Model Theory of the Exponentiation: Abstract

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In the following document we present an abstract in english of my thesis *Model Theory of the Exponentiation*.

1. Abstract

We present a comprenhensive introduction to Zilber exponential fields as an approach to solving Schanuel's Conjecture.

Schanuel's Conjecture. (1966) Given $z_1, ..., z_n$ complex numbers $\mathbb{Q}-$linearly independent then amongst the following $2n$ numbers:

$$z_1, ..., z_n, exp(z_1), ..., exp(z_n)$$

there are at least n which are algebraically independent [3].

The document is divided into five chapters as follows.

1.1. Preliminaries

In this chapter we introduce the minimum requirements needed to understand the thesis. The main topics that we introduce are: $L_{\omega_1,\omega}(Q)$ logic, Model Theory on $L_{\omega_1,\omega}$ and Algebraic Geometry. In the first section, we give basic definitions such as satisfaction and prove that $L_{\omega_1,\omega}(Q)$ logic is quite different by observing that Compactness fails. In the second section, we introduce models and ways to compare

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them such as $back-and-forth$ systems and elementary equivalence, give a complete proof of Scott Theorem and introduce what it is understood by definability of a set and of a property of a set. Finally in the third section, basic concepts of Algebraic Geometry are introduced and Zariski topology is studied.

1.2. Axiomatization of Pseudo-exponentiation

In this chapter following [4] we introduce the fundamental concepts of Zilber fields, the axioms of Zilber fields and we prove that every axiom (except the countable closure property) is expressible in $L_{\omega_1,\omega}$ and that the countable closure property is expressible in $L_{\omega_1,\omega}(Q)$. What it is understood by a Zilber field is the content of the following definition.

Zilber fields. A structure $\langle \mathfrak{F}, E\rangle$ is a Zilber field if:

1. $\mathfrak{F}$ is an algebraically closed field of characteristic zero.
2. E is an epimorphism from the additive group of $\mathfrak{F}$ to the multiplicative group of $\mathfrak{F}$ such that $Ker(E) \cong \mathbb{Z}$.
3. The Schanuel property holds, i.e., if $X \subseteq_{fin} F$ then:

   $$tr(X \cup E(X)) - lin(X) \geq 0.$$

   Where $tr(X)$ is the trascendence degree of X over $\mathbb{Q}$ and $lin(X)$ is the linear dimensión of the $\mathbb{Q}-$linear space generated by X.
4. The exponential algebraic closure $Ecl_{\mathfrak{F}}$ of every finite set is at most countable.
5. Every variety which is "big enough"(that means they satisfy the technical conditions of being rotund and free) has generic realizations over finite sets.

One of the main difficulties of this chapter is to write everything in $L_{\omega_1,\omega}(Q)$. See [8].

1.3. Zilber fields exist

In this chapter we construct a Zilber field from the bottom up. We start with the field $\mathbb{Q}$ with $Graph(E) = \{(0, 1)\}$ and with the help of Kirby's article [5] we finish with $\mathbb{Q}(\tau)^\sharp$ a countable Zilber field where τ is trascendental over $\mathbb{Q}$ (plays the role of $2\pi i$ in the complex case) and $\sharp$ corresponds to the further completion process. The completion process is divided in to two basic steps.

In the first step, we construct an algebraically closed field with *standard* exponentation that satisfies the Schanuel Conjecture through recursive application of three operations e, l, a in a way that the extension is *strong*. With e we extend the domain of the exponentation, with l we make the exponentation surjective and with a we close algebraically. At the end we have a field that satisfies the first four axioms. One very important fact that we have only proved partially is that the countable model obtained through this step is unique up to isomorphism.

In the second step, we add generic realizations to the "big enough"varieties in a way that the first four axioms still hold and moreover the fifth axiom hold. This part is the trickiest of all the construction and is the place where we see the necessity of the concepts of freeness and rotundity.

1.4. Quasiminimal Excellent Classes

In this chapter we introduce the concept of *quasiminimal excellent class* and prove the following categoricity theorem:

Theorem. If $\mathcal{C}$ is a quasiminimal excellent class defined in $L_{\omega_1,\omega}$ (except the countable closure proprety), the closure operator is defined in $L_{\omega_1,\omega}$ and has an infinite dimensional model then $\mathcal{C}$ is uncountable categorical.

Apart from the introduction given to pregeometries, this chapter is divided basically in to two parts following Kirby's article [6] and Baldwin's book [7]. In the first, we prove that in case a model of κ uncountable cardinal exists, then the model is unique, the proof is done first in the case of dimension $\aleph_1$ using $\aleph_0-$homogeneity and then in the case when the dimension is bigger than $\aleph_1$. In the latter, I discovered a small gap in the published proof but Kirby was able to solve it when asked.

In the second part, we prove that there are models of every uncountable cardinal. The way we do this is by taking a direct limit of structures in the class and constructing a chain in a way similar to that proposed by Kirby.

1.5. Zilber fields are uncountably categorical

In this final chapter, we prove that Zilber fields make up a quasiminimal excellent class, to do so we prove condition one of the definition of quasiminimal excellent class (see [8]) and then prove that $\aleph_0-$homogeneity holds following to some extent [4] and [1]. In this section we use the strong theorem that asserts that the quasiminimal excellence axiom holds once the conditions stated above hold[1] and finish with Zilber's conjecture.

Zilber's Conjecture. The countable Zilber field of the size of the continuum is isomorphic to the complex numbers with exponentation.

In particular, Zilber's conjecture implies Schanuel's Conjecture.

Note

The actual thesis is about 150 pages long and the only big detail missing is the proof that Zilber fields satisfy the second condition of $\aleph_0-$homogeneity in the general case. The idea is to finish the work by January, submit the thesis by February and to present it by April.

Referencias

[1] Martin Bays y Jonathan Kirby. Excellence and uncountable categoricity of Zilbers exponential fields. Submited, 2013.

[2] Jonathan Kirby. A Note on the Axioms of Zilbers Pseudo-Exponential Fields. *Notre Dame Journal of Formal Logic*, 54(3-4): 509-520,2013.

[3] Michel Waldschmidt. Diophantine Aproximation on Linear Algebraic Groups: Trascendence Properties of the Exponential Function in Several Variables. Springer: 1-21 , 2000.

[4] Boris Zilber. Pseudo-exponentation on algebraically closed fields of characteristic zero. *Annals of Pure and Applied Logic*, 132 (1): 67-95,2005.

[5] Jonathan Kirby. Finitely presented exponential fields. *Algebra and Number Theory* 7(4): 943-980, 2013.

[6] Jonathan Kirby. On Quasiminimal Excellent Classes. The Journal of Symbolic Logic, 75: 551-564, 2010.

[7] John T. Baldwin. Categoricity. University Lecture Series, AMS (50): 233, 2009.

[1] This theorem and the Thumbtack Lemma are the only two facts that are stated in the thesis but not proved.

[8] Marcos Mazari Armida. *Teoría de Modelos de la exponencial* (Model Theory of the Exponentation), Undergraduate thesis, UNAM, 2014.