. This is the case as every module is absolutely pure by the hypothesis on the ring. $\square$

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